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Diffraction of a plane TM-polarized optical wave on a non-absorbing medium with a periodic dielectric permeability variation

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Taking advantage of the coupled waves method, a system of linear differential equations with constant coefficients has been obtained which describes the diffraction of a plane TM-polarized optical wave on a grating. The electric field strength was used as a variable. For such a choice of a variable, only the coupling between the adjacent coupled waves appears in the system of equations, which results in a substantial simplification of the system.

Keywords: TM-polarization, grating, periodic medium, diffraction.

1. Introduction

Media with periodic variation of a certain optical parameter (refraction index, absorption coefficient, thickness) are used in many optical devices [1]–[4]. Therefore, there is a great number of papers devoted to the analysis of light diffraction on periodic structures. The monograph [4] published in 1981 lists 780 references. But in that monograph, light diffraction analysis is mainly made by means of approximate methods based on the solution of the wave equation, where the second derivative obtained in the equations is neglected as a consequence of the assumption that the wave amplitude varies very little as the wave propagates throughout a periodic medium. Kogelnik's theory [5] is the one known best among such theories. However, it is also an approximate one, and therefore it can only be used to analyze thick holograms with a small coefficient of the refraction index modulation. In monograph [4] and in paper [5], the analysis is made mainly for TE-polarized light. Approximate equations for description of TM-polarized light diffraction were obtained by means of indirect methods, by correcting appropriate equations for TE-polarization. New publications appeared afterwards, which focused attention on light diffraction on volume gratings, without neglecting second derivatives. But they also deal mainly with

TE-polarized light diffraction. Among those publications noteworthy are papers [6], [7], where precise systems of linear differential equations were obtained to describe diffraction on periodic structures for both TE- [6] and TM-polarizations [7]. But equations obtained in [6] are rather complicated since the choice of variables is not quite adequate. Based on the approach described in [8], linear equations were obtained [9], which turned out to be simpler than the corresponding systems in [6]. This system of equations proved to be applicable to the analysis of thin, intermediate and thick holograms [9]. It should also be mentioned that in the systems of equations obtained for TE-polarized light diffraction analysis [6], [8], [9], the variables sought are related to the electric field strength of the optical wave in a periodic medium, since for this case the assumption equation (Helmhlotz's equation) obtained, based on Maxwell's equations, is scalar. In [7] in order to obtain equations which describe TM-polarized light diffraction on periodic structures based on Maxwell's equations, an assumption vector wave equation was used with respect to the magnetic field strength of the optical wave, which has an extra term compared to Helmholtz's equation. This extra term results in a greater complexity of the corresponding systems of differential equations. In each equation of the TM-polarization system, the two extra sums appear, each having r terms (r being equal to the number of diffraction orders). This system will be especially complex for thin and intermediate thick gratings, and for a large refraction index modulation coefficient, when big values of diffraction orders have to be taken into account. Such a complex system of equations in [7] is due to the fact that the dielectric permeability is multiplied by the electric field strength in Maxwell's equations, while the variables in those equations are associated with the magnetic field strength. If the magnetic permeability varied with a periodic law, the approach used in [7] would lead to a simpler system of equations similar to those obtained in [6] for TE-polarization. Therefore, it seems that the precise system of equations used to describe TM-polarized optical wave diffraction on periodic structures will be simpler than that in [7] when the variables are associated with the electric field strength. Besides, the solutions obtained in [6], [7] and in part those in [4] consist in reducing the wave equation with periodic variation of the dielectric constant to Mathieu equation which foresees a two-dimensional periodicity. At the same time the actual gratings have finite thickness, and in two other dimensions, they may be assumed to have infinite dimensions (the thickness of gratings is much smaller than their size measured across). That means that the actual gratings can be assigned periodicity in one direction only. Therefore, the coupled wave equations [4], [6], [7] obtained, based on Floquet's theorem, are not quite adequate to the actual gratings. The coupled wave method used in [8], [9] to obtain equations describing TE-polarized light diffraction is not based on Floquet's theorem and can be used to describe diffraction on finite-thickness gratings. Thus, the purpose of this paper is to obtain, without using Floquet's theorem, a system of differential equations describing TM-polarized optical wave diffraction on periodic media and having variables associated with the electric field strength.

2. Theory of TM-polarized light diffraction on periodic non-absorbing structures

Let us assume that in a dielectric medium along a certain direction the dielectric constant varies according to the formula

$$\varepsilon(x, y, z) = \begin{cases} \varepsilon_1, & -\infty < z < 0, -\infty < x < \infty, -\infty < y < \infty, \\ \varepsilon_{20} + \tilde{\varepsilon}\cos(K_z z + K_x x) = \\ = \varepsilon_{20} + \tilde{\varepsilon}\cos\mathbf{Kr}, & 0 < z < T, -\infty < x < \infty, -\infty < y < \infty, \\ \varepsilon_3, & T < z < \infty, -\infty < x < \infty, -\infty < y < \infty \end{cases} \tag{1}$$

where ε_1 and ε_3 are the dielectric constants of the first and the third media, respectively, ε_{20} – constant component of the dielectric constant of the grating medium, $\tilde{\varepsilon}$ – amplitude of the variable component of the dielectric constant, K_x – projection of vector $\mathbf{K}$ onto the axis OX, K_z – projection of vector $\mathbf{K}$ onto the axis OZ, the modulus of vector $\mathbf{K}$ being related to the grating Λ as $K = 2\pi/\Lambda$. Figure 1 shows a schematic representation of non-diffracted and diffracted plane waves and a periodic volume grating.

The electric field strength vector $\mathbf{E}$ of a TM-polarized electromagnetic wave is in the plane of propagation and diffraction, *i.e.*, in the plane XZ. The magnetic field strength vector $\mathbf{H}$ is perpendicular to the plane XZ and coincides with the direction of the axis OY. A non-diffracted plane wave containing a wave vector $\mathbf{k}_0$ propagates at an angle θ_0 to the axis OZ. At an angle θ_i the wave diffracts into the i-th order. For instance, Fig. 1 shows a wave diffracting into the first order at an angle θ_1 with wave vector $\mathbf{k}_1$. Grating vector $\mathbf{K}$ is at angle ϕ to the axis OZ.

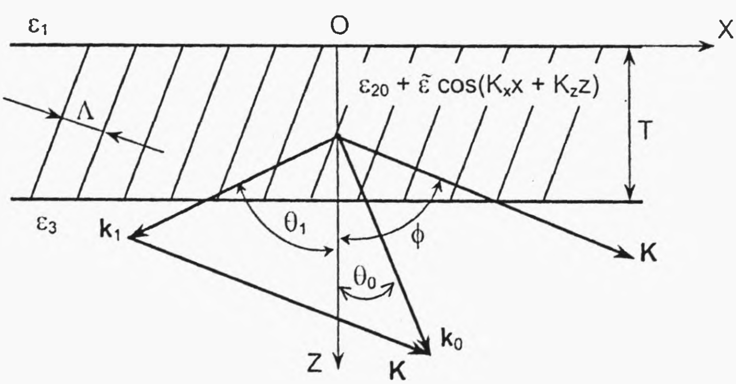

Fig. 1. Schematic representation of incident and diffracted plane waves and a periodic volume grating (T is the thickness of grating in the direction of axis OZ).

Equations describing TM-wave propagation through a periodic medium can be obtained from the systems of equations that are given in [10] and have the following form:

$$\begin{cases} \operatorname{rot} \mathbf{E} = -j\dfrac{\omega}{c}\mathbf{H}, \\ \operatorname{rot} \mathbf{H} = j\varepsilon_2\dfrac{\omega}{c}\mathbf{E} \end{cases} \tag{2}$$

where ε_2 is the dielectric constant which varies along a certain direction according to Eq. (1), c – the speed of light, ω – the angular frequency.

Based on systems (2), we can write the second-order equation as follows:

$$\operatorname{rot}\operatorname{rot}\mathbf{E} = \varepsilon_2\frac{\omega^2}{c^2}\mathbf{E}. \tag{3}$$

The solution to Eq. (3)

$$\mathbf{E}(x, z) = \sum_i C_i[-\mathbf{l}\cos\theta_i A_{i,x}(z) + \mathbf{n}\sin\theta_i A_{i,z}(z)]\exp[-j(k_{i,x}x + k_{i,z}z)] \tag{4}$$

where $\mathbf{l}$, $\mathbf{n}$ are individual vectors directed along the axes OX and OZ, respectively, C_i is the normalization factor, $-\cos\theta_i A_{i,x}(z)$, $\sin\theta_i A_{i,x}(z)$ are electric field strength amplitude projections on the axes OX and OZ, respectively, $k = |\mathbf{k}_i| = 2\pi n_{20}/\lambda$ is a modulus of the wave vector $\mathbf{k}_i$ of plane waves into which the electromagnetic field in a periodic medium is decomposed, $k_{i,x} = k\sin\theta_i$, $k_{i,z} = k\cos\theta_i$ are wave vector projections on the axes OX and OZ, respectively, $\exp(-j\mathbf{k}_i\mathbf{r}) = \exp[-j(k_{i,x}x + k_{i,z}z)]$.

As a matter of fact, the solution to Eq. (3) has been represented as a sum of plane waves that propagate in a periodic medium and whose amplitude is a function of z-coordinate, with the electric field strength vector projections being described by different functions. If $\tilde{\varepsilon} = 0$, then the solution to Eq. (3) can be represented as a plane wave with a constant amplitude.

In order to find the normalization factor C_i we use the first equation of system (2) and find magnetic field strength for a constant-amplitude plane wave propagating at angle θ_i, when $\tilde{\varepsilon} = 0$. Thus the magnetic field strength for a plane wave is

$$\begin{aligned} \mathbf{H}_i &= j\frac{c}{\omega}\operatorname{rot}\mathbf{E}_i = j\frac{c}{\omega}\operatorname{rot}\{C_i[-\mathbf{l}\cos\theta_i + \mathbf{n}\sin\theta_i]\exp[-j(k_{i,x}x + k_{i,z}z)]\} \\ &= \mathbf{m}C_i\sqrt{\varepsilon_{20}}\exp(-j\mathbf{k}_i\mathbf{r}) \end{aligned} \tag{5}$$

where $\mathbf{m}$ is the individual vector directed along the axis OY.

Let us find the projection of Poynting vector [10] for a plane wave on the axis OZ and assume it to be unity

$$\langle\mathbf{S}\rangle_{i,z} = \operatorname{Re}\left\{\frac{c}{4\pi}[\mathbf{E}_i\mathbf{H}_i^*]_z\right\} = |C_i|^2\frac{c\sqrt{\varepsilon_{20}}}{4\pi}\cos\theta_i = 1. \tag{6}$$

Thus the normalization factor C_i is

$$C_i = 2\sqrt{\frac{\pi}{cn_{20}\cos\theta_i}} \tag{7}$$

where $n_{20} = \sqrt{\varepsilon_{20}}$.

Such a representation of the normalization factor C_i is due to the fact that the diffraction efficiency of a hologram can be expressed by a z-component of Poynting vector in the third medium if z-component of an incident plane wave on the grating in the first medium is equal to unity [2].

Substituting expression (4) into the left-hand side of the Eq. (3), and we have

$$\begin{aligned}\mathrm{rot\,rot}\mathbf{E} &= \\ &= \sum_i C_i\{\mathbf{l}[\cos\theta_i\ddot{A}_{i,x} - 2jk\cos^2\theta_i\dot{A}_{i,x} - jk\sin^2\theta_i\dot{A}_{i,z} - k^2\cos^3\theta_i A_{i,x} \\ &- k^2\sin^2\theta_i\cos\theta_i A_{i,z}] + \mathbf{n}[jk\cos\theta_i\sin\theta_i\dot{A}_{i,x} + k^2\cos^2\theta_i\sin\theta_i A_{i,x} \\ &+ k^2\sin^3\theta_i A_{i,z}]\}\exp(-j\mathbf{k}_i\mathbf{r}).\end{aligned} \tag{8}$$

Substituting expression (4) into the right-hand side of the Eq. (3), we obtain

$$\begin{aligned}\mathrm{rot\,rot}\mathbf{E} &= \frac{\omega^2}{c^2}\sum_i C_i\left\{\varepsilon_{20} + \frac{\tilde{\varepsilon}}{2}\exp(j\mathbf{Kr}) + \exp(-j\mathbf{Kr})\right\} \\ &\times(-\mathbf{l}\cos\theta_i A_{i,x} + \mathbf{n}\sin\theta_i A_{i,z})\exp(-j\mathbf{k}_i\mathbf{r}).\end{aligned} \tag{9}$$

By equalling the right-hand sides of Eqs. (8) and (9) and reducing similar terms we obtain the following expression:

$$\begin{aligned}&\sum_i C_i\{\mathbf{l}[\cos\theta_i\ddot{A}_{i,x} - 2jk\cos^2\theta_i\dot{A}_{i,x} - jk\sin^2\theta_i\dot{A}_{i,z} + k^2\cos\theta_i\sin^2\theta_i A_{i,x} \\ &- k^2\sin^2\theta_i\cos\theta_i A_{i,z}] + \mathbf{n}[jk\cos\theta_i\sin\theta_i\dot{A}_{i,x} + k^2\cos^2\theta_i\sin\theta_i A_{i,x} \\ &- k^2\cos^2\theta_i\sin\theta_i A_{i,z}]\}\exp(-j\mathbf{k}_i\mathbf{r}) = \\ &= a\sum_i C_i[\exp(j\mathbf{Kr}) + \exp(-j\mathbf{Kr})](-\mathbf{l}\cos\theta_i A_{i,x} + \mathbf{n}\sin\theta_i A_{i,z})\exp(-j\mathbf{k}_i\mathbf{r})\end{aligned} \tag{10}$$

where $a = (\tilde{\varepsilon}\omega^2)/(2c^2)$.

We calculate a vector product of Eq. (10) and a complex conjugate of Eq. (5), integrate it in the plane XY, divide the result by the integration domain area and take the limit of this expression for the integration area approaching infinity. Here we use the fact that the idealized grating has infinite dimensions in the plane XY. As a result we obtain the following system of equations:

$$\cos^2\theta_i \ddot{A}_{i,x} - 2jk\cos^3\theta_i \dot{A}_{i,x} - jk\sin^2\theta_i\cos\theta_i \dot{A}_{i,z} + k^2\cos^2\theta_i\sin^2\theta_i A_{i,x}$$

$$-k^2\cos^2\theta_i\sin^2\theta_i A_{i,z} = -a\left[\cos\theta_i\cos\theta_{i-1}\sqrt{\frac{\cos\theta_i}{\cos\theta_{i-1}}}A_{i-1,x}\exp(j\Delta_{i+}z)\right.$$

$$\left.+\cos\theta_i\cos\theta_{i+1}\sqrt{\frac{\cos\theta_i}{\cos\theta_{i+1}}}A_{i+1,x}\exp(-j\Delta_{i-}z)\right], \tag{11}$$

$$jk\cos\theta_i\sin^2\theta_i\dot{A}_{i,x} + k^2\cos^2\theta_i\sin^2\theta_i A_{i,x} - k^2\cos^2\theta_i\sin^2\theta_i A_{i,z}$$

$$= a\left[\sin\theta_i\sin\theta_{i-1}\sqrt{\frac{\cos\theta_i}{\cos\theta_{i-1}}}A_{i-1,z}\exp(j\Delta_{i+}z)\right.$$

$$\left.+\sin\theta_i\sin\theta_{i+1}\sqrt{\frac{\cos\theta_i}{\cos\theta_{i+1}}}A_{i+1,z}\exp(-j\Delta_{i-}z)\right] \tag{12}$$

where $\Delta_{i+} = K_z + k_{t,z} - k_{t-1,z}$, $\Delta_{i-} = K_z - k_{t,z} + k_{t+1,z}$.

Thus, we have obtained a system of equations describing TM-polarized wave diffraction on volume non-absorbing gratings, being a linear system of differential equations with variable coefficients. The system of Eqs. (11) and (12) corresponds to Eq. (10), provided the following condition is fulfilled:

$$K_x + k_{i,x} - k_{i-(1,x)} = 0. \tag{13}$$

This condition stems from the assumption that the grating spreads in the plane XY to infinity. On the other hand, the system of Eqs. (11), (12) is valid only if Eq. (13) is fulfilled. According to this condition we determine the direction of diffraction (diffraction angles θ_i) of plane waves with variable amplitudes into which the electromagnetic field is decomposed inside the grating. Knowing $k_{0,x}$ (determined by the direction of a non-diffracted wave), we can determine $k_{\pm1,x}$, then $k_{\pm2,x}$ and so on from Eq. (13). From $k_{i,x}$ we can calculate $k_{i,z}$ in accordance with the formula

$$k_{i,z} = \sqrt{k^2 - k_{i,x}^2}.$$

Thus, all the quantities are defined in the system of Eqs. (11), (12). It should be noted that in this system of equations $\cos\theta_i$ must be a real quantity, *i.e.*, for all diffracted waves there is a projection of Poynting vector on the axis OZ. So, the condition that $-1 < \sin\theta_i < 1$ must also be fulfilled. Since wave amplitudes for $i_{\min}$ and $i_{\max}$ are very small, the system of Eqs. (11) and (12) may be used in practice in most cases, for instance, to explain the properties of phase holograms. In order to take into account in the system of Eqs. (11), (12) the diffraction orders for which $\cos\theta_i$ is an imaginary number, further studies are necessary, as well as a probably somewhat different approach to the derivation of equations, because non-uniform waves do not normalize. If $\tilde{\varepsilon} \ll \varepsilon_{20}$, we may use a whole series of approximations. In this case, the amplitude $A_{i,x}(z) \approx A_{i,z}(z)$. Equalling the amplitudes $A_{i,x}(z) = A_{i,z}(z) = A_i(z)$ and deducing Eq. (12) from Eq. (11), for each i, one can obtain a simpler system of equations. Such a system of equations has been obtained in [11], and it has a relatively simple form

$$\frac{d^2 A_i}{dz^2}\cos^2\theta_i - 2jk_{i,z}\frac{dA_i}{dz} + a\cos(\theta_i - \theta_{i-1})\sqrt{\frac{\cos\theta_i}{\cos\theta_{i-1}}}A_{i-1}\exp(j\Delta_{i+}z)$$
$$+ a\cos(\theta_i - \theta_{i+1})\sqrt{\frac{\cos\theta_i}{\cos\theta_{i+1}}}A_{i+1}\exp(-j\Delta_{i-}z) = 0. \tag{14}$$

As we see, the system of Eq. (14) has half as many equations and is rather simple. Numerical calculations are necessary to determine the ratio $\tilde{\varepsilon}/\varepsilon_{20}$ for which the approximation (14) will not result in a serious error. To calculate diffraction on thick holograms, the parabolic approximation can be used, in which the second derivative in Eqs. (11) and (14) are neglected. If we neglect the second derivatives in Eq. (14) in the two-wave approximation, we obtain the equations listed in [2], [5]. As we see, the approximation of the system of Eqs. (11) and (12), when $\tilde{\varepsilon} \ll \varepsilon_{20}$, leads to the well -known systems of equations that describe approximately TM-polarization wave propagation in periodic media.

The coupled Eqs. (11) and (12) are linear with variable coefficients. This system can be transformed into the linear one with constant coefficients, if the following substitution for the variables is performed:

$$\begin{aligned} A_{i,x}(z) &= B_{i,x}(z)\exp(j\Delta_i z), \\ A_{i,z}(z) &= B_{i,z}(z)\exp(j\Delta_i z) \end{aligned} \tag{15}$$

where $\Delta_i = iK_z + k_{i,z}$.

Let us take the first derivative from the right-hand side and the left-hand side of expressions (15) and the second derivative from the first equation of (15). We obtain the following formulae:

$$\begin{aligned} \dot{A}_{i,x}(z) &= \dot{B}_{i,x}(z)\exp(j\Delta_i z) + j\Delta_i B_{i,x}(z)\exp(j\Delta_i z), \\ \dot{A}_{i,z}(z) &= \dot{B}_{i,z}(z)\exp(j\Delta_i z) + j\Delta_i B_{i,z}(z)\exp(j\Delta_i z), \\ \ddot{A}_{i,x}(z) &= \ddot{B}_{i,x}(z)\exp(j\Delta_i z) + 2j\Delta_i\dot{B}_{i,x}(z)\exp(j\Delta_i z) - \Delta_i^2 B_{i,x}(z)\exp(j\Delta_i z). \end{aligned} \tag{16}$$

Substituting (16) in (11) and (12), we obtain a linear system of differential equations with constant coefficients as follows:

$$\cos^2\theta_i\ddot{B}_{i,x} - 2jiK_z\cos^2\theta_i\dot{B}_{i,x} - jk\sin^2\theta_i\cos\theta_i\dot{B}_{i,z} - i^2K_z^2\cos^2\theta_iB_{i,x}$$
$$+ k^2\cos^2\theta_iB_{i,x} + ikK_z\sin^2\theta_i\cos\theta_iB_{i,z}$$
$$= -a\left[\cos\theta_i\cos\theta_{i-1}\sqrt{\frac{\cos\theta_i}{\cos\theta_{i-1}}}B_{i-1,x} + \cos\theta_i\cos\theta_{i+1}\sqrt{\frac{\cos\theta_i}{\cos\theta_{i+1}}}B_{i+1,x}\right], \tag{17}$$

$$jk\cos\theta_i\sin^2\theta_i\dot{B}_{i,x} - ikK_z\cos\theta_i\sin^2\theta_iB_{i,x} - k^2\cos^2\theta_i\sin^2\theta_iB_{i,z}$$
$$= a\left[\sin\theta_i\sin\theta_{i-1}\sqrt{\frac{\cos\theta_i}{\cos\theta_{i-1}}}B_{i-1,z} + \sin\theta_i\sin\theta_{i+1}\sqrt{\frac{\cos\theta_i}{\cos\theta_{i+1}}}B_{i+1,z}\right]. \tag{18}$$

In holography, the straight gratings are of great importance, *i.e.*, when $K_z = 0$. In this case, which we shall consider in more detail, the system of Eqs. (17), (18) is simplified and is written as follows:

$$\cos^2\theta_i\ddot{B}_{i,x} - jk\sin^2\theta_i\cos\theta_i\dot{B}_{i,z} + k^2\cos^2\theta_iB_{i,x}$$
$$= -a\left[\cos\theta_i\cos\theta_{i-1}\sqrt{\frac{\cos\theta_i}{\cos\theta_{i-1}}}B_{i-1,x} + \cos\theta_i\cos\theta_{i+1}\sqrt{\frac{\cos\theta_i}{\cos\theta_{i+1}}}B_{i+1,x}\right], \tag{19}$$

$$jk\cos\theta_i\sin^2\theta_i\dot{B}_{i,x} - k^2\cos^2\theta_i\sin^2\theta_iB_{i,z}$$
$$= a\left[\sin\theta_i\sin\theta_{i-1}\sqrt{\frac{\cos\theta_i}{\cos\theta_{i-1}}}B_{i-1,z} + \sin\theta_i\sin\theta_{i+1}\sqrt{\frac{\cos\theta_i}{\cos\theta_{i+1}}}B_{i+1,z}\right]. \tag{20}$$

Let us divide Eq. (19) by $\cos^2\theta_i$ and Eq. (20) by $j\cos\theta_i\sin^2\theta_i$. As a result, we obtain the following system of equations:

$$\ddot{B}_{i,x} - jk\frac{\sin^2\theta_i}{\cos\theta_i}\dot{B}_{i,z} + k^2B_{i,x} + a\left[\sqrt{\frac{\cos\theta_{i-1}}{\cos\theta_i}}B_{i-1,x} + \sqrt{\frac{\cos\theta_{i+1}}{\cos\theta_i}}B_{i+1,x}\right] = 0, \tag{21}$$

$$\dot{B}_{i,x} + jk\cos\theta_iB_{i,z} + d_iB_{i-1,z} + c_iB_{i+1,z} = 0 \tag{22}$$

where the coefficients d_i and c_i are given by the formulae:

$$d_i = j\frac{a\sin\theta_{i-1}}{k\sin\theta_i\sqrt{\cos\theta_i\cos\theta_{i-1}}}, \qquad c_i = j\frac{a\sin\theta_{i+1}}{k\sin\theta_i\sqrt{\cos\theta_i\cos\theta_{i+1}}}.$$

Taking the derivative of the Eq. (22), we obtain the following result:

$$\ddot{B}_{i,x} + jk\cos\theta_i \dot{B}_{i,z} + d_i \dot{B}_{i-1,z} + c_i \dot{B}_{i+1,z} = 0. \tag{23}$$

We subtract Eq. (23) from Eq. (21) and then taking into account Eq. (22), we obtain the following, linear first-order system of equations with constant coefficients:

$$\dot{B}_{i,z} + jk\cos\theta_i B_{i,x} + f_i B_{i-1,x} + g_i B_{i+1,x} + p_i \dot{B}_{i-1,z} + q_i \dot{B}_{i+1,z} = 0, \tag{24}$$

$$\dot{B}_{i,x} + jk\cos\theta_i B_{i,z} + d_i B_{i-1,z} + c_i B_{i+1,z} = 0 \tag{25}$$

where the coefficients f_i, g_i, p_i, q_i are respectively:

$$f_i = j\frac{a\sqrt{\cos\theta_i\cos\theta_{i-1}}}{k}, \quad g_i = j\frac{a\sqrt{\cos\theta_i\cos\theta_{i+1}}}{k},$$

$$p_i = -j\frac{\cos\theta_i d_i}{k}, \quad p_i = -j\frac{\cos\theta_i c_i}{k}.$$

We see that the linear system of the differential Eqs. (25), (26) is much simpler than the corresponding system of equations obtained in [7]. The equations of this system contain only variables or their derivatives with indices $i-1$, i, $i+1$. Therefore, it is convenient to solve such a system of equations by means of the standard software of Maple 6 type. It should be noted that the system of Eqs. (17), (18) can similarly be reduced to a first order linear system that will be similar to the system of Eqs. (25) and (26). It is noteworthy that using linear algebra methods one can reduce the subsystem of Eq. (24) to the following common form:

$$\dot{\mathbf{B}}_z = \mathbf{D}\mathbf{B}_x \tag{24a}$$

where $\dot{\mathbf{B}}_z$ is the column vector with components equal to $\dot{B}_{i,z}$, $\mathbf{D}$ – the square matrix with constant elements, $\mathbf{B}_x$ – the column vector with components equal to $B_{i,z}$; the dimensional representation of vector and square matrix is $i_{\max} - i_{\min} + 1$. But in this form, the subsystem of Eq. (24a) in the right-hand side will contain a linear combination of all $B_{i,x}$, where i varies from $i_{\min}$ to $i_{\max}$.

For a complete solution of the diffraction problem, especially when $\varepsilon_1 \neq \varepsilon_{20} \neq \varepsilon_3$ and $\tilde{\varepsilon}$ are not much smaller than ε_{20}, it is necessary to find exact initial conditions. This can be done in accordance with the method described in [6], [7].

In many practical problems that arise in holography condition $\tilde{\varepsilon} \ll \varepsilon_{20}$ is fulfilled and therefore one can use approximate initial conditions $B_{0,x}(0) = 1$, $B_{0,z}(0) = 1$ and $B_{i,x}$, $B_{i,z}$ equal to zero for all $i \neq 0$.

According to [10], the i-th component of Poynting vector projection on the axis OZ of the respective coupled wave is expressed as follows:

$$\langle \mathbf{S} \rangle_{i,\,z}(z) = \mathrm{Re}\left\{ \frac{c}{4\pi} [\mathbf{E}_i \mathbf{H}_i^*]_z \right\}. \tag{26}$$

In our case of $\mathbf{E}_i(x, z) = C_i[-\mathbf{l}\cos\theta_i B_{i,\,x} + \mathbf{n}\sin\theta_i B_{i,\,z}]\exp(-ik\sin\theta_i x)$, the magnetic field strength is (according to Eq. (25))

$$\begin{aligned} \mathbf{H}_i(x, z) &= j\frac{c}{\omega}\mathrm{rot}\mathbf{E}_i \\ &= \mathbf{m}\frac{c}{\omega}C_i[-kB_{i,\,z} + j\cos\theta_i d_i B_{i-1,\,z} + j\cos\theta_i c_i B_{i+1,\,z}]\exp(-jk\sin\theta_i x). \end{aligned} \tag{27}$$

After substituting $\mathbf{E}_i(x, z)$ and $\mathbf{H}_i(x, z)$ into formula (26), we obtain the following expression:

$$\langle \mathbf{S} \rangle_{i,\,z}(z) = \mathrm{Re}[B_{i,\,x}B_{i,\,z}^* + p_i B_{i,\,x}B_{i-1,\,z}^* + q_i B_{i,\,x}B_{i+1,\,z}^*]. \tag{28}$$

Direct substitution of initial conditions into Eq. (28) yields the following expressions for $\langle \mathbf{S} \rangle_{i,\,z}(0)$:

$$\langle \mathbf{S} \rangle_{0,\,z}(0) = 1, \quad \langle \mathbf{S} \rangle_{i,\,z}(0) = 0 \quad \text{for} \quad i \neq 0. \tag{29}$$

The notion of diffraction efficiency can be derived for each coupled wave as follows:

$$\eta_i(z) = \langle \mathbf{S} \rangle_{i,\,z}(z) = \mathrm{Re}[B_{i,\,x}B_{i,\,z}^* + p_i B_{i,\,x}B_{i-1,\,z}^* + q_i B_{i,\,x}B_{i+1,\,z}^*]. \tag{30}$$

From the results of [12] for the given initial conditions $\sum_{i_{\min}}^{i_{\max}} \eta_i(z) = 1$, which corresponds to the energy conservation law when light propagates in a non-absorbing medium with periodic variation of refraction index.

3. Numerical analysis of TM-polarized light diffraction on a non-absorbing thick phase hologram

Light diffraction analysis was conducted by means of numerical solution of the system of Eqs. (24), (25) using the Runge–Kutte method for holograms having the refraction index variation period Λ equal to 0.438 μm. This period for a mean refraction index of $n_0 = 1.577$ ($\varepsilon_{20} = 2.487$) corresponds to Bragg angle inside the medium $\theta_{\mathrm{B}} = 0.325$ rad. The analysis was made for a four-wave approximation, when $i_{\min} = -1$, $i_{\max} = 2$. To this end, a system of equations consisting of eight first-order differential equations was solved numerically. As the solution was obtained, all diffraction efficiencies η_i were sought, their sum was found, and $|B_{1,\,x}(z)|^2$ and $|B_{1,\,z}(z)|^2$ were calculated, with which the diffraction efficiency of the first-order diffraction η_1

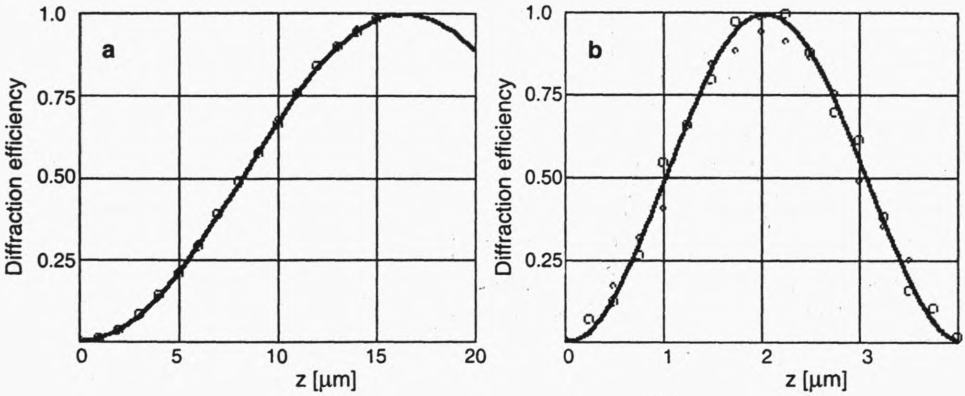

Fig. 2. Plot of diffraction efficiency $\eta_1(z)$ for $\tilde{\varepsilon} \approx 0.0505$ (**a**) and $\tilde{\varepsilon} \approx 0.404$ (**b**).

was compared. First, the diffraction efficiencies were calculated in relation to the coordinate z according to formula (30) for $\theta_0 = \theta_B$ and for the following amplitudes of a variable term of dielectric constant $\tilde{\varepsilon} = 0.0505$ and $\tilde{\varepsilon} = 0.404$, which are shown in Figs. 2**a** and **b**, respectively. The sum of the diffraction efficiencies along the whole interval of solutions differed from unity by less than 10^{-5}. This error results from the calculation accuracy, and appears when the calculation is done with 10 nonzero digits. When the calculation is done with 15 nonzero digits this error is less than 10^{-9}. This means that the proposed system of differential equations corresponds exactly to the energy conservation law [12]. In Fig. 2, the line is the plot of $\eta_1(z)$, the large circles are $|B_{1,x}(z)|^2$, the small circles are $|B_{1,z}(z)|^2$.

For $\tilde{\varepsilon} = 0.0505$ (which corresponds to the refraction index variation amplitude $\tilde{n} = 0.016$) $|B_{1,x}(z)|^2$ and $|B_{1,z}(z)|^2$ are very close to $\eta_1(z)$, and the difference in the corresponding coordinates in Fig. 2**a** is practically undistinguishable. In this case, the diffraction efficiencies $\eta_{-1}(z)$ and $\eta_2(z)$ are less than 0.0005. Therefore, in order to calculate holographic characteristics of thick holograms with parameters that correspond to Fig. 1**a**, one may use simpler theories, such as parabolic approximation [8]. In the parabolic approximation it is reasonable to make an additional simplification: $B_{i,x} \approx B_{i,z} \approx B_i$ [12]. The difference between $\eta_1(z)$ obtained by solving the accurate system of equations, and $\eta_1(z)$, calculated by means of parabolic approximation, is less than 0.0003 within the interval 0–20 μm for $\tilde{\varepsilon} = 0.0505$. For $\tilde{\varepsilon} = 0.404$ (which corresponds to the refraction index variation amplitude $\tilde{n} = 0.128$) $|B_{1,x}(z)|^2$, $|B_{1,z}(z)|^2$ and $\eta_1(z)$ differ between themselves, and this difference is clearly seen in Fig. 2**b**. Diffraction efficiencies $\eta_{-1}(z)$ and $\eta_2(z)$ are less than 0.02. The difference between $\eta_1(z)$ obtained from an accurate solution and that calculated from a parabolic approximation reaches 0.02. Therefore, in order to analyze phase holograms for large values of $\tilde{\varepsilon}$, one should use an accurate system of equations, which is especially important for small Bragg angles [12].

Figure 3 shows the dependence of the diffraction efficiency $\eta_1(z)$ of a 16 μm thick hologram on the propagation angle θ_0 of a non-diffracted beam for TE- and

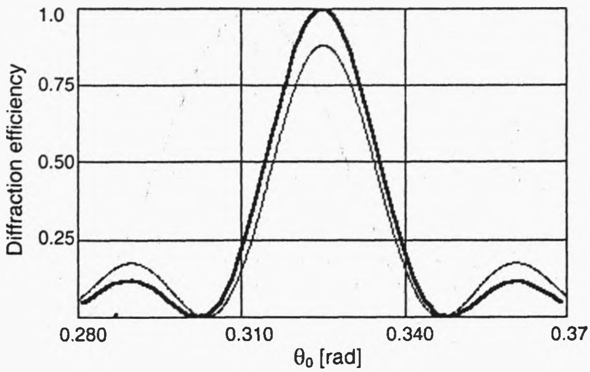

Fig. 3. Relationship between the diffraction efficiency and the angle θ_0 for TM-polarization (thick line) and for TE-polarization (thin line).

TM-polarization. The other data are the same as those in Fig. 2**a**. The calculation for TE-polarization was done by means of the parabolic approximation. For TM-polarization the calculations were made by means of accurate Eqs. (24), (25) and using parabolic approximation. In the case of TM-polarization the difference in the behaviour of the curves obtained by the two methods is not shown in Fig. 3.

As can be seen from Fig. 3, the angular relationship of diffraction efficiency for TM-polarization is typical of thick holograms. Since for this polarization the hologram parameters are selected, so that $\eta_1(z) \approx 1$ for $\theta_0 = \theta_B$, respective efficiency for TE-polarization is less than 1. The diffraction efficiency for TE-polarization will be equal to unity either for a smaller value of $\tilde{\varepsilon}$, or for a smaller hologram thickness which can be determined by formulae listed in [5]. Such angle relationships of diffraction efficiencies are typical of thick holograms [5].

4. Conclusions

A system of differential equations has been obtained to describe propagation of TM-polarized light in periodic non-absorbing structures. This system of differential equations with variables associated with electric field strength has been derived without using Floquet's theorem. The obtained system of equations is valid only for a certain correlation between the projections of diffracted wave vectors $k_{i,x}$ and the projection of the reverse grating K_x, which is equal to $K_x + k_{i,x} - k_{i-1,x} = 0$. The electric field inside the periodic structure is represented as a sum of normalized plane waves with variable amplitudes depending upon component z, each plane wave being described by two variable amplitudes that correspond to electric field strength projections on the axes OX and OZ. For condition $\tilde{\varepsilon} \ll \varepsilon_{20}$ one may use various simplifications and the system of differential equations obtained can be reduced to the well-known solutions.

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