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On the Partially Coherent Near and Far Field Diffraction

In the paper the partially coherent diffraction in the case of both near field and far field approximations is investigated. The concept of quasi-stationarity of the mutual coherence function is presented.

The generalized Schell's theorem is formulated. The latter expresses the intensity distribution in the partially coherent diffraction pattern by the convolution of the corresponding intensity distribution in coherent diffraction pattern and the Fourier transform of the mutual coherence function.

1. Introductory remarks

In the paper the diffraction of partially coherent light under paraxial approximation is dealt by applying the Kirchhoff-Fresnel diffraction integrals and assuming the quasi-monochromacy. This implies that it is enough to deal with the spatial parts of the optical signal $U(\boldsymbol{P})$ only. Thus the Mutual Coherence Function (MCF) can be defined [1] as

$$\Gamma(\boldsymbol{P}', \boldsymbol{P}'') = \langle U(\boldsymbol{P}')\cdot U^*(\boldsymbol{P}'')\rangle. \quad (1)$$

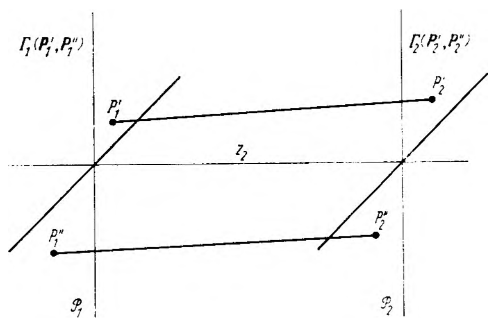

Fig. 1. Propagation of the MCF in a free space

According to simplified notation used in this equation and throughout the whole text (see Fig. 1) the point in the $\mathscr{P}$ plane is denoted by $P = (x, y)$ and described by radius vector $\boldsymbol{P}$

$$\boldsymbol{P} = [x, y].$$

The distance between two points P', P'' is

$$\boldsymbol{W} = \boldsymbol{P}'-\boldsymbol{P}'' = [x'-x'', y'-y''] \quad (2a)$$

similarly:

$$\boldsymbol{U} = \boldsymbol{P}'+\boldsymbol{P}'' = [x'+x'', y'+y'']. \quad (2b)$$

Scalar multiplication gives

$$\boldsymbol{P}'\cdot\boldsymbol{P}'' = x'x''+y'y'',$$

so

$$\boldsymbol{P}^2 = x^2+y^2. \quad (2c)$$

By differential $d\boldsymbol{P}$ we mean

$$d\boldsymbol{P} = dx\cdot dy. \quad (2d)$$

The following simplified models of optical elements are applied:

The diffraction screen is assumed to be a transparency of amplitude transmittance $t(\boldsymbol{P})$:

$$t(\boldsymbol{P}) = \begin{cases} t(\boldsymbol{P}) & \text{inside the diffraction aperture } \Sigma \\ 0 & \text{outside } \Sigma \end{cases} \quad (3)$$

Lens is treated as a thin, nonaberrant phase transparency of transmittance $t_L(\boldsymbol{P})$:

$$t_L(\boldsymbol{P}) = \exp\left(\frac{-ik}{2f}\boldsymbol{P}^2\right). \quad (4)$$

This approximation allows to extend to infinity the region of integration in the Kirchhoff-Fresnel integrals.

2. Quasi-stationarity

When discussing the propagation of the MCF, as well as the diffraction or imaging in partially coherent light it is usually assumed that the MCF is spatially stationary, i.e. it has the form [2], [3]:

$$\Gamma(\boldsymbol{P}', \boldsymbol{P}'') = \Gamma(\boldsymbol{P}'-\boldsymbol{P}''). \quad (5)$$

Such an assumption is not true in general. To justify this statement let us consider the propagation of the MCF in a free space (Fig. 1).

Under paraxial approximation we can write [2]:

$$\Gamma_2(\boldsymbol{P}_2', \boldsymbol{P}_2'') = \frac{\exp\left[\frac{ik}{2z_2}(\boldsymbol{P}_2'^2-\boldsymbol{P}_2''^2)\right]}{\lambda^2 z_2^2}\iint \Gamma_1(\boldsymbol{P}_1', \boldsymbol{P}_1'')\times$$
$$\times\exp\left[\frac{ik}{2z_2}(\boldsymbol{P}_1'^2-\boldsymbol{P}_1''^2)\right]\times$$
$$\times\exp\left[-\frac{ik}{z_2}(\boldsymbol{P}_1'\boldsymbol{P}_2'-\boldsymbol{P}_1''\boldsymbol{P}_2'')\right]d\boldsymbol{P}_1'\,d\boldsymbol{P}_2'', \quad (6)$$

where $k = 2\pi/\lambda$.

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Assuming stationarity (5) and changing the variables

$$P_1'-P_1'' = U,$$
$$P_1'+P_1'' = W \qquad (7)$$

the equation (6) becomes:

$$\Gamma_2(P_2',P_2'') = \frac{\exp\left[\frac{ik}{2z_1}(P_2'^2-P_2''^2)\right]}{2\lambda_2 z_2^2}\iint \Gamma_1(U)\exp\left(\frac{ik}{2z_2}UW\right)\exp\left\{-\frac{ik}{2z_2}[P_2'(W+U)-P_2''(W-U)]\right\}dWdU$$

$$= \frac{\exp\left[\frac{ik}{2z_2}(P_2'^2-P_2''^2)\right]}{2\lambda^2 z_2^2}\int \Gamma_1(U)\exp\left[-\frac{ik}{2z_2}U(P_2'+P_2'')\right]\int \exp\left[-2\pi W\left(\frac{P_2'-P_2''-U}{2\lambda z_2}\right)\right]dWdU$$

$$= \frac{\exp\left[\frac{ik}{2z^2}(P_2'^2-P_2''^2)\right]}{2\lambda^2 z_n^2}\int \Gamma_1(U)\exp\left[-\frac{ik}{2z_2}U(P_2'+P_2'')\right]\delta\left(\frac{P_2'-P_2''-U}{2\lambda z_2}\right)dU = \frac{1}{\lambda z_2}\Gamma_1(P_2'-P_2''). \qquad (8)$$

The last line can be expressed in the form:

$$\Gamma_2(P_2'-P_2'') = C\Gamma_1(P_2'-P_2''). \qquad (8a)$$

This equation states that apart of the multiplicative constant the MCF does not change. Such a constancy of the MCF during propagation is in contradiction to the experiment.

Instead of stationarity let us assume that the MCF is *quasi-stationary* in space, i.e. it has the form:

$$\Gamma(P',P'') = \mathring{\Gamma}(P'-P'')\exp\left[\frac{ik}{2z}(P'^2-P''^2)\right]. \qquad (9)$$

Thus, the MCF consists of a stationary part $\mathring{\Gamma}$ and a quadratic phase factor. This form is suggested by a well known form of the MCF [2] generated by a completely incoherent, flat source S of the intensity distribution $I_S(P)$ (Fig. 2).

$$\Gamma_2(P_2',P_2'') = \frac{\exp\left[\frac{ik}{2z_2}(P_2'^2-P_2''^2)\right]}{\lambda^2 z_2^2} I_S(P_1)\times$$
$$\times\exp\left[\frac{-ik}{z_1}P_1(P_2'-P_2'')\right]dP_1 \qquad (10)$$

The assumption of quasi-stationarity leads to the results consistent with the experiment. As an example

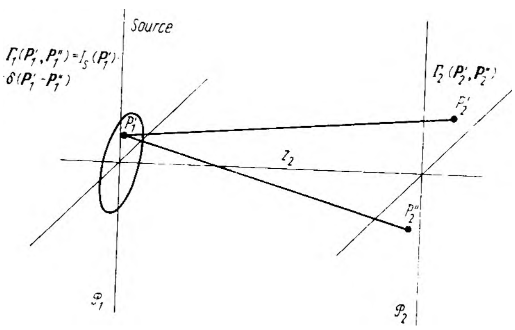

Fig. 2. Generation of the partially coherent field by the flat incoherent source

let us repeat the above calculations in the case of free propagation of the MCF.

Inserting (9) into (6) and changing variables in analogous manner we can obtain:

$$\Gamma_2(P_2',P_2'') = \frac{\exp\left[\frac{ik}{2z_2}(P_2'^2-P_2''^2)\right]}{\lambda^2 z_2^2}\iint \mathring{\Gamma}_1(P_1'-P_1'')\exp\left[\frac{ik}{2}\left(\frac{1}{z_1}+\frac{1}{z_2}\right)\times\right.$$
$$\left.\times(P_1'^2-P_1''^2)\right]\exp\left[-\frac{ik}{z_2}(P_1'P_2'-P_1''P_2'')\right]dP_1'dP_1''$$

$$= \frac{\exp\left[\frac{ik}{2z_2}(P_2'^2-P_2''^2)\right]}{2\lambda^2 z_2^2}\int \Gamma_1(U)\exp\left[-\frac{ik}{2z_2}U(P_2'+P_2'')\right]\int \exp\left[-\frac{ik}{2z_2}W\left(P_2'-P_2''-U\frac{z_1+z_2}{z_1 z_2}\right)\right]dWdU$$

$$= \frac{z_1}{(z_1+z_2)z_2\lambda}\mathring{\Gamma}\left[\frac{z_1}{z_1+z_2}(P_2'-P_2'')\right]\exp\left[\frac{ik}{2(z_1+z_2)}(P_2'^2-P_2''^2)\right]. \qquad (11)$$

Equation (11) shows that in free propagation the quasi-stationarity is conserved. In this case the form of the MCF remains constant, but the function is changed in scale.

Propagation through a lens does not change a character of the MCF either, since this process causes only multiplying by a quadratic phase factor:

$$\Gamma_{\text{a.l.}}(\boldsymbol{P}', \boldsymbol{P}'') = \Gamma_{\text{b.l.}}(\boldsymbol{P}', \boldsymbol{P}'')t_L(\boldsymbol{P}')t_L^*(\boldsymbol{P}'') = \Gamma_{\text{b.l.}}(\boldsymbol{P}', \boldsymbol{P}'')\exp\left[-\frac{ik}{2f}(\boldsymbol{P}'^2-\boldsymbol{P}''^2)\right] = \mathring{\Gamma}_{\text{b.l.}}(\boldsymbol{P}'-\boldsymbol{P}'')\times$$
$$\times\exp\left[\frac{ik}{2}\left(\frac{1}{z_2}-\frac{1}{f}\right)(\boldsymbol{P}'^2-\boldsymbol{P}''^2)\right], \quad (12)$$

where $\Gamma_{\text{b.l.}}$ denotes the MCF immediately before the lens,

$\Gamma_{\text{a.l.}}$ denotes the MCF after passing the lens.

3. Near field diffraction in partially coherent light

Let us consider a diffraction on a transparency of transmittance $t(\boldsymbol{P}_2)$ placed in a $\mathscr{P}_2$ plane (Fig. 3).

Let the transparency be illuminated by the light characterized by the MCF equal to $\Gamma(\boldsymbol{P}_2', \boldsymbol{P}_2'')$. Immediately after the transparency the MCF becomes:

$$\Gamma(\boldsymbol{P}_2', \boldsymbol{P}_2'')t(\boldsymbol{P}_2')t^*(\boldsymbol{P}_2'') \quad (13)$$

propagation from the $\mathscr{P}_2$ plane to $\mathscr{P}_4$ is described by the Fresnel diffraction integral [1]:

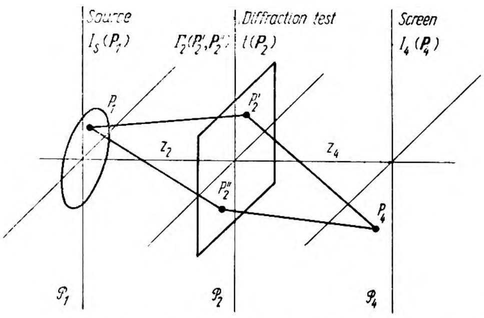

Fig. 3. Near field diffraction on the test $t(\boldsymbol{P}_2)$

$$\Gamma(\boldsymbol{P}_4', \boldsymbol{P}_4'') = \frac{\exp\left[\frac{ik}{2z_4}(\boldsymbol{P}_4'^2-\boldsymbol{P}_4''^2)\right]}{\lambda^2 z_4^2}\iint \Gamma(\boldsymbol{P}_2', \boldsymbol{P}_2'')t(\boldsymbol{P}_2')t^*(\boldsymbol{P}_2'')\times$$
$$\times\exp\left[\frac{ik}{2z_4}(\boldsymbol{P}_2'^2-\boldsymbol{P}_2''^2)\right]\exp\left[-\frac{ik}{z_4}(\boldsymbol{P}_2'\boldsymbol{P}_4'-\boldsymbol{P}_2''\boldsymbol{P}_4'')\right]d\boldsymbol{P}_2'd\boldsymbol{P}_2''. \quad (14)$$

To obtain the expression for the intensity distribution on the $\mathscr{P}_4$ plane it is enough to set $\boldsymbol{P}_4' = \boldsymbol{P}_4'' = \boldsymbol{P}_4$. Then

$$I(\boldsymbol{P}_4) = \frac{1}{\lambda^2 z_4^2}\iint \Gamma(\boldsymbol{P}_2', \boldsymbol{P}_2'')t(\boldsymbol{P}_2')t^*(\boldsymbol{P}_2'')\exp\left[\frac{ik}{2z_4}(\boldsymbol{P}_2'^2-\boldsymbol{P}_2''^2)\right]\exp\left[-\frac{ik}{z_4}\boldsymbol{P}_4(\boldsymbol{P}_2'-\boldsymbol{P}_2'')\right]d\boldsymbol{P}_2'd\boldsymbol{P}_2''. \quad (15)$$

Transformation of the variables

$$\boldsymbol{P}_2'-\boldsymbol{P}_2'' = \boldsymbol{U}, \quad (16)$$

$$\boldsymbol{P}_2'+\boldsymbol{P}_2'' = \boldsymbol{W}$$

gives

$$I(\boldsymbol{P}_4) = \frac{1}{2\lambda^2 z_4^2}\int \mathring{\Gamma}_2(\boldsymbol{U})\left\{\int t\left(\frac{\boldsymbol{W}+\boldsymbol{U}}{2}\right)t^*\left(\frac{\boldsymbol{W}-\boldsymbol{U}}{2}\right)\exp\left(\frac{ik}{2}\frac{z_2+z_4}{z_2 z_4}\boldsymbol{U}\boldsymbol{W}\right)d\boldsymbol{W}\right\}$$
$$\times\exp\left(\frac{-ik}{z_4}\boldsymbol{P}_4\boldsymbol{U}\right)d\boldsymbol{U} = \hat{\mathring{\Gamma}}\left(\frac{\boldsymbol{P}_4}{\lambda z_4}\right)\otimes\hat{R}_t(\boldsymbol{P}_4), \quad (17)$$

where $\otimes$ — denotes convolution,

$\hat{}$ — denotes the Fourier transform.

$$\hat{R}_t(\boldsymbol{P}_4) = \frac{1}{2\lambda^2 z_4^2}\iint t\left(\frac{\boldsymbol{W}+\boldsymbol{U}}{2}\right)t^*\left(\frac{\boldsymbol{W}-\boldsymbol{U}}{2}\right)\exp\left(\frac{ik}{2}\frac{z_2+z_4}{z_2 z_4}\boldsymbol{U}\boldsymbol{W}\right)d\boldsymbol{W}\exp\left(-\frac{ik}{z_4}\boldsymbol{P}_4\boldsymbol{U}\right)d\boldsymbol{U}\times$$
$$= \frac{1}{\lambda^2 z_4^2}\iint t(\boldsymbol{P}_2')t^*(\boldsymbol{P}_2'')\exp\left[\frac{ik}{2}\frac{z_2+z_4}{z_2 z_4}(\boldsymbol{P}_2'^2-\boldsymbol{P}_2''^2)\right]\exp\left[-\frac{ik}{z_4}\boldsymbol{P}_4(\boldsymbol{P}_2'-\boldsymbol{P}_2'')\right]d\boldsymbol{P}_2'd\boldsymbol{P}_2''$$
$$= \frac{1}{\lambda^2 z_4^2}\left|\int t(\boldsymbol{P}_2)\exp\left(\frac{ik}{2}\frac{z_2+z_4}{z_2 z_4}\boldsymbol{P}_2^2\right)\exp\left(-\frac{ik}{z_4}\boldsymbol{P}_2\boldsymbol{P}_4\right)d\boldsymbol{P}_2\right|^2 \quad (18)$$

The meaning of $\hat{R}_t(\boldsymbol{P}_4)$ is clear, if we notice that the intensity distribution $I_{\text{coh}}(\boldsymbol{P}_4)$ in a diffraction pattern on a transparency $t(\boldsymbol{P}_2)$ due to a point source located in $\boldsymbol{P}_1 = 0$ can be obtained by inserting a spherical wave diverging from the centre of the $\mathscr{P}_1$ plane into the Kirchhoff-Fresnel integral.

This spherical wave is:

$$U(\boldsymbol{P}_2) = U_0 \exp\left(\frac{ik}{2z_2}\, \boldsymbol{P}_2^2\right). \tag{19}$$

The Kirchhoff-Fresnel integral is:

$$U(\boldsymbol{P}_4) = \frac{\exp\left(\frac{ik}{2z_4}\right)\boldsymbol{P}_4^2}{\lambda z_4} \iint U(\boldsymbol{P}_2)t(\boldsymbol{P}_2) \exp\left(\frac{ik}{2z_4}\,\boldsymbol{P}_2^2\right)\left(\frac{ik}{z_4}\,\boldsymbol{P}_2\boldsymbol{P}_4\right)d\boldsymbol{P}_2. \tag{20}$$

That is: after substitution of (19) into (20) and setting $I(\boldsymbol{P}_4) = U(\boldsymbol{P}_4)\cdot U^*(\boldsymbol{P}_4)$:

$$I_{\text{coh}}(\boldsymbol{P}_4) = \frac{1}{\lambda^2 z_4^2} \iint t(\boldsymbol{P}_2')t^*(\boldsymbol{P}_2'')\,U_0 U_0^* \exp\left[\frac{ik}{2z_2}\,\boldsymbol{P}_2'^2 - \boldsymbol{P}_2''^2)\right] \times$$
$$\times \exp\left[\frac{ik}{2z_4}\,(\boldsymbol{P}_2'^2 - \boldsymbol{P}_2''^2)\right]\exp\left[-\frac{ik}{z_4}\,\boldsymbol{P}_4(\boldsymbol{P}_2' - \boldsymbol{P}_2'')\right]d\boldsymbol{P}_2'\,d\boldsymbol{P}_2'' = \frac{|U_0|}{\lambda^2 z_4^2}\Bigg|^2 \int t(\boldsymbol{P}_2) \times$$
$$\times \exp\left(\frac{ik}{2}\,\frac{z_4+z_2}{z_4 z_2}\,\boldsymbol{P}_2^2\right)\exp\left(-\frac{ik}{z_4}\,\boldsymbol{P}_4\boldsymbol{P}_2\right)d\boldsymbol{P}_2\Bigg|^2 = |U_0|^2\cdot\hat{R}_t(\boldsymbol{P}_4). \tag{21}$$

Therefore with an accuracy to a multiplicative constant

$$I_{\text{p.coh}}(\boldsymbol{P}_4) = \hat{\Gamma}_2\left(\frac{\boldsymbol{P}_4}{\lambda z_2}\right) \otimes I_{\text{coh}}(\boldsymbol{P}_4). \tag{22}$$

4. Far field diffraction in partially coherent light

As it is usually done the far field diffraction can be realized by placing a lens after a diffraction screen and observing a diffraction pattern in a back focal plane of the lens (Fig. 4). Propagation of the MCF from the transparency in the $\mathscr{P}_2$ plane to the lens ($\mathscr{P}_3$ plane) is described by (see (14)):

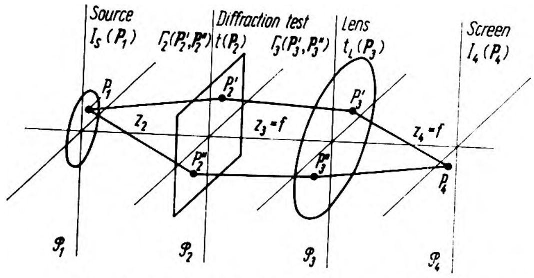

Fig. 4. Far field diffraction on the test $t(\boldsymbol{P}_2)$

$$\Gamma_3(\boldsymbol{P}_3', \boldsymbol{P}_3'') = \frac{\exp\left[\frac{ik}{2z_3}\,(\boldsymbol{P}_3'^2 - \boldsymbol{P}_3''^2)\right]}{\lambda^2 z_3^2} \iint \Gamma_2(\boldsymbol{P}_2', \boldsymbol{P}_2'')t(\boldsymbol{P}_2')t^*(\boldsymbol{P}_2'')\exp\left[\frac{ik}{2z_3}\,(\boldsymbol{P}_2'^2 - \boldsymbol{P}_2''^2)\right] \times$$
$$\times \exp\left[-\frac{ik}{z_3}\,(\boldsymbol{P}_2'\boldsymbol{P}_3' - \boldsymbol{P}_2''\boldsymbol{P}_3'')\right] d\boldsymbol{P}_2'\,d\boldsymbol{P}_2''. \tag{23}$$

The lens causes the multiplication of $\Gamma_3(\boldsymbol{P}_3', \boldsymbol{P}_3'')$ by (see (11)):

$$t_L(\boldsymbol{P}_3')t_L^*(\boldsymbol{P}_3'')\ \exp\left[-\frac{ik}{2f}\,(\boldsymbol{P}_3'^2 - \boldsymbol{P}_3''^2)\right].$$

Then the propagation to the $\mathscr{P}_4$ plane follows:

$$\Gamma_4(\boldsymbol{P}_4', \boldsymbol{P}_4'') = \frac{\exp\left[\frac{ik}{2z_4}\,(\boldsymbol{P}_4'^2 - \boldsymbol{P}_4''^2)\right]}{\lambda^2 z_4^2} \iint \Gamma_3(\boldsymbol{P}_3', \boldsymbol{P}_3'')\exp\left[-\frac{ik}{2f}\,\boldsymbol{P}_3'^2 - \boldsymbol{P}_3''^2)\right] \times$$
$$\times \exp\left[\frac{ik}{2z_4}\,(\boldsymbol{P}_3'^2 - \boldsymbol{P}_3''^2)\right]\exp\left[-\frac{ik}{z_4}\,\boldsymbol{P}_3'\boldsymbol{P}_4' - \boldsymbol{P}_3''\boldsymbol{P}_4'')\right] d\boldsymbol{P}_3'\,d\boldsymbol{P}_3''. \tag{24}$$

Inserting (23) into (24) and setting $z_3 = z_4 = f$ we have:

$$\Gamma_4(\boldsymbol{P}_4', \boldsymbol{P}_4'') = \frac{\exp\left[\frac{ik}{2z_4}\,(\boldsymbol{P}_4'^2 - \boldsymbol{P}_4''^2)\right]}{\lambda^4 f^4} \iiiint \Gamma_2(\boldsymbol{P}_2', \boldsymbol{P}_2'')t(\boldsymbol{P}_2')t^*(\boldsymbol{P}_2'')\exp\left[\frac{ik}{2f}\,(\boldsymbol{P}_2'^2 - \boldsymbol{P}_2''^2)\right] \times$$
$$\times \exp\left[\frac{ik}{2f}\,(\boldsymbol{P}_3'^2 - \boldsymbol{P}_3''^2)\right]\exp\left\{-\frac{ik}{f}\,[\boldsymbol{P}_3'(\boldsymbol{P}_2' + \boldsymbol{P}_4') - \boldsymbol{P}_3''(\boldsymbol{P}_2'' + \boldsymbol{P}_4'')]\right\} d\boldsymbol{P}_2'\,d\boldsymbol{P}_2''\,d\boldsymbol{P}_3'\,d\boldsymbol{P}_3''. \tag{25}$$

The intensity distribution in the $\mathscr{P}_4$ plane is $I(P_4) = \Gamma_4(P_4, P_4)$, hence,

$$I(P_4) = \frac{1}{\lambda^4 f^4} \iiiint \Gamma_2(P_2', P_2'') t(P_2') t^*(P_2'') \exp\left[\frac{ik}{2f}(P_3'^2 - P_3''^2)\right] \times$$
$$\times \exp\left[\frac{ik}{2f}(P_2'^2 - P_2''^2)\right] \exp\left\{-\frac{ik}{f}[P_3'(P_4+P_2') - P_3''(P_4+P_2'')]\right\} \times$$
$$\times dP_2' dP_2'' dP_3' dP_3'' = \frac{1}{\lambda^4 f^4} \iint \Gamma(P_2' P_2'') t(P_2') t^*(P_2'') \exp\left[\frac{ik}{2f}(P_2'^2 - P_2''^2)\right] \times$$
$$\times \iint \exp\left[\frac{ik}{2f}(P_3'^2 - P_3''^2)\right] \exp\left\{-\frac{ik}{f}[P_3'(P_4+P_2') - P_3''(P_4+P_2'')]\right\} dP_3' dP_3' \times dP_2' dP_2''. \quad (26)$$

Integration over $P_3' P_3''$ is easy to perform after introducing new variables:

$$P_3' - P_3'' = U, \qquad P_3' + P_3'' = W. \quad (27)$$

$$\iint \exp\left[\frac{ik}{2f}(P_3'^2 - P_3''^2)\right] \exp\left\{-\frac{ik}{f}[P_3'(P_4+P_2') - P_3''(P_4+P_2'')]\right\} dP_3' dP_3''$$
$$= \frac{1}{2} \int \exp\left[-\frac{ik}{2f} U(2P_4 + P_2' + P_2'')\right] \int \exp\left[\frac{ik}{2f} W(U - P_2' + P_2''\right] dW dU$$
$$= \lambda f \exp\left[-\frac{ik}{2f} P_2'^2 - P_2''^2)\right] \exp\left[-\frac{ik}{f} P_4(P_2' - P_2'')\right]. \quad (28)$$

The intensity distribution $I(P_4)$ can be obtained by inserting (27) into (26) and assuming quasi-stationarity of the MCF in the $\mathscr{P}_2$ plane (9). Integration over $P_2' P_2''$ can be performed easily after changing the variables

$$P_2' - P_2'' = U, \qquad P_2' + P_2'' = W, \quad (29)$$

so:

$$I(P_4) = \frac{1}{\lambda^3 f^3} \iint \mathring{\Gamma}_2(P_2' - P_2'') t(P_2') t^*(P_2'') \exp\left[\frac{ik}{2}\left(\frac{1}{z_2} + \frac{1}{f}\right)(P_2'^2 - P_2''^2)\right] \exp\left[-\frac{ik}{2f}(P_2'^2 - P_2''^2)\right] \times$$
$$\times \exp\left[-\frac{ik}{2f}(P_2' - P_2'') P_4\right] dP_2' dP_2'' = \frac{1}{2\lambda^3 f^3} \int \mathring{\Gamma}_2(U) \left\{ \int \exp\left(\frac{ik}{2z_2} UW\right) t\left(\frac{W+U}{2}\right) t^*\left(\frac{W-U}{2}\right) dW \right\} \times$$
$$\times \exp\left(-\frac{ik}{f} P_4 U\right) dU = \mathring{\Gamma}_2\left(\frac{P_4}{\lambda f}\right) \otimes \hat{R}_t(P_4). \quad (30)$$

Similarly to (17) let us write

$$\hat{R}_t(P_4) = \frac{1}{2\lambda^3 f^3} \iint \exp\left(\frac{ik}{2z_2} UW\right) t\left(\frac{W+U}{2}\right) t^*\left(\frac{W-U}{2}\right) dW \exp\left(-\frac{ik}{f} P_4 U\right) dU$$
$$= \frac{1}{\lambda^3 f^3} \left| \int t(P_2) \exp\left(\frac{ik}{2z_2} P_2^2\right) \exp\left(-\frac{ik}{f} P_2 P_4\right) dP_2 \right|^2. \quad (31)$$

The last equation expresses the intensity distribution in a diffraction pattern on a transparency $t(P_2)$, due to a point source located in the centre of the $\mathscr{P}_1$ plane. To check this statement it is enough to insert spherical diverging wave (19) into the Kirchhoff-Fresnel integral describing coherent propagation through a lens:

$$U(P_4) = \frac{\exp\left(\frac{ik}{2z_4} P_4^2\right)}{\lambda^2 z_3 z_4} \iint U(P_2) t(P_2) \exp\left[\frac{ik}{2}\left(\frac{1}{z_3} + \frac{1}{z_4} - \frac{1}{f}\right) P_3^2\right] \times$$
$$\times \exp\left(\frac{ik}{2z_3} P_2^2\right) \exp\left[-ikP_3\left(\frac{P_4}{z_4} + \frac{P_2}{z_3}\right)\right] dP_2 dP_3, \quad (32)$$

but $z_3 = z_4 = f$, and $I(\boldsymbol{P}_4) = U(\boldsymbol{P}_4)U^*(\boldsymbol{P}_4)$ hence,

$$I_{\text{coh}}(\boldsymbol{P}_4) = \left| \frac{1}{\lambda^2 f^2} \iint U_0 t(\boldsymbol{P}_2) \exp\left[\frac{ik}{2}\left(\frac{1}{z_2}+\frac{1}{f}\right)\boldsymbol{P}_2^2\right] \exp\left(\frac{ik}{2f}\boldsymbol{P}_3^2\right) \times \right.$$
$$\left. \times \exp\left[-\frac{ik}{f}\boldsymbol{P}_3(\boldsymbol{P}_4+\boldsymbol{P}_2)\right] d\boldsymbol{P}_2 d\boldsymbol{P}_3 \right|^2 = \frac{|U_0|^2}{\lambda^4 f^4} \iint t(\boldsymbol{P}_2') \times$$
$$\times t^*(\boldsymbol{P}_2'') \exp\left[\frac{ik}{f}\left(\frac{1}{z_2}+\frac{1}{f}\right)(\boldsymbol{P}_2'^2 - \boldsymbol{P}_2''^2)\right] \times$$
$$\times \iint \exp\left[\frac{ik}{2f}(\boldsymbol{P}_3'^2 - \boldsymbol{P}_3''^2)\right] \exp\left\{-\frac{ik}{f}[\boldsymbol{P}_3'(\boldsymbol{P}_4+\boldsymbol{P}_2') - \boldsymbol{P}_3''(\boldsymbol{P}_4+\boldsymbol{P}_2'')]\right\} \times d\boldsymbol{P}_2' d\boldsymbol{P}_2'' d\boldsymbol{P}_3' d\boldsymbol{P}_3''. \quad (33)$$

Referring to the equation (28):

$$I_{\text{coh}}(\boldsymbol{P}_4) = \frac{|U_0|^2}{\lambda^3 f^3} \iint t(\boldsymbol{P}_2') t^*(\boldsymbol{P}_2'') \exp\left[\frac{ik}{2}\left(\frac{1}{z_2}+\frac{1}{f}\right)(\boldsymbol{P}_2'^2 - \boldsymbol{P}_2''^2)\right] \times$$
$$\times \exp\left[-\frac{ik}{2f}(\boldsymbol{P}_2'^2 - \boldsymbol{P}_2''^2)\right] \exp\left[-\frac{ik}{f}\boldsymbol{P}_4(\boldsymbol{P}_2' - \boldsymbol{P}_2'')\right] d\boldsymbol{P}_2' d\boldsymbol{P}_2''$$
$$= \frac{|U_0|^2}{\lambda^3 f^3} \left| \int t(\boldsymbol{P}_2) \exp\left(\frac{ik}{2z_2}\boldsymbol{P}_z^2\right) \exp\left(-\frac{ik}{f}\boldsymbol{P}_4 \boldsymbol{P}_2\right) d\boldsymbol{P}_2 \right|^2. \quad (34)$$

Consequently (28) becomes:

$$I_{\text{p.coh}}(\boldsymbol{P}_4) = \hat{\mathring{\Gamma}}_2\left(\frac{\boldsymbol{P}_4}{f}\right) \otimes I_{\text{coh}}(\boldsymbol{P}_4). \quad (35)$$

5. Conclusions

The equation

$$I_{\text{p.coh}}(\boldsymbol{P}) = \hat{\mathring{\Gamma}}_{\text{test}}\left(\frac{\boldsymbol{P}}{\lambda z}\right) \otimes I_{\text{coh}}(\boldsymbol{P}) \quad (36)$$

is valid for a partially coherent diffraction in both cases: i.e. in near field (Fresnel) diffraction, as well as in far field (Fraunhofer) diffraction. Paraxial approximation is the only important one. By analogy to the Schell's theorem [4] the equation (36) can be called „the generalized Schell's theorem", and formulated as follows:

An intensity distribution in a diffraction pattern on a test of transmittance $t(\boldsymbol{P})$ in partially coherent paraxial diffraction is proportional to the convolution of the intensity distribution in a diffraction pattern on the same test, due to a point source and a Fourier transform of a stationary part of a Mutual Coherence Function in a test plane.

If the test sample is illuminated by a complete incoherent source, then (according to (10))

$$\mathring{\Gamma}(\boldsymbol{P}'-\boldsymbol{P}'') \simeq \hat{I}_{\text{source}}\left(\frac{\boldsymbol{P}'-\boldsymbol{P}''}{\lambda z_2}\right); \quad (37)$$

consequently

$$\hat{\mathring{\Gamma}}_{\text{test}}\left(\frac{\boldsymbol{P}}{\lambda z_4}\right) \simeq I_{\text{source}}\left(-\boldsymbol{P}\frac{z_4}{z_2}\right). \quad (38)$$

In such case the intensity distribution in the diffraction pattern is a convolution of the intensity distribution in the diffraction pattern, due to a point source and the intensity distribution on a source. This means that each point of the source generates its own diffraction pattern shifted with respect to such pattern generated by the neighbouring point of the source. All these patterns superpose incoherently. This causes "bluring" of the resultant diffraction pattern — the effect being more visible as the dimensions of the source extends.

Such understanding of partially coherent diffraction phenomenon may be helpful in investigation of the visibility or resolution in the image given by a hologram (especially that of Fourier type) reconstructed by an incoherent source.

О частично когерентной дифракции близкого и далекого полей

В статье обсуждается частично когерентная дифракция в случаях аппроксимации близкого и далекого полей. Представлено понятие квазистационарности функции взаимной когерентности. Сформулирована обобщенная теорема Шелла. Она определяет распределение интенсивности в дифракционном спектре свертыванием соответствующего распределения интенсивности в когерентном дифракционном спектре и преобразованием Фурье взаимокогерентной функции.

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