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Matrix approach for ray tracing through holographic lens

E. JAGOSZEWSKI

Institute of Physics, Technical University of Wrocław, Wybrzeże Wyspiańskiego 27, 50–370 Wrocław, Poland.

Matrix algebra is ideally suited to handle problems of transformation of the coordinates of rays passing the system, and is a powerful tool for analysis of quality of the optical systems. In this paper, we consider how matrices may be used to describe ray propagation through optical system containing a holographic lens.

1. Introduction

In a similar way as an optical system which consists of rotationally symmetric refracting (or reflecting) surfaces, a holographic system can also be described by matrices composed of elements representing intervals and the focusing powers (in this case, the diffracting powers). This matrix method is useful in elucidating fundamental paraxial properties, and permits many results to be derived mechanically, especially for lens combinations. If we take the optical axis to be along the z-axis of the Cartesian coordinate system, then the intersection points of rays at a certain plane perpendicular to the optical axis can be specified in the xy-plane by the vectors that contain information of the position and direction of each ray. It would be convenient that we can find the coordinates of the ray at any other plane normal to the optical axis, by means of successive operators acting on the initial ray coordinate vectors. These operators can be represented by matrices. The advantage of this matrix formalism is that any ray propagating through the holographic system can be treated by successive matrix multiplications, which can be easily programmed on a computer. Such a representation of geometrical optics is elegant and powerful, and is used in optical system design.

Ray tracing through a holographic system resolves itself into propagation from one plane to the next and to operation of diffraction at the holographic surfaces. A ray crosses each constant plane: $z = \text{const}$ at a point (x,y) with direction cosines (l,m,n), as shown in Fig. 1. The coordinates of the intersection points of meridional rays lying in the plane that includes the axis are then $(x,0)$. Paraxial rays are those in the limit of having small angles and small distances from the axis. A ray originating in the input plane: $z_0 = \text{const}$, at point (x_0,y_0) intersects the surface of holographic optical element at point (x,y), where:

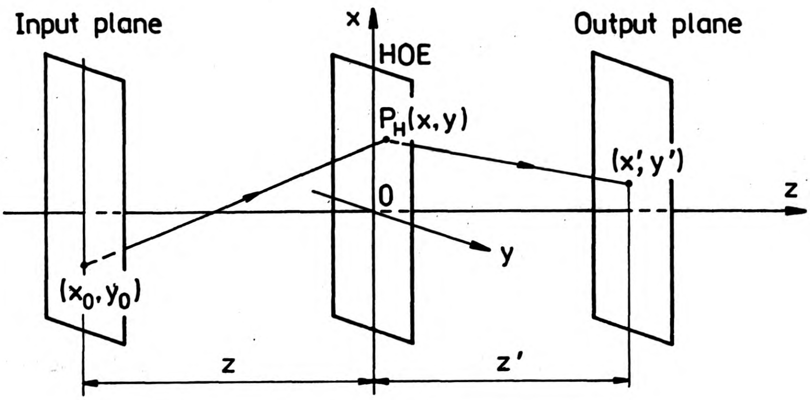

Fig. 1. Ray tracing through an elementary holographic system

$$x = x_0 + \frac{l}{n} z_0, \quad y = y_0 + \frac{m}{n} z_0.$$

If u is the slope (angle between the axis and the ray), then for the meridional rays we have

$$x = x_0 + z_0 \tan u_0, \quad x_I = x + z_I \tan u_I$$

where $(x_I, 0, z_I)$ are the coordinates of the intersection point of the ray at the output plane $z_I = \text{const}$. We see that for the transfer operation of paraxial rays, the outgoing heights of the intersection points depend linearly on the incoming heights and slopes, *i.e.*, a ray that enters the k-th surface of the system at height x_k and angle u_k leaves the next plane with the height

$$x_{k+1} = \frac{\partial x_{k+1}}{\partial x_k} x_k + \frac{\partial x_{k+1}}{\partial u_k} u_k. \tag{1}$$

Analogously for the power operation by an optical element, the outgoing angles depend linearly on the incoming heights and angles

$$u_{k+1} = \frac{\partial u_{k+1}}{\partial x_k} x_k + \frac{\partial u_{k+1}}{\partial u_k} u_k, \tag{2}$$

because for each focusing element (see Fig. 1) the slope of the paraxial ray tracing can be written in the form

$$u_{k+1} = -\frac{x_k}{f} + u_k.$$

The partial derivatives in the above equations depend on the structure of the system and on the axial locations of the input and output planes. They are determined by tracing rays through the system, and are constant for a given system. The linearity is the basis of the matrix treatment, since Eqs. (1) and (2) can be written in the matrix form

$$\begin{bmatrix} x_{k+1} \\ u_{k+1} \end{bmatrix} = \begin{pmatrix} \dfrac{\partial x_{k+1}}{\partial x_k} & \dfrac{\partial x_{k+1}}{\partial u_k} \\ \dfrac{\partial u_{k+1}}{\partial x_k} & \dfrac{\partial u_{k+1}}{\partial u_k} \end{pmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix}. \tag{3}$$

2. Ray tracing

The optical system design is usually based on the data obtained by ray tracing through the system [1]. In this paper, ray tracing through a holographic optical element is treated as a generalized case of tracing the rays through a diffraction grating. As we know, for the diffraction grating inserted in the second plane of the system shown in Fig. 1, the relationship between the direction cosines of the diffracted and the incident rays are given by the equations

$$\begin{aligned} l' &= l \pm \frac{\lambda}{d}\cos\varphi, \\ m' &= m \pm \frac{\lambda}{d}\cos\varphi, \\ n' &= \pm\sqrt{1-l'^2-m'^2}, \end{aligned} \tag{4}$$

where: λ is the wavelength of the incident light, d – the grating spacing, φ – the angle in the xy-plane between the normal to the grating lines and the x-axis.

The sign choice in the first two equations is used for the plus (+) order or the minus (−) order diffracted wave front; the sign choice in the last equation of (4) is used to select the z-direction of propagation of the wave front. Simultaneously, Eqs. (4) that determine the spatial frequency of the grating can be applied to define the diffracting power of the grating, *i.e.*,

$$\frac{1}{d_x} = \frac{l'-l}{\lambda} \quad \text{in the } xz\text{-plane,}$$

and

$$\frac{1}{d_y} = \frac{m'-m}{\lambda} \quad \text{in the } yz\text{-plane.}$$

A holographic optical element (HOE) is described by the transfer phase function that refers to the intensity distribution in the interference pattern between the object and reference recording wave fronts. When the HOE is illuminated by a wave front identical with the wave front of former reference beam, one of the diffracted beams that are formed is a duplicate of the former object recording beam. Therefore, this property of HOE's is used to determine the diffracting power. Let the holographic focusing element be recorded in the xy-plane inserted in the origin of coordinate system (as shown in Fig. 1) by two spherical wave fronts. The centres of the object and reference waves are situated at the points: $P_O(x_O, y_O, -z_O)$ and $P_R(-R \sin u, 0, -R\cos u)$, respectively, where R is the distance of P_R from origin of the coordinate system along a line making the angle u with the z-axis. In accordance with the fundamental property of such a holographic optical element which is illuminated by a point source of light of wavelength λ placed at $P_R(-R \sin u, 0, -R\cos u)$, a reconstructed wave front appears as a duplicate of the former object wave front to the right of the HOE. In this way the direction cosines of the ray incident at point $P_H(x,y,0)$ of the hologram from point source P_R are as follows:

$$l = \left(\frac{x}{R} - \sin u\right)\left(1 - \frac{2x}{R}\sin u + \frac{x^2+y^2}{R^2}\right)^{-1/2},$$

$$m = \frac{y}{R}\left(1 - \frac{2x}{R}\sin u + \frac{x^2+y^2}{R^2}\right)^{-1/2},$$

$$n = \left(1 - \frac{2x}{R}\sin u + \frac{x^2+y^2}{R^2}\right)^{-1/2}\cos u. \quad (5)$$

After diffraction this ray appears to come from the point $P_O(x_O, y_O, -z_O)$ with the direction cosines

$$l' = (x-x_O)[(x-x_O)^2+y^2+z_O^2]^{-1/2},$$
$$m' = y[(x-x_O)^2+y^2+z_O^2]^{-1/2},$$
$$n' = z_O[(x-x_O)^2+y^2+z_O^2]^{-1/2}. \quad (6)$$

We see that the diffracted light corresponds here to first order diffraction in the direction which would reconstruct the original object. Equations (5) and (6) can then be used to describe the diffraction power of the HOE at the point of intersection of the hologram surface by the incident and the diffracted rays. When a holographic optical element is considered as a general element in an optical system, computerized ray tracing requires the calculation of both the direction and amplitude of the diffracted rays.

3. Ray transfer matrices

In ray tracing through an optical system, we have four types of transformations of ray coordinates:

– transformation describing the reflection of rays at a reflecting surface,

– transformation describing the refraction of light passing through a refracting surface,

– transformation describing the diffraction effect of light at a diffracting surface, and

– transformation describing the translation of the rays from one to the next surface in a homogeneous medium.

Restricting ourselves to one transverse direction (for example, the x-direction), the ray at a given plane (z = const) may be specified by its height in x-direction from the optical axis and by its slope u that it makes with the optical axis. Thus, the two quantities, height and slope, represent the ray at a given plane in the examined optical system.

Reflection and refraction are governed by Snell's law which states that the refracted (or reflected) ray lies in the plane defined by the incident ray and the normal to the point of incidence. The relationship between the incidence and the refraction (reflection) angles can be described by Snell's law in the vector form

$$N'(\mathbf{r}' \times \mathbf{S}) = N(\mathbf{r} \times \mathbf{S}) \tag{7}$$

where $\mathbf{r}$ and $\mathbf{r}'$ are the unit vectors of the incident and the refracted (reflected) rays at the point of incidence, respectively. $\mathbf{S}$ is the unit vector of the normal to the refracting (reflecting) surface at the ray intersection point T, N, and N' are the refractive indices of the medium before and after refraction. For reflection $N' = -N$, thus we can treat reflection as a special case of refraction. For simplicity, we consider ray tracing only in the xy-plane. The optical direction cosines before and after refraction are then defined by the relations:

$$L = Nl, \quad L' = N'l',$$

and the coordinates at the point of refraction of the surface are always the same: $x' = x$. Thus, the transformation equation for refraction [2], [3] is given by

$$\begin{bmatrix} L' \\ x' \end{bmatrix} = \begin{pmatrix} 1 & -A \\ 0 & 1 \end{pmatrix} \begin{bmatrix} L \\ x \end{bmatrix} \tag{8}$$

where $A = \dfrac{N'\cos i' - N\cos i}{\rho}$ is the power of the refracting surface, ρ is the radius of curvature of the spherical surface, i, i' are the incident and refraction angles, respectively.

The purpose of this paper is to describe the ray tracing through a diffracting surface and to derive the equation for holographic optical element in the matrix form, analogously as we have shown for the refraction surface. Let a holographic optical element be inserted in a right-handed coordinate system having its origin at the centre (or at the vertex in case of curved surface) of hologram. For curved diffraction surface defined by the equation

$$F(x,y,z) = 0,$$

the unit vector normal to the surface at the point of incidence $P_H(x,y,z)$ is determined by the surface derivatives

$$\nabla F(x,y,z) = \mathbf{i}\frac{\partial F}{\partial x} + \mathbf{j}\frac{\partial F}{\partial y} + \mathbf{k}\frac{\partial F}{\partial z},$$

and the direction cosines of the normal vector are given by

$$\cos\alpha = \frac{\partial F}{\partial x}\left[\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2\right]^{-1/2},$$

$$\cos\beta = \frac{\partial F}{\partial y}\left[\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2\right]^{-1/2},$$

$$\cos\gamma = \frac{\partial F}{\partial z}\left[\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2\right]^{-1/2}.$$

The unit vectors $\mathbf{r}$ and $\mathbf{r}'$ of the incident and diffracted rays define the incident and diffraction angles, respectively. They are taken to be positive when the z-axis has to be rotated counterclockwise through the angle of less than $\pi/2$ to bring it into coincidence with the ray.

The ray tracing through a holographic optical element resolves itself into determining the direction of rays leaving a generalized holographic grating, and has been discussed in scientific literature [4]. Many papers tried to emulate the diffraction properties of the element by means of mathematical analogy between a holographic and a conventional optical element. Usually, the holographic optical element is created by the interference of two perfect wave fronts, where both reference and object beams originate from single point sources. It is relatively simple to trace rays through such a hologram, since one can easily compute the exact directional cosines of each beam that is incident at a given point of the optical surface. It is also possible to obtain a simple vector equation for ray tracing [4] which applies to any shape of holographic optical element. This equation is applicable to holographic lenses, holographic gratings, and is given by

$$\mathbf{S}\times[(\mathbf{r}_I-\mathbf{r}_C)\pm\frac{\lambda}{\lambda_0}(\mathbf{r}_O-\mathbf{r}_R)] = 0 \tag{9}$$

where $\mathbf{S}$ is the unit vector along the local normal to the hologram surface at the incident point P_H (see Fig. 2), $\mathbf{r}_O$, $\mathbf{r}_R$ are the unit vectors along the rays emerging

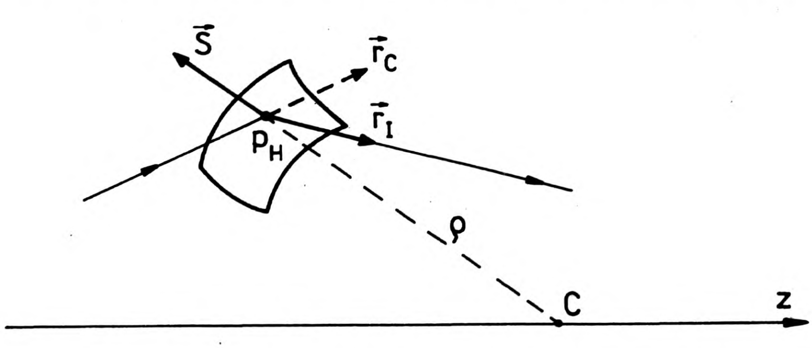

Fig. 2. Unit vectors $\mathbf{r}_C, \mathbf{r}_I$ along the readout rays at the incident point P_H; S is the unit vector along the local normal to the holographic surface, and ρ is the curvature radius of this curved surface

from the object and reference points at P_H, and $\mathbf{r}_C$, $\mathbf{r}_I$ are the unit vectors along the reconstruction and the diffracted (image) rays at P_H; λ_0, λ are the wavelengths used in construction and reconstruction, respectively. Therefore the grating equations hold for the incident ray impinging on the optical element and are given in components form by

$$l_I = l_C \pm \frac{\lambda}{2\pi}\frac{\partial\Phi(x,y)}{\partial x},$$
$$m_I = m_C \pm \frac{\lambda}{2\pi}\frac{\partial\Phi(x,y)}{\partial y},$$
$$n_I = \pm\sqrt{1-l_I^2-m_I^2} \tag{10}$$

where the components of the unit vectors $\mathbf{r}_I$, $\mathbf{r}_C$, $\mathbf{r}_O$, $\mathbf{r}_R$, determine the direction cosines of the respective rays. The function $\Phi(x, y)$ defines the transfer phase function of the holographic optical element, and the partial derivatives of this function are evaluated in the incident point of the HOE. In agreement with the definition of matrices for the conventional elements, the matrix equation for the holographic element takes the form

$$\begin{bmatrix} l' \\ x' \end{bmatrix} = \frac{1}{k}\begin{pmatrix} k & -\frac{1}{x}\frac{\partial\Phi}{\partial x} \\ 0 & k \end{pmatrix}\begin{bmatrix} l \\ x \end{bmatrix} \tag{11}$$

where $k = 2\pi/\lambda$ is the wave number of beam used in the diffraction process. The transfer phase function specifies the transparency of the holographic optical element whose transmittance is given by

$$t(x,y) = \frac{1}{2} + \left(\frac{1}{2}\right)\cos[\Phi(x,y)].$$

When only one pair of an input and its desired output wave fronts is given, we obtain the trivial phase function: $\Phi(x,y) = \Phi_{\text{out}} - \Phi_{\text{in}}$. Generally, we have a set of continuously different input phase distributions and another set of corresponding, desired output distributions. Therefore, it would be impossible to obtain a holographic optical element which converts every input wave front into each desired output wave front. The design is to determine parameters of an optimal holographic element that make the differences between the emergent and desired wave fronts as small as possible over the set of inputs. If we consider a complete system containing a diffracting surface (*i.e.*, holographic element), as shown in Fig. 3, then the matrix describing the transformation of the coordinates of a ray from the first to the last plane is given by

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ a_I & 1 \end{pmatrix}\begin{pmatrix} 1 & -\frac{1}{kx}\frac{\partial\Phi}{\partial x} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -a_O & 1 \end{pmatrix},$$

where a_O and a_I are the distances along the rays in the object and image spaces, respectively. Now carrying out the multiplication, we have

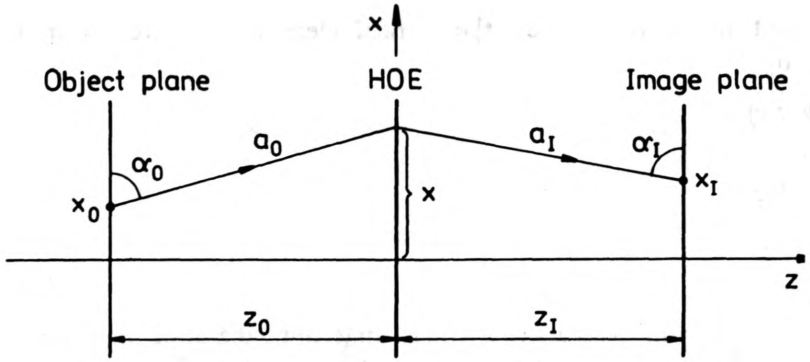

Fig. 3. Object–image relationship; in the sign convention the distances are $a_O \langle 0, a_I \rangle 0$

$$\begin{bmatrix} l' \\ x' \end{bmatrix} = \begin{pmatrix} 1+\frac{a_0}{kx}\frac{\partial \Phi}{\partial x} & -\frac{1}{kx}\frac{\partial \Phi}{\partial x} \\ a_I - a_O + \frac{a_I a_O}{kx}\frac{\partial \Phi}{\partial x} & 1-\frac{a_I}{kx}\frac{\partial \Phi}{\partial x} \end{pmatrix} \begin{bmatrix} l \\ x \end{bmatrix} \tag{12}$$

where in the coordinate column $x' = x_I$ for $x = x_O$. It follows that

$$x_I = \left(a_I + \frac{a_O a_I}{kx}\frac{\partial \Phi}{\partial x}\right) l - \frac{a_I}{k}\frac{\partial \Phi}{\partial x}.$$

We require that x_I should be independent of l, since $x_I = M x_O$ for any angle u_O, M is the linear magnification. The lower left-hand element of the system matrix in Eq. (12) should be equal to zero

$$a_I - a_O + \frac{a_O a_I}{kx}\frac{\partial \Phi}{\partial x} = 0,$$

hence

$$a_I = a_O \bigg/ \left(1 + \frac{a_O}{kx}\frac{\partial \Phi}{\partial x}\right).$$

Further,

$$M = 1 - \frac{a_I}{kx}\frac{\partial \Phi}{\partial x} = \frac{1}{1 + \frac{a_O}{kx}\frac{\partial \Phi}{\partial x}},$$

which is due to the fact that the determinant value of the matrix A must be equal to unity. For the axial conjugate points, $x_O = x_I = 0$, and

$$l' = \left(1 + \frac{a_O}{kx}\frac{\partial \Phi}{\partial x}\right) l,$$

i.e.,

$$M\sin u_I = \sin u_O,$$

where u_O and u_I are the angles which the ray forms with the optical axis in the object and image spaces, respectively.

4. Conclusion

The ray tracing through an optical system containing a holographic optical element has been discussed. We introduced the diffraction matrix which describes the transformation of two coordinates of the incoming ray. One can treat the direction of the ray and the position of a point on the ray by asking where a point in space is imaged after passing the system. This treatment can be extended to the analysis of more complicated systems.

References

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