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Probabilistic Model for Shape Errors of Spherical Surfaces in Optical Elements

In this paper a probabilistic model for shape errors of spherical surfaces in optical elements is described, and the formulae allowing to determine the tolerances are given.

The errors in shape of the spherical surfaces of optical elements (toricity of the surface) result in perturbations in the rotational symmetry of the respective optical systems. In such systems the classical aberrations are different in both the main crosses, and consequently, the astigmatism on the axis of the system is also observed. Independently an anamorphosis appears. These are the so-called first order aberrations which are usually employed in estimation of the shape error tolerances for optically acting surfaces. Anamorphosis of the image being of a practical importance only for measuring instruments of great field of view, the surface shape tolerances are usually determined from admissive value of astigmatism on the optical axis. The axial astigmatism of the system depends on the value of deformations and on the spatial configuration of their directions. This is confirmed by a well-known fact that the axial astigmatism may be compensated during the mounting operations by rotating the elements around the optical axis of the system. For small deformations — which are typically assumed when technological defects are considered — the axial astigmatism of the system is given by the formula

$$\Delta s_p = \frac{4\lambda}{n_p u_p^2}\Big[\sum_{i=1}^{p} A_i^2 \Delta N_i^2 + 2\sum_{i=1}^{p}\sum_{k=i+1}^{p} A_i A_k \Delta N_i \Delta N_k \cos 2\Delta\varphi_{i,k}\Big]^{1/2} \tag{1}$$

(see [1]), where

Δs_p — axial astigmatism of the system,
λ — light wavelength,
n_p — refractive index in the image space,
u_p — image aperture angle of the system,
p — number of system surfaces,
A_i — a coefficient dependent on constructional parameters of the system.

A_i is evaluated from the formula

$$A_i = \frac{|n_i'-n_i|\cdot h_i^2}{\O_i^2}, \tag{2}$$

where

n_i', n_i — the respective image and object refractive indices at the i-th surface,
h_i — height of incidence of the aperture ray on the i-th surface of the system,
$\O_i$ — acting diameter of the i-th surface,
$\Delta\varphi_{i,k} = \varphi_i - \varphi_k$ — angle of mutual spatial position of i-th and k-th deformations, respectively,
ΔN_i — value of deformation of the surfaces weighted ovalisation of interference fringes,
φ_i, φ_k — directional azimuths of positions of i-th and k-th deformations, respectively.

If the compensating possibilities of the axial astigmatism connected with the spatial position of the deformation directions are not taken into consideration, i.e. if the directions of the deformations are assumed to lie in the same plane (to be consistent), then the axial astigmatism achieves its maximum value, and according to (1) it amounts to

$$\Delta s_p = \frac{4\lambda}{n_p u_p^2}\sum_{i=1}^{p} A_i \Delta N_i. \tag{3}$$

The above relation (1) allows to determine the axial astigmatism of the system, provided that the values of deformation ΔN_i, the azimuths of deformation directions $\Delta\varphi_{i,k}$ and the working conditions and constructional parameters of the optical system are known. When, however the determination of tolerances is based on admissive astigmatism Δs_p on the axis of the optical system then an additional condition should be introduced, namely $p-1$ additional relations connecting the tolerances ΔN_i with each other. Usually, the method of equal influences is employed by assuming that for the case under

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consideration

$$A_i \Delta N_i = \text{const.} \tag{4}$$

In a general case the coefficient A_i may take values $A_{1i} \cdot K_i$, where K_i is a factor correcting the weight of the particular tolerances depending on the processing difficulties, economical effects and so on.

Considering compensation potentialities of axial astigmatism due to the spatial orientation of the deformation directions, as well as a low probability for elements of maximal working deviation to appear in the system, it is reasonable to determine the tolerances by the calculus of probability. According to a probabilistic nomenclature the axial astigmatism of the system is a random variable dependent on two random variables: the deformation value ΔN_i and the spatial configuration of the deformation directions φ_i. This suggests a geometrical (vectorial) summing of the partial astigmatisms. To simplify further considerations a notion of rotational vector of deformation is introduced (see also deformation vector). Its magnitude characterises the value of deformation of the spherical surface, while its direction determines the angular position of this deformation in the plane perpendicular to the axis of optical system (Fig. 1).

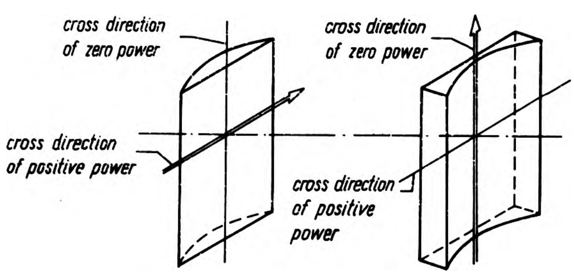

Fig. 1. An explanation of the principle of determining the direction of the rotational vector of deformation of the spherical surfaces denoted by an arrow (for instance, the direction of maximal power of the surface)

By applying the notion of the rotational vector of deformation, and by taking account of the condition $A_i \Delta N_i = \delta_i$ the formulae (1) can be transformed into the form

$$\Delta s_p = \frac{4\lambda}{n_p u_p^2}\left[\left(\sum_{i=1}^{p} \delta_i \cos\varphi_i\right)^2 + \left(\sum_{i=1}^{p} \delta_i \sin\varphi_i\right)^2\right]^{1/2}. \tag{5}$$

It should be mentioned that the present meaning of $A_i \Delta N_i = \delta_i$ has a statistical sense, i.e. $\delta_i \in (0, \delta_{\max})$ and $A_i \Delta N_{i\max}\, \delta_{\max} = \text{const.}$

During the mounting operations the mutual positioning of the deformation vectors (and more strictly their projections on the plane perpendicular to the optical axis) takes the form shown in Fig. 2. Hence, it follows that the axial astigmatism of the system is

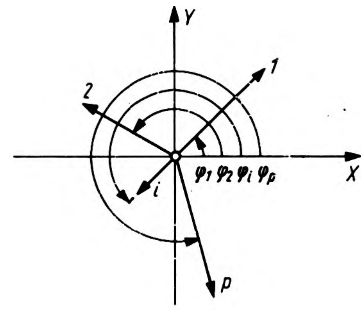

Fig. 2.

a geometrical sum dependent on the deformation vectors while its modulus (5) depends on the square root of the two simple (linear) sums of the squared projections of the deformation vectors. By applying to these sums a central theorem of the probability calculus it may be concluded that the distribution of the sums will tend to a normal one.

In other words, it may be assumed that a two--dimensional normal distribution with the centre at the point 0 is spread over an XOY plane (see Fig. 2), in which a rotational radius of the deformation vector moves. As the practical experience indicates that the probability distribution of the deformation vector azimuth φ_i is uniform, it may be proved mathematically [2] that the radius of the deformation vector is subjected to the Rayleigh distribution (distribution χ (2) i.e. disribution chi with two degrees of freedom) with density function

$$f(x) = \frac{1}{s^2} \cdot x \cdot \exp\left(-\frac{x^2}{2s^2}\right)$$

with the mean value

$$E(x) = \sqrt{\frac{\pi}{2}}\, s \approx 1.25 \cdot s,$$

and variance

$$D^2(x) = \left(2 - \frac{\pi}{2}\right) s^2 \approx 0.43 \cdot s^2,$$

where s is a distribution parameter. The graphical form of the distribution is given in Fig. 3. It may be

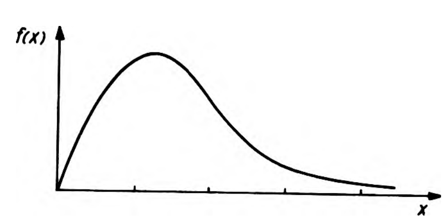

Fig. 3. Density function of Rayleigh distribution

proved moreover that the sums in relation (5) being assumed to be accurately consistent with the normal (and not asymptotic) distribution the Rayleigh distribution will also occur for the deformation, though the parameters of these distributions differ from one another. This case is assumed to be a standard for shape errors of spherical surfaces of the optical elements within the considered model. A typical case will be the tolerance ΔN_2 presented in Fig. 4. In the case of narrow tolerances this distribution will approach the uniform distribution (tolerance ΔN_1 in Fig. 4), while for wide tolerances it will be close to a normal distribution (tolerance ΔN_3 in Fig. 4).

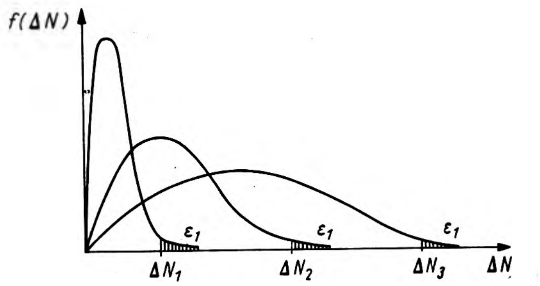

Fig. 4. The forms of Rayleigh distribution for different parameters ε_1 — the assumed defectivity of a practical spread of the working deviations

In accordance with the accepted probabilistic model the tolerance is determined by evaluating for separate surfaces of the system the upper limit of the interval $[0, \delta_{\max}]$ within which the variable $\delta_i = A_i \Delta N_i$ changes randomly, and such that the probability of occurance of a defective system with the axial astigmatism greater than its admissive value Δs_p does not exceed the assumed value ε. Therefore, the following statistical relation should be solved

$$P\left(\sqrt{\left(\sum_{i=1}^{p} \delta_i \cos\varphi_i\right)^2 + \left(\sum_{i=1}^{p} \delta_i \sin\varphi_i\right)^2} \leqslant \frac{n_p u_p^2 \Delta s_p}{4\lambda}\right) = 1-\varepsilon. \tag{6}$$

An additional assumption that δ_i has a Rayleigh distribution allows to consider a standard case of the model. Then the random variables $\delta_i \cos\varphi_i$, $\delta_i \sin\varphi_i$ are subject to a normal distribution of parameters $N(0, s)$, while

$$\sum_{i=1}^{p} \delta_i \cos\varphi_i, \quad \sum_{i=1}^{p} \delta_i \sin\varphi_i$$

have the same distribution but of parameters $N(0, \sqrt{p}\, s)$. The whole square root expression follows the Rayleigh distribution [3] of the density function

$$f(\sigma) = \frac{1}{ps^2} \cdot \sigma \cdot \exp\left(-\frac{\sigma^2}{2ps^2}\right),$$

where

$$\sigma = \frac{n_p u_p^2 \Delta s_p}{4\lambda}$$

The deformations of particular surfaces have Rayleigh distribution of density function

$$f(\delta) = \frac{1}{s^2} \cdot \delta \cdot \exp\left(-\frac{\delta^2}{2s^2}\right).$$

The unknown distribution parameter s may be determined in the following way

$$p\left(0 \leqslant \sigma \leqslant \frac{n_p u_p^2 \Delta s_p}{4\lambda}\right) = \frac{1}{ps^2} \cdot \int_0^{\frac{n_p u_p^2 \Delta s_p}{4\lambda}} \sigma \cdot \exp\left(-\frac{\sigma^2}{2ps^2}\right) d\sigma = 1-\exp\left(-\frac{n_p^2 u_p^4 \Delta s_p^2}{32\lambda^2 p s^2}\right).$$

It is requested that the probability of occurance of a correct system be great, i.e.

$$1-\exp\left(-\frac{n_p^2 u_p^4 \Delta s_p^2}{32\lambda^2 p s^2}\right) = 1-\varepsilon.$$

Hence, after elementary rearangements

$$s = \frac{n_p u_p^2 \Delta s_p}{4\lambda} \sqrt{\frac{-1}{4.6 \cdot p \cdot \log\varepsilon}}. \tag{7}$$

After the parameter s is determined the distribution is already uniquely specified and therefore an arbitrary statistical problem may be solved within the given model. In our case we look for the upper limit of the random variable interval $\delta_i \in (0, \delta_{\max})$, which fulfills the dependence (6). Hence

$$p(0 \leqslant \delta_i \leqslant \delta_{\max}) = \frac{1}{s^2} \int_0^{\delta_{\max}} \delta \cdot \exp\left(\frac{\delta^2}{2s^2}\right) d\delta = 1-\exp\left(-\frac{\delta_{\max}^2}{2s^2}\right).$$

By assuming a priori the defectiveness ε_1 of the practical spread of the working deviations (Fig. 4) we get

$$1-\exp\left(-\frac{\delta_{\max}}{2s^2}\right) = 1-\varepsilon_1.$$

Hence

$$\delta_{\max}^2 = s\sqrt{-4.6 \log\varepsilon_1}. \tag{8}$$

By inserting the formulae (7) into (8) and taking account of $\delta_i = A_i \Delta N_i$ we obtain finally that for spherical surfaces represented by ovality of the interference fringes the tolerance of shape errors is given by the following expression

$$\Delta N_i = \frac{n_p u_p^2 \Delta s_p}{4\lambda A_i} \sqrt{\frac{\log\varepsilon_1}{p \log\varepsilon}}. \tag{9}$$

Now, we will consider the general case of an arbitrary (but determined) distribution of deformations of the spherical surface. Then the random variables

$$\delta_i \cos\varphi_i,\ \delta_i \sin\varphi_i$$

have no more the normal distribution, which however still remains the limit to which the sums of random variables tend usually.

Let the standard deviation of these distributions in the standard and general cases differ by a factor K, i.e. the factor K determines the influence of the kind of deformation on the parameter s of the axial astigmatism. Hence the random variables

$$\sum_{i=1}^{p} \delta_i \cos\varphi_i,\ \sum_{i=1}^{p} \delta_i \sin\varphi_i$$

have the normal distribution of parameters $N(0, K\sqrt{p}\, s)$. Then the distribution of axial astigmatism has the form of Rayleigh distribution [3] of density function.

$$f(\sigma) = \frac{1}{K^2 p s^2} \sigma \exp\left(-\frac{\sigma^2}{2K^2 p s^2}\right).$$

Hence, analogically to the standard case the parameter s of the distribution of axial astigmatism amounts to

$$s = \frac{n_p u_p^2 \Delta s_p}{4\lambda K} \sqrt{\frac{-1}{4.6 p \log\varepsilon}}.$$

In this case for determining the tolerance the following reasoning is made. If in both the standard and general cases the axial astigmatism distributions differ only by the parameter, then similarly to the method used to determine tolerance in the standard case it may be assumed, that in the general case the deformation has also the Rayleigh distribution but with a different parameter. Hence, by analogy to the standard case the tolerance amounts to

$$\Delta N_i = \frac{n_p u_p^2 \Delta s_p}{4\lambda A_i K} \sqrt{\frac{\log\varepsilon_1}{p \log\varepsilon}}.$$

In each probabilistic model the value of ε_1 must be established. Thus, it is suggested to assume $\varepsilon_1 = 0.023$ (by analogy to the defectiveness of six-standard field of tolerance in case of a normal distribution).

If we assume that $\varepsilon = \varepsilon_1$ then

$$\Delta N_i = \frac{n_p u_p^2 \Delta s_p}{4\sqrt{p}\lambda A_i K}.$$

By employing additionally the Maréchal cryterion [4] for the axial astigmatism we get

$$\Delta N_i = \frac{1}{4\sqrt{2p} A_i K},$$

where again

- Δs_p — axial astigmatism of the system (admissive value),
- n_p — refractive index in the image space of the system,
- u_p — image aperture angle,
- A_i — coefficient determined from (2),
- p — number of surface,
- ε — defectiveness of the series of systems mounted ($\varepsilon = 0.01$ signifies 1%),
- ε_1 — a priori accepted defectiveness of the practical spread of the working deviation elements,
- K — coefficient depending upon the kind of the deformation distribution ΔN_i.

The value of the coefficient K — according to the assumption — may be determined by comparing the parameters of axial astigmatism distributions occurring for an arbitrary (general case) and Rayleigh (standard case) distributions of deformations. E. g. for a uniform distribution the value of the coefficient K amounts to 1.5.

The tolerances ΔN_i obtained from the above formulae for an enlarger objective of the triplet type of local length $f = 55$ mm and the relative aperture $1{:}K = 1{:}4.5$ given as an example are the following: the admissive value of axial astigmatism $\Delta s_p = 0.046$ (Maréchal cryterion), the surface deformation distribution is uniform, the defectiveness $\varepsilon = 0.01$; $\Delta N_1 = 0.93$, $\Delta N_2 = 1.02$, $\Delta N_3 = 0.45$, $\Delta N_4 = 0.45$, $\Delta N_5 = 0.83$, $\Delta N_6 = 0.79$.

Finally it should be mentioned that since the tolerances obtained have been determined by using the near-axial cryterion (axial astigmatism) then it should be verified whether the so deformed surfaces do not exceed the tolerances of the ray that result from the aberrational analysis of the systems composed of ideal spherical surfaces.

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Вероятностная модель для погрешностей формы сферических поверхностей оптических элементов

В статье описана вероятностная модель для погрешностей формы сферических поверхностей оптических элементов и приведены зависимости, позволяющие определять в ней допуски.

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