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Transformation of polarization state of the light using wave plates with arbitrary phase difference: half wave plates

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The practical problem of the lack of a quarter-wave plate for an arbitrary wavelength has been solved in our earlier work. The possibility of substituting this plate with a combination of two (or more) retardation plates with phase shift different from 90° has been shown. However, there are some special cases in which another wave plate commonly used in polarization optics is needed, *i.e.*, a half-wave plate. In the present paper some formulae have been derived for several of the most important applications of these retardation plates in measuring setup.

Keywords: polarized light, polarization, wave plate, phase difference.

1. Introduction

The synthesis and analysis of light with arbitrary polarization state is one of the most important problems of polarization optics. In all measuring methods, in which polarized light is used, a generation of precisely controlled polarization state of the light is necessary as well as transformation of the analyzed light – changes of azimuth and ellipticity angles [1].

The basic tools used in polarization optics are always the same – a polarizer (in practice, a linear one) and two birefringent plates: one with phase difference between first and second eigenwaves equal to 90° ($\lambda/4$ in wavelength) – a quarter-wave plate, and second with phase difference between first and second eigenwaves equal to 180° ($\lambda/2$ in wavelength) – a half-wave plate. A half-wave plate is commonly used in some special cases and some most important of them will be discussed in this paper, *e.g.*:

– how to change the azimuth angle of linearly polarized light;

– how to change the sign of the ellipticity of light, without changing its azimuth angle;

– how to transform the light with a given polarization state into the light with orthogonal polarization state.

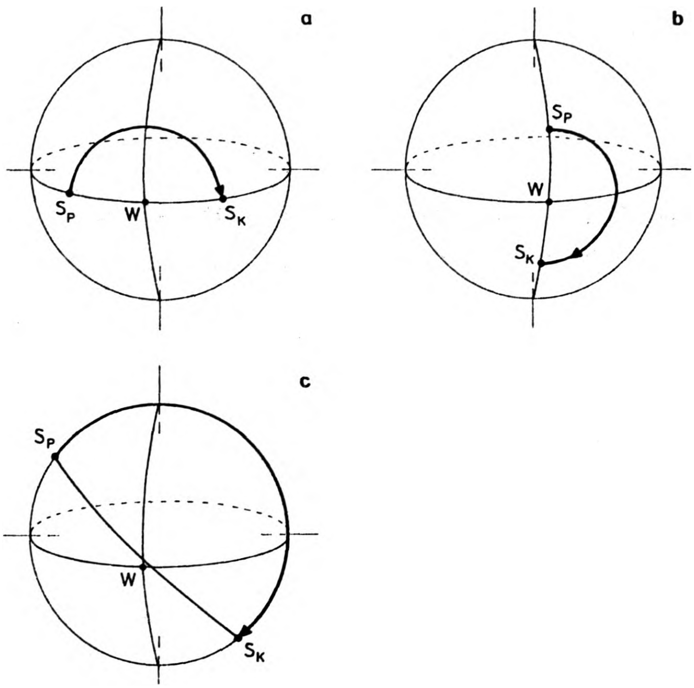


Fig. 1. Three most important cases of using the half-wave plates in optical setups: change of the azimuth angle of linearly polarized light (**a**); change of the sign of the ellipticity of light (**b**); change of the polarization state of light into perpendicular one (**c**).

The half-wave plate is easy to use for those purposes if only its phase difference is equal exactly to 180°. Figure 1 shows all the above-mentioned cases of using half -wave plate. The mismatch of phase shifts causes mismatch in parameters of the final polarization state of light which could be treated as errors in many cases, but sometimes the difference between an "ideal" half-wave plate and the one which is at our disposal, is too big and could not be treated as an "error". In a modern optical laboratory, a setup of good quality quarter- and half-wave plates for different wavelengths might be necessary, which is usually expensive and hard to achieve. As was shown in our earlier paper [2], due to the optical dispersion of birefringence of materials used to make retardation plates, the difference in phase shift in wave-plate made for He-Ne laser light (632.8 nm) used with light of sodium lamp (589.3 nm) reaches about 10°. This means that such a phase plate will be practically unacceptable to use as a half-wave plate (in the meaning described above). We propose a way of solving this problem by using two (or more) phase plates with an arbitrary phase difference. In our further considerations, similarly as in [2], we decided to use the Poincaré sphere formalism [3], [4], which let us find the parameters of polarized light in a more intuitive way than solving a system of complicated matrix equations.

2. Transformation of the azimuth angle of linearly polarized light

In this section, we analyze how to change the azimuth angle of linearly polarized light having at least two wave plates whose phase shifts are different from 180°. The polarization state of the input light is represented by point S_p with the azimuth angle α_p (Fig. 2). The first plate W_1 (with the azimuth angle α_1 and phase difference γ_1) transforms this state to the point S' (with the azimuth angle α' and ellipticity angle ϑ'), while the second plate W_2 (with the azimuth angle α_2 and phase difference γ_2) to the point S_k (with the azimuth angle α_k) on the equator of the sphere.

Based on the description given in Fig. 2, the following formulae for spherical triangle $W_1S'A$ could be easily written:

$$\sin \overset{\frown}{W_1S'} \sin \sphericalangle S'W_1A = \sin \overset{\frown}{AS'}, \tag{1a}$$

$$\sin \overset{\frown}{W_1A} \tan \sphericalangle S'W_1A = \tan \overset{\frown}{AS'}, \tag{2a}$$

$$\tan \overset{\frown}{W_1S'} \cos \sphericalangle S'W_1A = \tan \overset{\frown}{W_1A} \tag{3a}$$

where $\overset{\frown}{W_1S'} = \overset{\frown}{W_1S_p} = 2(\alpha_1 - \alpha_p)$, $\overset{\frown}{AS'} = 2\vartheta'$, $\overset{\frown}{W_1A} = 2(\alpha' - \alpha_1)$, $\sphericalangle S'W_1A = 180° - \gamma_1$. The same formulae for spherical triangle $W_2S'A$ are:

$$\sin \overset{\frown}{W_2S'} \sin \sphericalangle S'W_2A = \sin \overset{\frown}{AS'}, \tag{1b}$$

$$\sin \overset{\frown}{W_2A} \tan \sphericalangle S'W_2A = \tan \overset{\frown}{AS'}, \tag{2b}$$

$$\tan \overset{\frown}{W_2S'} \cos \sphericalangle S'W_2A = \tan \overset{\frown}{W_2A} \tag{3b}$$

where $\overset{\frown}{W_2S'} = \overset{\frown}{W_2S_k} = 2(\alpha_2 - \alpha_k)$, $\overset{\frown}{W_2A} = 2(\alpha' - \alpha_2)$, $\sphericalangle S'W_2A = 180° - \gamma_2$.

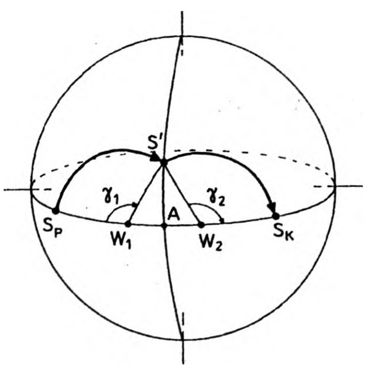


Fig. 2. Transformation of the azimuth angle of linearly polarized light.

In a general case, when $\gamma_1 \neq \gamma_2$, the system of equations obtained from spherical geometry analysis could not be solved in an analytical way and desired quantities α_1 and α_2 (azimuth angles of two retardation plates) could be obtained using numerical calculations. The analytical solution could be easily obtained when $\gamma_1 = \gamma_2 \equiv \gamma$, which fortunately happens most commonly in optical laboratory (the wave plates, which have the same retardations for one wavelength, have also the same retardation for another wavelength). In this case, one can immediately obtain from Eqs. (1a), (1b) and Eqs. (2a), (2b):

$$\alpha' = \frac{\alpha_p + \alpha_k}{2} \tag{4}$$

(the "intermediate" polarization state S' lies exactly in the middle of the arc $W_1 W_2$ on the Poincaré sphere), and

$$\alpha_1 - \alpha_p = \alpha_k - \alpha_2 \tag{5}$$

(the position of the first retardation plate W_1 in relation to input state of the light S_p and the position of the second retardation plate W_2 in relation to desired output state of the light S_k are the same).

Using the set of equations described above one can obtain the following quadratic equation:

$$(T_{kp}C)X^2 + (C+1)X - T_{kp} = 0 \tag{6}$$

where $T_{kp} = \tan(\alpha_k - \alpha_p)$, $C = \cos(180° - \gamma)$ (known values) and $X = \tan 2(\alpha_1 - \alpha_p)$ (calculated value). One can easily obtain from this equation the desired quantity $\alpha_1 - \alpha_p$ as well as the conditions for the existence of possible solutions for this quantity. The question is: which solution could be chosen (two possible solutions of quadratic equations and periodicity of tangent function) but they are common for any polarization calculations and the authors of the present paper are convinced that detailed explanations are unnecessary.

3. Transformation of the sign of the ellipticity of light, without changing its azimuth angle

This section is devoted to the problem of how to change the rotation of the light polarization state, that is the sign of the ellipticity of light without changing its azimuth angle. For retardation plate with phase difference γ equal exactly to 180° the solution is simple and intuitive, as shown in Fig. 1**b**. For plates with phase difference γ different from 180° the problem is also easy to solve using the Poincaré sphere formalism and Fig. 3 shows the right way. The polarization state of the input light is represented by point S_p with the azimuth angle α_p. The retardation plate W (with the azimuth angle α

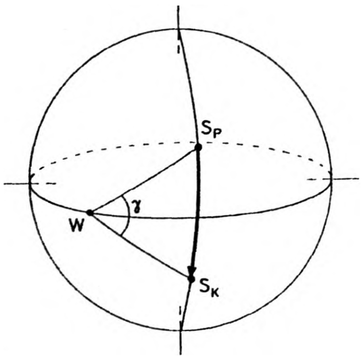


Fig. 3. Transformation of the ellipticity sign of light.

and phase difference γ) transforms this state to the point S_k (with the azimuth angle α_k). The following formulae for spherical triangle WS_pA could be written:

$$\sin \overset{\frown}{WA} \tan \sphericalangle S_p WA = \tan \overset{\frown}{AS_p} \tag{7}$$

where $\overset{\frown}{AS_p} = 2\vartheta_p$, $\overset{\frown}{WA} = 2(\alpha_p - \alpha)$, $\sphericalangle S_pWA = \gamma/2$. Note that in this case, only one retardation phase plate is needed, and its position according to the azimuth angle of input light could be easily calculated from Eq. (7)

$$\sin 2(\alpha_p - \alpha) = \frac{\tan 2\vartheta}{\tan \gamma/2}. \tag{8}$$

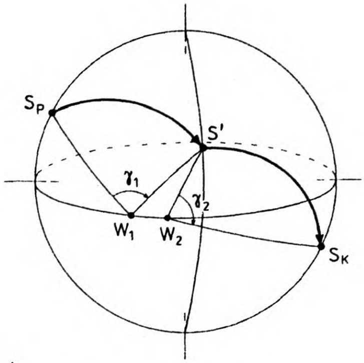
Moreover, the conditions for the existence of possible solutions for the azimuth angle α_p are also immediately obtained from Eq. (7)

$$|\tan 2\vartheta| \le \left|\tan \frac{\gamma}{2}\right|, \tag{9}$$

which means that only for retardation plate with phase difference $\gamma \le 180°$ all possible polarization states could be transformed into states with opposite ellipticity.

4. Transformation of the polarization state of light into the orthogonal one

In this case, the system of equations, obtained from spherical geometry analysis is also not easy to solve in analytical way even if $\gamma_1 = \gamma_2 \equiv \gamma$. There is no symmetry in general case due to the fact that retardation plates are linear, and the points W_1 and W_2 which represent first eigenvectors of these plates always should be placed on the equator of the sphere (Fig. 4). However, another simplifying assumption could be made that the transformation from the point S_p (which represents the input light with azimuth angle α_p and ellipticity angle ϑ_p) to the point S_k (which represents the output light with



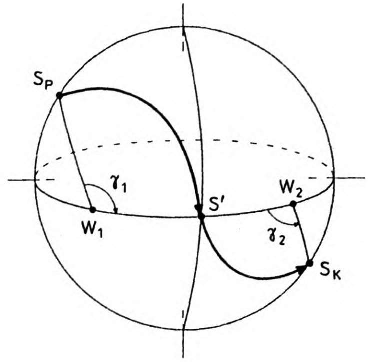


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Fig. 4. Transformation of the polarization state of light into the orthogonal one.

Fig. 5. Transformation of the polarization state of light into the orthogonal one using two wave plates with the same retardations $\gamma_1 = \gamma_2 \equiv \gamma$.

azimuth angle α_k and ellipticity angle ϑ_k; note that $\alpha_k = \alpha_p + 90°$ and $\vartheta_k = -\vartheta_p$ due to the fact that S_k is perpendicular to S_p) is made through the intermediate point S' (with the azimuth angle α'), which lies on the equator of the Poincaré sphere. Due to this limiting assumption we can easily obtain the formulae for azimuth angles α_1 and α_2 of retardation plates W_1 and W_2, respectively. We take the advantage of our earlier paper [2] in which such situations as transformation from elliptically polarized light into linear one as well as the opposite transformation were described. The scheme of computation is shown in Fig. 5. Also in this case we have made our calculations for two identical retardation plates, *i.e.* $\gamma_1 = \gamma_2 \equiv \gamma$ (see the comments in Sec. 2). The symmetry of the problem makes our computations easier. It readily follows from Fig. 5 that in this case the distance (on the Poincaré sphere) between the first retardation plate W_1 and polarization state of the input light S_p is equal to the distance between the second plate W_2 and output light S_k as shown by formula (5)

$$\alpha_1 - \alpha_p = \alpha_k - \alpha_2. \tag{10}$$

Let us note that α_2 denotes the azimuth angle of the second eigenvector of second retardation plate W_2, which simplifes our considerations. The following formula could be written for spherical triangle $W_1 S_p A$:

$$\sin \overset{\frown}{W_1 A} \tan \sphericalangle S_p W_1 A = \tan \overset{\frown}{A S_p} \tag{11}$$

where: $\overset{\frown}{AS_p} = 2\vartheta_p$, $\overset{\frown}{W_1 A} = 2(\alpha_1 - \alpha_p)$, $\sphericalangle S_p W_1 A = 180° - \gamma$ (note that this is the same formula as in Eq. (7)). The position of the first retardation plate W_1 according to the azimuth angle of input light could be easily calculated from Eq. (11)

$$\sin 2(\alpha_1 - \alpha_p) = \frac{\tan 2\vartheta}{\tan(180° - \gamma)} \tag{12}$$

and position (azimuth angle) of the second plate W_2 could be obtained from Eq. (10).

5. Conclusions

The main objective of this paper consisted in solving the practical problem, which could arise in optics laboratory, namely, lack of a half-wave plate for an arbitrary wavelength. It has been shown how to make use of retardation plates with phase shift different from 180° in some special cases, in which half-wave plates are usually used. Sometimes, it appears necessary to use more than one plate, as shown in Sections 2 and 4. All the formulae obtained are valid for arbitrary phase differences of the component plates and may be useful in some cases, especially if these differences are close to 180° (due to the solving conditions).

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