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Point source scattering by a metallic half-plane

S. Asghar, Tasawar Hayat

Mathematics Department, Quaid-i-Azam University, Islamabad, Pakistan.

The diffraction of an E-polarized and an H-polarized spherical wave (emanating from a point source) by an imperfectly conducting half-plane (on which impedance boundary conditions are imposed) is studied. This consideration is important because the point source is regarded as fundamental radiating device. The two independent problems are solved by a method due to Jones based on the Wiener–Hopf technique.

1. Introduction

The diffraction of electromagnetic waves from perfectly conducting obstacles, on which the tangential component of electric field vanishes, has been discussed by many authors. Lee [1] considered the problem of diffraction by an open-ended waveguide and calculated the field inside two parallel, semi-infinite, perfectly staggered plates. However, Mittra and Lee [2] presented their results in the case of non-staggered plates and calculated the field within and outside the waveguide. All these considerations were based upon the fundamental assumption of perfect conductivity. In practice, however, obstacles which would have perfect conductivity are unlikely to be encountered. It is therefore desirable to discuss the problems of diffraction from obstacles having impedance boundary conditions. Subsequently, employing these impedance boundary conditions, solutions have been obtained for the problem of diffraction by half-plane or metallic sheets (Senior [3], [4] and Williams [5]), and for the problem of diffraction by a wedge (Williams [6], [7] and Senior [8]). In 1959, Malinzhinets [9] studied scalar diffraction by a wedge with different face impedances and obtained a complete solution by means of a function theoretic technique. When the wedge angle is zero, the solution for a half-plane with different face impedances is obtained. The electromagnetic diffraction with different techniques has also been discussed in papers [10]–[16].

In this paper, we examine the diffraction of a spherical electromagnetic wave by an imperfectly conducting half-plane. This consideration is important in the sense that point sources are regarded as better substitutes for real sources. The integral transforms, Wiener–Hopf technique [17] and asymptotic methods are employed to obtain the integral representation of the diffracted field. These integrals are normally difficult to handle and are only amenable to solution using asymptotic approximations. The far field is calculated at the end using the steepest descent method [18].

2. Formulation of the problem

Let (x', y', z') define right-handed rectangular Cartesian coordinates with origin 0 and z'-axis parallel to the edge of imperfectly conducting half-plane. The metallic plane $y' = 0$, $-\infty < z' < \infty$, $x' \geqslant 0$, has finite conductivity, vanishing thickness and satisfies the impedance boundary conditions [19]. The region exterior to the half-plane $x' \geqslant 0$ is assumed to be a vacuum with permeability and dielectric constants μ_0 and ε_0, respectively. The time dependence is assumed to be of harmonic nature of the type $e^{-i\omega t}$ and is suppressed throughout. In the H-mode the electric field is polarized parallel to the z'-axis, *i.e.*, $E_{x'} = E_{y'} = 0$ and, consequently, $H_{z'} = 0$. We consider a point source to be located at the position (x'_0, y'_0, z'_0). The wave equation satisfied by $E_{z'}$ in presence of a point source having unit strength is

$$\left(\frac{\partial^2}{\partial x'^2}+\frac{\partial^2}{\partial y'^2}+\frac{\partial^2}{\partial z'^2}+\tilde{k}^2\right)E_{z'} = \delta(x'-x'_0)\delta(y'-y'_0)\delta(z'-z'_0) \tag{1}$$

where $\tilde{k} = \omega(\varepsilon_0\mu_0)^{1/2} = 2\pi/\lambda$, λ is the wavelength in meters and ω is the frequency. In the E-mode the magnetic field is polarized parallel to the z'-axis, *i.e.*, $H_{x'} = H_{y'} = E_{z'} = 0$ and $H_{z'}$ satisfies Eq. (1).

The boundary conditions associated with a metal of high but finite conductivity in H-mode and E-mode, respectively, are given by [19]. Subsequent analysis will be much simplified if the variables are non-dimensional. Thus, on writing $x' = lx$, $y' = ly$, $z' = lz$, $x'_0 = lx_0$, $y'_0 = ly_0$, $z'_0 = lz_0$, $E_{z'} = 1/l\,E_z$, $H_{z'} = 1/l\,H_z$, Eq. (1) becomes

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}+k^2\right)E_z = \delta(x-x_0)\delta(y-y_0)\delta(z-z_0), \tag{2}$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}+k^2\right)H_z = \delta(x-x_0)\delta(y-y_0)\delta(z-z_0), \tag{3}$$

where $k = l\tilde{k}$ is non-dimensional.

On the metallic plate, occupying the position $x \geqslant 0$, we have the boundary conditions [19]

$$E_z(x,0^{\pm},z) = \pm i\delta\frac{\partial}{\partial y}E_z(x,0^{\pm},z), \tag{4}$$

$$\frac{\partial}{\partial y}H_z(x,0^{\pm},z) = \pm i\mu H_z(x,0^{\pm},z) \tag{5}$$

where $\delta\,(=1/kn)$ and $\mu\,(=-k/n)$ are non-dimensional and n is the complex refractive of the metal. Thus, Eqs. (2) and (4), together with Eqs. (3) and (5), specify two boundary value problems which, whilst mathematically identical, will be treated as independent problems. The independent treatment, which will be presented in Sect. 3 and 4, respectively, is necessary because approximate solutions are given which depend on the fact that both $|\delta|$ and $|\mu|$ are small quantities. The electric and magnetic field components (E_z and H_z, respectively) and their respective derivatives

$\left(\frac{\partial}{\partial y}E_z \text{ and } \frac{\partial}{\partial y}H_z\right)$ are continuous on $x<0$, *i.e.*,

$$\left.\begin{aligned}\Phi(x,0^+,z) &= \Phi(x,0^-,z)\\ \frac{\partial}{\partial y}\Phi(x,0^+,z) &= \frac{\partial}{\partial y}\Phi(x,\ 0^-,z)\end{aligned}\right\} x<0. \tag{6}$$

In Equations (6), Φ is equal to E_z and H_z in H- and E-modes, respectively.

3. Solution of the *H*-mode problem

In this section, we consider the diffraction of an incident H-mode spherical wave from a half-plane. It is convenient to split the total field E_z as

$$E_z = E_z^{inc} + \varphi(x,y,z) \tag{7}$$

where E_z^{inc} is the solution of inhomogeneous wave equation (2) that corresponds to the incident wave, and φ is the solution of homogeneous wave equation (2) that corresponds to the diffracted field.

In addition, we insist that $\varphi(x,y,z)$ represents an outward travelling wave as $r=(x^2+y^2+z^2)^{1/2}\to\infty$ and satisfies the edge conditions [18]:

$$\left.\begin{aligned}\varphi(x,0,z) &= O(1)\\ \frac{\partial\varphi}{\partial y}(x,0,z) &= O(x^{-1/2})\end{aligned}\right\} \text{ as } x\to 0^-. \tag{8}$$

The Fourier transform and its inverse over the variable z are defined as

$$\left.\begin{aligned}\psi_t(x,y,\zeta) &= \int_{-\infty}^{\infty} E_z(x,y,z)e^{-ik\zeta z}dz\\ E_z(x,y,z) &= \frac{k}{2\pi}\int_{-\infty}^{\infty}\psi_t(x,y,\zeta)e^{ik\zeta z}d\zeta\end{aligned}\right\}. \tag{9}$$

In Equation (9), the transform parameter is taken as $k\zeta$ (ζ is non-dimensional). For analytic convenience, k is assumed to be complex and has a small positive imaginary part. The decomposition (9) is common in other field theories as well, for example, Fourier optics [20], [21]. Transforming Eqs. (2), (7) and the boundary conditions (4) and (6), we obtain

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k^2\gamma^2\right]\psi_t(x,y,\zeta) = \tilde{a}\delta(x-x_0)\delta(y-y_0), \tag{10}$$

$$\left.\psi_t(x,0^\pm,\zeta) = \pm i\delta\frac{\partial}{\partial y}\psi_t(x,0^\pm,\zeta)\right\} x>0, \tag{11}$$

$$\left.\begin{aligned}\psi_t(x,0^+,\zeta) &= \psi_t(x,0^-,\zeta)\\ \frac{\partial}{\partial y}\psi_t(x,0^+,\zeta) &= \frac{\partial}{\partial y}\psi_t(x,0^-,\zeta)\end{aligned}\right\} x<0, \tag{12}$$

$$\psi_t = \psi_0 + \psi(x,y,\zeta) \tag{13}$$

where:

$$\gamma^2 = (1-\zeta^2), \quad \tilde{a} = e^{-ik\zeta z_0}. \tag{13a,b}$$

Using Equation (13), Equations (10)−(12) can be written as:

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k^2\gamma^2\right]\psi_0(x,y,\zeta) = \tilde{a}\delta(x-x_0)\delta(y-y_0), \tag{14}$$

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k^2\gamma^2\right]\psi(x,y,\zeta) = 0, \tag{15}$$

subject to the boundary conditions

$$\left(1 \mp i\delta\frac{\partial}{\partial y}\right)[\psi_0(x,0,\zeta)+\psi(x0^{\pm},\zeta)] = 0, \; x>0, \tag{16}$$

$$\left.\begin{aligned} &\frac{\partial}{\partial y}\psi_t(x,0^+,\zeta) = \frac{\partial}{\partial y}\psi_t(x,0^-,\zeta) \\ &\psi_t(x,0^+,\zeta) = \psi_t(x,0^-,\zeta) \end{aligned}\right\} \; x<0. \tag{17}$$

It is convenient to introduce half-range Fourier transform on x as:

$$\begin{aligned} \bar{\psi}_+(\alpha,y,\zeta) &= \int_0^\infty \psi(x,y,\zeta)e^{i\alpha x}dx \\ \bar{\psi}_-(\alpha,y,\zeta) &= \int_{-\infty}^0 \psi(x,y,\zeta)e^{i\alpha x}dx \end{aligned} \tag{18}$$

where $\bar{\psi}_+(\alpha,y,\zeta)$ is regular for $\mathrm{Im}\alpha > -\mathrm{Im}k\gamma$, $\bar{\psi}_-(\alpha,y,\zeta)$ is regular for $\mathrm{Im}\alpha < \mathrm{Im}k\gamma$ and the inverse transform, which lies along the real line, is

$$\psi(x,y,\zeta) = (2\pi)^{-1/2}\int_{-\infty}^{\infty}(\bar{\psi}_+ + \bar{\psi}_-)e^{-i\alpha x}d\alpha. \tag{19}$$

The solution of Equation (14) can be written as

$$\begin{aligned} \psi_0 &= \frac{\tilde{a}}{4\pi i}\int_{-\infty}^{\infty}\frac{e^{-i\alpha(x-x_0)+i(k^2\gamma^2-\alpha^2)^{1/2}|y-y_0|}}{(k^2\gamma^2-\alpha^2)^{1/2}}d\alpha, \\ &= -\frac{\tilde{a}}{4i}H_0^{(1)}(k\gamma[(x-x_0)^2+(y-y_0)^2]^{1/2}). \end{aligned} \tag{20}$$

Now changing variables:

$$x_0 = r_0\cos\vartheta_0, \quad y_0 = r_0\sin\vartheta_0, \quad \pi \leqslant \vartheta_0 \leqslant 3\pi/2,$$

in Eq. (20), and letting $r_0 \to \infty$, we obtain, using the asymptotic form for the Hankel function

$$\psi_0 = \tilde{b}(\zeta)e^{-ik\gamma(x\cos\vartheta_0 + y\sin\vartheta_0)} \tag{21}$$

where ϑ_0 is the angle of incidence from the x-axis and

$$\tilde{b}(\zeta) = -\frac{\tilde{a}}{4i}\sqrt{\frac{2}{\pi k\gamma r_0}}\, e^{i(k\gamma r_0 - \pi/4)}. \tag{22}$$

Equation (15) is transformed to yield

$$\frac{d^2}{dy^2}\bar{\psi}(\alpha,y,\zeta) + \tilde{\gamma}^2\bar{\psi}(\alpha,y,\zeta) = 0, \tag{23}$$

and solution of Eq. (23) satisfying radiation condition is given by

$$\bar{\psi}(\alpha,y,\zeta) = \begin{cases} \tilde{A}_1(\alpha)e^{i\tilde{\gamma}y}, & y \geqslant 0, \\ \tilde{A}_2(\alpha)e^{-i\tilde{\gamma}y}, & y \leqslant 0 \end{cases} \tag{24}$$

where $\tilde{\gamma}^2 = (k^2\gamma^2 - \alpha^2)$, and to determine the two unknown functions of α it is necessary to transform the boundary conditions (16) and (17). These give:

$$\bar{\psi}_+(\alpha,0^+,\zeta) - i\delta\bar{\psi}'_+(\alpha,0^+,\zeta) + \frac{i\tilde{b}(\zeta)(1 - k\gamma\delta\sin\vartheta_0)}{(\alpha - k\gamma\cos\vartheta_0)} = 0, \tag{25a}$$

$$\bar{\psi}_+(\alpha,0^-,\zeta) + i\delta\bar{\psi}'_+(\alpha,0^-,\zeta) + \frac{i\tilde{b}(\zeta)(1 + k\gamma\delta\sin\vartheta_0)}{(\alpha - k\gamma\cos\vartheta_0)} = 0, \tag{25b}$$

$$\left.\begin{aligned} \bar{\psi}_-(\alpha,0^+,\zeta) &= \bar{\psi}_-(\alpha,0^-,\zeta) = \bar{\psi}_-(\alpha,0,\zeta) \\ \bar{\psi}'_-(\alpha,0^+,\zeta) &= \bar{\psi}'_-(\alpha,0^-,\zeta) = \bar{\psi}'_-(\alpha,0,\zeta) \end{aligned}\right\} \tag{25c}$$

where the prime denotes differentiation with respect to y. From Eqs. (24) and (25c), it is known that

$$\bar{\psi}_+(\alpha,0^+,\zeta) + \bar{\psi}_-(\alpha,0,\zeta) = \tilde{A}_1(\alpha), \tag{25d}$$
$$\bar{\psi}_+(\alpha,0^-,\zeta) + \bar{\psi}_-(\alpha,0,\zeta) = \tilde{A}_2(\alpha), \tag{25e}$$
$$\bar{\psi}'_+(\alpha,0^+,\zeta) + \bar{\psi}'_-(\alpha,0,\zeta) = i\tilde{\gamma}\tilde{A}_1(\alpha), \tag{25f}$$
$$\bar{\psi}'_+(\alpha,0^-,\zeta) + \bar{\psi}'_-(\alpha,0,\zeta) = -i\tilde{\gamma}\tilde{A}_2(\alpha). \tag{25g}$$

From Equations (25d)–(25g), we have

$$\tilde{A}_1(\alpha) = J_+(\alpha,0,\zeta) + \frac{1}{i\tilde{\gamma}}J'_+(\alpha,0,\zeta), \tag{26a}$$

$$\tilde{A}_2(\alpha) = -J_+(\alpha,0,\zeta) + \frac{1}{i\tilde{\gamma}}J'_+(\alpha,0,\zeta), \tag{26b}$$

$$\bar{\psi}'_+(\alpha,0^+,\zeta) + \bar{\psi}'_-(\alpha,0,\zeta) = i\tilde{\gamma}[\bar{\psi}_+(\alpha,0^+,\zeta) + \bar{\psi}_-(\alpha,0,\zeta)], \tag{26c}$$

$$\bar{\psi}'_+(\alpha,0^-,\zeta) + \bar{\psi}'_-(\alpha,0,\zeta) = -i\tilde{\gamma}[\bar{\psi}_+(\alpha,0^-,\zeta) + \bar{\psi}_-(\alpha,0,\zeta)], \tag{26d}$$

where:

$$J_+(\alpha,0,\zeta) = \frac{1}{2}[\bar{\psi}_+(\alpha,0^+,\zeta) - \bar{\psi}_+(\alpha,0^-,\zeta)], \tag{27a}$$

$$J'_+(\alpha,0,\zeta) = \frac{1}{2}[\bar{\psi}'_+(\alpha,0^+,\zeta) - \bar{\psi}'_+(\alpha,0^-,\zeta)]. \tag{27b}$$

On eliminating $\bar{\psi}'_+(\alpha,0^+,\zeta)$ between Equations (25a) and (26c), $\bar{\psi}'_+(\alpha,0^-,\zeta)$ from Eqs. (25b) and (26d), and then adding the resulting equations we obtain

$$\bar{\psi}'_-(\alpha,0,\zeta) - i\tilde{\gamma}\tilde{L}(\alpha)J_+(\alpha,0,\zeta) - \frac{\tilde{b}(\zeta)k\gamma\sin\vartheta_0}{(\alpha - k\gamma\cos\vartheta_0)} = 0, \tag{28}$$

where

$$\tilde{L}(\alpha) = (1 + 1/\delta\tilde{\gamma}).$$

Similarly, after eliminating $\bar{\psi}_+(\alpha,0^+,\zeta)$ between Equations (25a) and (26c), $\bar{\psi}_+(\alpha,0^-,\zeta)$ from Eqs. (25b) and (26d), and after subtracting the two resulting equations, we get

$$-i\alpha\bar{\psi}_-(\alpha,0,\zeta) + \delta\tilde{L}(\alpha)J'_+(\alpha,0,\zeta) - \frac{\tilde{b}(\zeta)}{(\alpha - k\gamma\cos\vartheta_0)} = 0. \tag{29}$$

Now, for Equations (26a,b), we have to solve Equations (28) and (29). For the solution we make the following factorization:

$$\tilde{L}_-(\alpha) = \tilde{L}_+(\alpha)\tilde{L}_-(\alpha) \tag{30}$$

where: $\tilde{L}_+(\alpha)$ is regular in the upper half-plane and $\tilde{L}_-(\alpha)$ is regular in the lower half-plane. The explicit forms of $\tilde{L}_\pm(\alpha)$ are given in the Appendix. Now, after using Eq. (30) and $\tilde{\gamma}^2 = (k^2\gamma^2 - \alpha^2)$, Eq. (28) can be rewritten as

$$\frac{\bar{\psi}'_-(\alpha,0,\zeta)}{\tilde{L}(\alpha)\sqrt{k\gamma - \alpha}} - i\sqrt{k\gamma + \alpha}\,\tilde{L}_+(\alpha)\,J_+(\alpha,0,\zeta) - \frac{\tilde{b}(\zeta)k\gamma\sin\vartheta_0}{(\alpha - k\gamma\cos\vartheta_0)\sqrt{k\gamma - \alpha}\,\tilde{L}_-(\alpha)} = 0. \tag{31}$$

We write

$$\frac{-\tilde{b}(\zeta)k\gamma\sin\vartheta_0}{(\alpha - k\gamma\cos\vartheta_0)\sqrt{k\gamma - \alpha}\,\tilde{L}_-(\alpha)} = \tilde{\varphi}_+(\alpha) + \tilde{\varphi}_-(\alpha) \tag{32}$$

where:

$$\tilde{\varphi}_+(\alpha) = \frac{\tilde{b}(\zeta)k\gamma\sin\vartheta_0}{(k\gamma\cos\vartheta_0 - \alpha)\tilde{L}_-(k\gamma\cos\vartheta_0)\sqrt{k\gamma - k\gamma\cos\vartheta_0}}, \tag{33a}$$

$$\tilde{\varphi}_-(\alpha) = \frac{\tilde{b}(\zeta)k\gamma\sin\vartheta_0}{(k\gamma\cos\vartheta_0 - \alpha)}\left\{\frac{1}{\tilde{L}_-(\alpha)\sqrt{k\gamma - \alpha}} - \frac{1}{\tilde{L}_-(k\gamma\cos\vartheta_0)\sqrt{k\gamma - k\gamma\cos\vartheta_0}}\right\} \tag{33b}$$

are regular in the upper and lower half-planes, respectively. With the help of Eqs. (32) and (33a,b), Eq. (31) can be arranged as

$$-i\sqrt{k\gamma+\alpha}\,\tilde{L}_+(\alpha)\,J_+(\alpha,0,\zeta)+\tilde{\varphi}_+(\alpha) = -\frac{\tilde{\psi}'_-(\alpha,0,\zeta)}{\tilde{L}_-(\alpha)\sqrt{k\gamma-\alpha}}-\tilde{\varphi}_-(\alpha). \tag{34}$$

We note that the left- and the right-hand sides in Equation (34) are regular in the upper and lower half-planes, respectively. Thus, on adopting the usual Wiener–Hopf procedure and using the extended form of Liouville's theorem [17], both sides can be equated to zero. Thus, on equating the left-hand side of Eq. (34) to zero and using Eq. (33a), we obtain

$$J_+(\alpha,0,\zeta) = \frac{\tilde{b}(\zeta)k\gamma\sin\vartheta_0(\sqrt{k\gamma-k\gamma\cos\vartheta_0})^{-1}}{i\sqrt{k\gamma+\alpha}\,\tilde{L}_+(\alpha)(k\gamma\cos\vartheta_0-\alpha)\,\tilde{L}_-(k\gamma\cos\vartheta_0)}. \tag{35}$$

For the Wiener–Hopf equation (29), a similar analysis can be repeated to arrive at

$$J'_+(\alpha,0,\zeta) = \frac{\tilde{b}(\zeta)}{\delta\tilde{L}_-(k\gamma\cos\vartheta_0)(\alpha-k\gamma\cos\vartheta_0)\,\tilde{L}_+(\alpha)}. \tag{36}$$

From Equations (28a), (37) and (38), we have

$$\tilde{A}_1(\alpha) = \frac{i\tilde{b}(\zeta)}{\tilde{L}_-(k\gamma\cos\vartheta_0)(\alpha-k\gamma\cos\vartheta_0)\,\tilde{L}_+(\alpha)}\left\{\frac{k\gamma\sin\vartheta_0}{\sqrt{k\gamma+\alpha}\sqrt{k\gamma-k\gamma\cos\vartheta_0}}-\frac{1}{\tilde{\gamma}\delta}\right\}. \tag{37}$$

Finally, the diffracted field can be written with the help of Equations (19), (24) and (37). Thus,

$$\psi(x,y,\zeta) = \tilde{b}(\zeta)\int_{-\infty}^{\infty}\tilde{f}(\alpha)e^{i\tilde{\gamma}y-i\alpha x}d\alpha \tag{38}$$

where

$$\tilde{f}(\alpha) = \frac{i(2\pi)^{-1/2}}{\tilde{L}_-(k\gamma\cos\vartheta_0)(\alpha-k\gamma\cos\vartheta_0)\,\tilde{L}_+(\alpha)}\left\{\frac{k\gamma\sin\vartheta_0}{\sqrt{k\gamma+\alpha}\sqrt{k\gamma-k\gamma\cos\vartheta_0}}-\frac{1}{\tilde{\gamma}\delta}\right\}. \tag{39}$$

We find that for $x>0$, the reflected wave as given by

$$\mathrm{RF} = R\tilde{b}(\zeta)e^{-ik\gamma(x\cos\vartheta_0-y\sin\vartheta_0)} \tag{39a}$$

can be calculated from Eq. (38) by deforming the contour in the upper half-plane, when the pole $\alpha = k\gamma\cos\vartheta_0$ is captured and we obtain the reflection coefficient R as given by

$$R = (1-k\gamma\delta\sin\vartheta_0)/(1+k\gamma\delta\sin\vartheta_0). \tag{39b}$$

In order to determine the diffracted field the integral appearing in Equation (38) can be evaluated asymptotically by using the steepest descent method [18]. For that reason, we introduce $x = r\cos\vartheta$, $y = r\sin\vartheta$ and deform the contour by the transformation $\alpha = -k\gamma\cos(\vartheta+ip)$ $(0<\vartheta<\pi,\ -\infty<p<\infty)$. Hence for large $k\gamma r$

$$\psi(x,y,\zeta) = -b(\zeta)\left[\frac{2\pi k\gamma}{r}\right]^{1/2}\sin\vartheta\tilde{f}(-k\gamma\cos\vartheta)e^{i(k\gamma r-\pi/4)}. \tag{40}$$

Now, from Equations (9), (22) and (40), we have

$$\varphi(x,y,z)=\frac{-k\sin\vartheta}{4\pi\sqrt{rr_0}}\int_{-\infty}^{\infty}\tilde{f}(-k\gamma\cos\vartheta)e^{ik[\gamma(r+r_0)+\zeta(z-z_0)]}d\zeta. \tag{41}$$

In order to solve the integral appearing in Equation (41), we put $r+r_0=r_{12}\sin\sigma$, $z-z_0=r_{12}\cos\sigma$ and using the transformation $\zeta=\cos(\sigma+iq)$, which changes the contour of integration over ζ into a hyperbola passing through the point $\cos\sigma$. The integral is then solved asymptotically using saddle point method and the resulting expression is given by

$$\varphi(x,y,z)=\frac{-k\sin\vartheta\sin\sigma\tilde{f}(-k\cos\vartheta\sin\sigma)}{\sqrt{2rr_0r_{12}\pi k}}e^{i(kr_{12}-\pi/4)} \tag{42}$$

where $\tilde{f}(-k\cos\vartheta\sin\sigma)$ is given by Eq. (39).

4. Solution of the *E*-mode problem

In this section, the boundary value problem (Eqs. (3), (5) and (6)) considered is that of the diffraction of an incident *E*-mode spherical wave from metallic half-plane. It is appropriate to split the total field H_z as

$$H_z=H_z^{inc}+\rho(x,y,z)$$

where H_z^{inc} is the solution of inhomogeneous wave equation (3) that corresponds to the incident wave and ρ is the solution of homogeneous wave equation (3) that corresponds to the diffracted field. Now, following the same method of solution as in Sect. 3, we have

$$\rho(x,y,z)=\frac{-k\sin\vartheta_0\sin\sigma\tilde{f}(-k\cos\vartheta\sin\sigma)}{\sqrt{2rr_0r_{12}\pi k}}e^{i(kr_{12}-\pi/4)} \tag{43}$$

where

$$\hat{f}(\alpha)=\frac{i(2\pi)^{-1/2}}{\hat{L}_-(k\gamma\cos\vartheta_0)(\alpha-k\gamma\cos\vartheta_0)\hat{L}_+(\alpha)}\left\{\frac{k\gamma\sin\vartheta_0}{\sqrt{k\gamma+\alpha}\sqrt{k\gamma-k\gamma\cos\vartheta_0}}+\frac{\mu}{\tilde{\gamma}}\right\},$$

$\hat{L}(\alpha)=\hat{L}_+(\alpha)\hat{L}_-(\alpha)=(1-\mu/\tilde{\gamma})$ and explicit form of $\hat{L}_\pm(\alpha)$ are given in the Appendix.

5. Conclusions

We have solved a canonical diffraction problem of a spherical electromagnetic wave by a metallic half-plane. This problem has practical advantages since point sources are regarded as better substitutes for real sources than line sources. We address this problem using an analytical approach based on the Wiener–Hopf method. A key attribute of such an approach is that it is not fundamentally numerical in nature and thus allows additional insight into the methematical and physical structure of the diffracted field. Expression for the reflection coefficient is obtained. Of particular interest is the possibility of obtaining in Eq. (39b) no reflection at a particular angle. This occurs when the numerator is zero. For this case,

$$k\gamma\delta\sin\vartheta_0 = 1,$$

or

$$\vartheta_0 = \sin^{-1}(n/\gamma).$$

At this angle, which is called the Brewster angle, there is no reflected wave when the incident wave is parallel (or vertically) polarized. In this case, E is parallel to the plane of incidence and H is parallel to the reflecting surface.

From Equations (42) and (43), it is interesting to note that the mathematical problem considered in this paper by taking $\delta = \infty$ and $\mu = 0$ in H- and E-modes, respectively, is identical to the problem which arises when an acoustic wave is diffracted from an imperfectly rigid (hard) barrier. We observe that when the complex refractive index of the metal becomes infinite, the half-plane is perfectly conducting and $\delta = 0$. In this case, there is no absorption of energy. Thus the consideration of a spherical wave in this paper represents somewhat more generalized model in the theory of diffraction and quite a few physical situations can be obtained as a special case by choosing suitable parameters.

Appendix

In this Appendix, we present briefly the factorization of the function

$$\tilde{L}(\alpha) = 1 + \frac{\delta_1}{(k^2\gamma^2 - \alpha^2)^{1/2}} = \tilde{L}_+(\alpha)\,\tilde{L}_-(\alpha) \tag{A1}$$

where $\delta_1 = 1/\delta$.

Using the results of NOBLE [17, p. 164], Equation (A1) can be written in the form

$$\tilde{L}(\alpha) = 1 + \delta_1[g_+(\alpha) + g_-(\alpha)] = [1 + \delta_1 g_+(\alpha)][1 + \delta_1 g_-(\alpha)]. \tag{A2}$$

In Equation (A2),

$$g_\pm(\alpha) = \frac{1}{\pi(k^2\gamma^2 - a^2)^{1/2}}\cos^{-1}(\pm\alpha/k\gamma).$$

Now, combining Equations (A1) and (A2), we obtain

$$\tilde{L}_\pm(\alpha) = 1 + \delta_1\pi^{-1}(k^2\gamma^2 - \alpha^2)^{-1/2}\cos^{-1}(\pm\alpha/k\gamma).$$

Similarly,

$$\hat{L}_\pm(\alpha) = 1 - \mu\pi^{-1}(k^2\gamma^2 - \alpha^2)^{-1/2}\cos^{-1}(\pm\alpha/k\gamma).$$

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