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The influence of the layer thickness on the coupling efficiency of plane waveguide with periodically variable refractive index**

Coupling equations for the plane waveguide with periodically changing refractive index have been derived. A weak modulation of the refractive index has been assumed. Also it has been accepted that both the media surrounding the waveguide have the same refractive index equal to the average refractive index of the waveguide. The solution of coupling equation for a purely phase and purely amplitude modulation has been given and the influence of the layer thickness on the coupling value examined.

Consider a plane waveguide, which is unlimited in the direction of x and y axes, respectively, and has the thickness s — called layer thickness — along the z axis. The coupling may be great but not great enough to change considerably the amplitude along the way equal to one wavelength of the radiation used. This allows to neglect the second derivatives of the amplitude. Let us assume, that the refractive index changes periodically

$$n = n_0+\eta\cos(\boldsymbol{\beta}\cdot\boldsymbol{r}+\varphi). \tag{1}$$

The vector $\boldsymbol{\beta}$ is perpendicular to the planes $n =$ const. A medium of this kind may be realized, for instance, with the help of accoustic waves propagating in a dielectric medium. The change of the refractive index in an acoustic field propagates with the velocity of sound. This motion is not taken into account in the equation (1), which may be treated as being caused by the field at a given time. The refractive index distribution determined by equation (1) may be obtained, for instance, by taking a photo of a spatial interference pattern. After bleaching the distribution of the refractive index is equivalent to that of n (eq. (1)).

This is a three-dimensional distribution of the phase grating in the dielectric material. The geometry of the problem is shown in fig. 1. Besides the vector $\boldsymbol{\beta}$, the vector $\boldsymbol{k}_2$ of incident plane, and vector $\boldsymbol{k}_s$ of the scattered wave have been also marked.

Hereafter we will assume that the media surrounding the layer have the refractive indices n_0. This considerably simplifies the discussion (allowing to neglect the reflection and refraction at the boundary surfaces) and does not restrict the generality of the derived conclusions. The coefficient η is assumed to be small, hence eq. (1) can be written in the form

$$n^2 \cong n_0^2+2\eta n_0\cos(\boldsymbol{\beta}\cdot\boldsymbol{r}+\varphi). \tag{2}$$

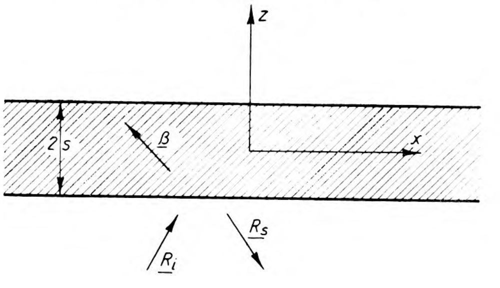

Fig. 1. Plane waveguide with periodically changing refractive index

Equation (2) may be generalized by introducing a complex refractive index

$$n_0 = n'-in''. \tag{3}$$

We assume that modulation of the refractive index $|\eta|$ is weak, and that the losses are defined in such a way that the amplitude along the way λ changes only slightly ($n'' \ll n'$). This assumptions enable to write the equation (2) in the form

$$n^2 \cong n'^2-2in'n''-2\eta n' \cos(\boldsymbol{\beta}\cdot\boldsymbol{r}). \tag{4}$$

The restricting conditions are almost always satisfied, otherwice the absorption becomes so strong that the Bragg condition is no more valid. A medium of refractive index determined by equation (4) may play the part of a spatial grating of phase, amplitude or mixed type. For the wave falling in accordance with the geometry shown in fig. 1 the resulting field may be written in the form

$$\Psi = A(z)\exp(-i\boldsymbol{k}_i\cdot\boldsymbol{r})+B(z)\exp(-i\boldsymbol{k}_s\cdot\boldsymbol{r}), \tag{5}$$

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where the first term corresponds to the incident wave, and the second one to the diffracted wave. By inserting the wave function defined by eq. (5) into the wave equation and neglecting the second derivatives we get [1, 2]:

$$\left[-2ik_{iz}\frac{\partial A}{\partial z}-2in'n''k_0^2A+\right.$$
$$\left.+\eta n'k_0^2B\exp\big(i(\boldsymbol{k}_i-\boldsymbol{k}_s-\boldsymbol{\beta})\cdot\boldsymbol{r}\big)\right]\exp(-i\boldsymbol{k}_i\cdot\boldsymbol{r})+$$
$$+\left[-2ik_{sz}\frac{\partial B}{\partial z}-2in'n''k_0^2B+\right.$$
$$\left.+\eta n'k_0^2A\exp\big(i(\boldsymbol{k}_s-\boldsymbol{k}_i+\boldsymbol{\beta})\cdot\boldsymbol{r}\big)\right]\exp(-i\boldsymbol{k}_s\cdot\boldsymbol{r})+$$
$$+\eta n'k_0^2\big[A\exp\big(-i(\boldsymbol{k}_i+\boldsymbol{\beta})\cdot\boldsymbol{r}\big)+$$
$$+B\exp\big(-i(\boldsymbol{k}_s-\boldsymbol{\beta})\cdot\boldsymbol{r}\big)\big]=0, \tag{6}$$

where k_{iz} and k_{sz} are the z-components of the vectors $\boldsymbol{k}_i$ and $\boldsymbol{k}_s$. If the Bragg condition is fulfilled the exponent in the third term of the first and second squared brackets disappears. Let us multiply the both sides of (6) by $\exp(i\boldsymbol{k}_i\cdot\boldsymbol{r})$ and integrate over the whole space. The integration along the z-axis is here reduced to that along the layer thickness. Thus the integration path is short as compared to the distance along which $A(z)$ and $B(z)$ change considerable, but it covers simultaneously many periods of oscillation. Keeping in mind that it follows that all the

$$\iiint\exp\big(i(\boldsymbol{k}_s'-\boldsymbol{k}_s)\cdot\boldsymbol{r}\big)dxdydz \tag{6a}$$
$$=(2\pi)^3\delta(k_{sx}'-k_{sx})\cdot\delta(k_{sy}'-k_{sy})\cdot\delta(k_{sz}'-k_{sz}),$$

we obtain that all the terms containing an exponential expression of quick oscillation will approximately cancel each other. This is valid for all the terms contained in the second squared bracket and for the first term in the third squared bracket. When Bragg condition holds, the second term in the third squared bracket is, however, different from zero and its neglecting as done by MARCUSE [2, 3] is unjustified. The integrad of the remaining terms will be equal to zero for all x and y if the integrad is equal to zero, i.e. if

$$\frac{\partial A}{\partial z}+\frac{n'n''k_0^2}{k_{iz}}A=\frac{\eta n'k_0^2}{ik_{iz}}B. \tag{7}$$

By multiplying the equation (6) by $\exp(i\boldsymbol{k}_s\cdot\boldsymbol{r})$ and integrating over the whole space we get the relation

$$\frac{\partial B}{\partial z}+\frac{n'n''k_0^2}{k_{sz}}B=\frac{\eta n'k_0^2}{ik_{sz}}A. \tag{7a}$$

For a constant refractive index ($\eta=0$) there is no coupling and the both equations (7) and (7a) are identical. Then the solution of equation (7) is given by the following function

$$A=A_0\exp\big(-a_i(z+s)\big),$$

where

$$a_i=\frac{n'n''k_0^2}{k_{iz}} \tag{8}$$

and the solution of the equation (7a) is

$$B=B_0\exp\big(-a_s(z+s)\big),$$

where

$$a_s=\frac{n'n''k_0^2}{k_{sz}}. \tag{8a}$$

In this case the both waves are evanescent. For $\eta\neq 0$ equations (7) and (7a) take the form

$$\frac{\partial A}{\partial z}+a_iA=-i\frac{k_0}{k_{iz}}bB,$$
$$\frac{\partial B}{\partial z}+a_iB=-i\frac{k_0}{k_{sz}}bA, \tag{9}$$

where

$$b=\eta n'k_0'. \tag{9a}$$

By excluding B from the second equation we obtain

$$\frac{\partial^2A}{\partial z^2}+(a_i+a_s)\frac{\partial A}{\partial z}+\left(a_i\cdot a_s+\frac{k^0b^2}{k_{iz}k_{sz}}\right)A=0. \tag{10}$$

In this equation the $\partial^2A/\partial z^2$ may not be neglected, because the quantities a_i, a_s, and b are small.

The solution of equation (10) my be sought in the form $A=\exp(\alpha z)$. After substitution we obtain a quadratic equation for α

$$\alpha^2+(a_i+a_s)\alpha+\left(a_ia_s+\frac{k_0^2b^2}{k_{iz}k_{sz}}\right)=0. \tag{11}$$

By solving it with respect to α we obtain

$$\alpha_\pm=-\frac{1}{2}(a_i+a_s)\pm$$
$$\pm\frac{1}{2}\sqrt{(a_i-a_s)^2-4\frac{k_0^2b^2}{k_{iz}k_{sz}}}. \tag{11a}$$

Thus solution of equation (10) will be given by the function

$$A=c\exp\big(\alpha_+(z+s)\big)+d\exp\big(\alpha_-(z+s)\big). \tag{12}$$

From the first of equations (9) we get

$$B=i\frac{k_{iz}}{k_0b}\big[(\alpha_++a_i)c\exp\big(\alpha_+(z+s)\big)+$$
$$+(\alpha_-+a_i)d\exp\big(\alpha_-(z+s)\big)\big]. \tag{13}$$

In order to examine the expression obtained let us divide the possible cases into two groups. The cases in which the scattered wave appears at the same side as the incident wave, i.e. if

$$B(s) = 0, \tag{14}$$

and

$$A(-s) = A_0 \tag{15}$$

belong to the first group. This scattering is called the backward or reflection scattering. The second group comprises the cases scattering in which the scattered wave appears at the opposite side with respect to the incident wave. Then

$$B(-s) = 0,$$
$$A(-s) = A_0.$$

This scattering is called forward-scattering or transmission-scattering. For the first case (see eq. (14) we have

$$c = \frac{(\alpha_- + a_i)\exp 2\alpha_- s A_0}{(\alpha_- + a_i)\exp 2\alpha_- s - (\alpha_+ + a_i)\exp 2\alpha_+ s}, \tag{16}$$
$$d = \frac{(\alpha_+ + a_i)\exp 2\alpha_+ s A_0}{(\alpha_- + a_i)\exp 2\alpha_- s - (\alpha_+ + a_i)\exp 2\alpha_+ s}$$

while for the second one we get

$$c = \frac{\alpha_- + a_i}{\alpha_- - \alpha_+} A_0, \tag{17}$$
$$d = -\frac{\alpha_+ + a_i}{\alpha_- - \alpha_+} A_0.$$

For a stricktly wave grating $n' = 0$, thus $a_i = 0$, $a_s = 0$. The equation (11a) may be simplified to the form

$$\alpha_\pm = \pm i\,\delta,$$
$$\delta = \frac{k_0 b}{\sqrt{k_{iz}\cdot k_{sz}}}. \tag{18}$$

The magnitude of amplitudes may be evaluated from equations (12) and (13). For the second case we get from (17)

$$A(z) = A_0 \cos\delta(z+s),$$
$$B(z) = -i\sqrt{\frac{k_{iz}}{k_{sz}}} A_0 \sin\delta(z+s), \tag{19}$$

for

$$-s \leqslant z \leqslant s.$$

Thus exchange of power occurs periodically. The effectivity will be equal to 100% if there exists only a scattered wave $(A(s) = 0)$, i.e. for the thickness satisfying the relation

$$2\delta s = (2n+1)\frac{\pi}{2},$$
$$n = 0, 1, 2, \ldots. \tag{20}$$

Generally, the amount of the transmitted power depends upon the coupling and the thickness of the layer s. For the very small thickness ($2\delta s \ll 1$) the solution of equation (19) may be simplified to the form

$$B(s) = -2is\sqrt{\frac{k_{iz}}{k_{sz}}}\,\delta A_0$$
$$= -2i\frac{n' k_0^2 \eta'}{k_{sz}} A_0 s. \tag{21}$$

For the first case (backward-scattering) the direction of the scattered wave is reverse to that of the incident wave

$$k_{iz}\cdot k_{sz} < 0.$$

Parameter δ is thus imaginary

$$\alpha_\pm = \pm|\delta|, \tag{22}$$

and the amplitude is represented as follows

$$A(z) = \frac{ch|\delta|(z-s)}{ch 2|\delta| s} \quad \text{for} \quad -s \leqslant z \leqslant s,$$
$$B(z) = i\sqrt{\left|\frac{k_{iz}}{k_{sz}}\right|}\frac{sh|\delta|(z-s)}{ch 2|\delta| s}. \tag{23}$$

For this scattering there is no power oscillation and the complete power transfer is impossible. The power transfer is the better the greater is the thickness. In this problem the thickness is not critical as it was the case for forward-scattering. Now, consider the amplitude gratings with both the components of the refractive index admitted. To avoid amplification we assume

$$n'' \geqslant |\eta|.$$

Now, the coefficient η is imaginery and $b^2 < 0$. To scatter the transmission

$$k_{iz}\cdot k_{sz} > 0.$$

For this case the equation (11a) is written as follows

$$\alpha_\pm = -a \pm \gamma,$$
$$a = \frac{1}{2}(a_i + a_s), \tag{24}$$
$$\gamma = \frac{1}{2}\sqrt{(a_i - a_s)^2 + 4\left|\frac{k_0^2 b^2}{k_{iz} k_{sz}}\right|}.$$

The amplitudes calculated from the equations (12) and (13) amount to

$$A(z) = \exp\left(-a(z+s)\right)\Big\{ch[\gamma(z+s)]-$$

$$-\frac{a_i-a_s}{2\gamma} sh[\gamma(z+s)]\Big\} A_0. \quad (25)$$

$$B(z) = \frac{ik_0 b}{\gamma k_{sz}} \exp\left(-a(z+s)\right) sh[\gamma(z+s)] A_0.$$

From this equations it is visible that the full exchange of power is impossible.

Consider the normal incidence ($|k_i| = k_{iz}$). As we are interested in the scattering satisfying the Bragg condition

$$\boldsymbol{k}_s = \boldsymbol{k}_i \pm \boldsymbol{\beta},$$

then the components x and y of the vectors $\boldsymbol{k}_s$ are identical with the respective components of the vector $\boldsymbol{\beta}$. For arbitrarily oriented planes of constant refractive index

$$k_{sz} = \frac{1}{m} k_{iz},$$

while

$$0 < \frac{1}{m} \leqslant 1. \quad (26)$$

The coefficient of proportionality has been written in the form $1/m$, which will be convenient in further considerations. This leads to the following values of a_s, a, and γ (eq. (26)):

$$a_s = ma_i,$$

$$a = \frac{1}{2} a_i(1+m), \quad (27)$$

$$\gamma = \frac{a_i}{2} \sqrt{(1-m)^2 + 4\frac{|\eta|}{n''} m}.$$

Since $n'' \leqslant \eta$ then $n'' = |\eta|$ will be the best assumption. For this value of the imaginary part of the refractive index we obtain

$$\gamma = \frac{a_i}{2}(1+m). \quad (27a)$$

From (24) we may be evaluated $\alpha_\pm$

$$\alpha_+ = 0,$$

$$\alpha_- = a_i(i+m).$$

The amplitudes calculated from (25) — after small rearrangements — may be written in the form

$$A(z) = \exp\left(-\frac{1}{2} a_i(1+m)(z+s)\right)\Big[ch\frac{1}{2} a_i(1+$$

$$+m)(z+s) - \frac{2(1-m)}{1+m} sh\frac{1}{2} a_i(1+m)(z+s)\Big] A_0, \quad (28)$$

$$B(z) = i\frac{2mA_0}{1+m} \exp(-\frac{1}{2} a_i(1+m)(z+s) \times$$

$$\times sh\left[\frac{1}{2} a_i(1+m)(z+s)\right].$$

Let us calculate for which s the amplitude $B(s)$ is maximal

$$B(s) = \frac{2imA_0}{1+m} \exp(-a_i(1+$$

$$+m)s) sh[a_i(1+m)s]. \quad (28a)$$

For maximal scattering the following condition must be fulfilled

$$\frac{dB(s)}{ds} = \frac{2imA_0}{1+m}\Big\{-a_i(1+m)\exp(-a_i(1+$$

$$+m)s) sh[a_i(1+m)s] + a_i(1+m)\exp(-a_i(1+$$

$$+m)s) ch[a_i(1+m)s]\Big\} = 0.$$

Hence we obtain

$$sh[a_i(1+m)s] = ch[a_i(1+m)s]. \quad (29)$$

This condition is satisfied for infinitely great thickness. Then

$$B_m(s) = \frac{im}{1+m} A_0. \quad (30)$$

From (29) and (30) it may be seen that the maximal value of amplitude of scattering may be reached not only for very great thickness, but also that it depends on the orientation of periodical changes of refractive index. Thus for instance, if the planes of constant refractive index are identical with $z =$ constant planes, $m = 1$, and the diffusion efficiency is equal to about 25%. The amplitude grating does not give the complete effectivity, and the advantageous conditions (great thickness and suitable orientation) are not always possible to satisfy. Generally speaking, for the amplitude gratings the scattered wave is difficult distinguish from that incident.

Now, consider the reflection scattering ($k_{iz} \cdot k_{sz} < < 0$). The equations (11) become

$$\alpha_+ = -a+\delta,$$

$$\alpha_- = -a-\delta, \tag{31}$$

$$\delta = \frac{1}{2}\sqrt{(a_i+a_s)^2-4\left|\frac{k_0^2 b^2}{k_{iz}k_{sz}}\right|}.$$

The amplitudes may be presented as follows

$$A(z) = A_0 \exp(-a(z+3s)) \times$$

$$\times \frac{\delta ch\delta(z-s)-\frac{1}{2}(a_i-a_s)sh\delta(z-s)}{\delta ch 2\delta s+\frac{1}{2}(a_i-a_s)sh 2\delta s}, \tag{32}$$

$$B(z) = -\left|\frac{k_0 b}{k_{sz}}\right| A_0 \exp(-a(z+3s)) \times$$

$$\times \frac{sh\delta(z-s)}{\delta ch 2\delta s+\frac{1}{2}(a_i-a_s)sh 2\delta s}.$$

The scattered wave appears for $z = -s$. Let us calculate the extreme amplitude of scattering for normal incidence. Let

$$k_{sz} = -\frac{1}{m}k_{iz},$$

$$0 < \frac{1}{m} < 1. \tag{33}$$

From (31) we get

$$a_s = ma_i,$$

$$a = 1/2(1-m)a_i,$$

$$\delta = 1/2(1-m)a_i, \tag{34}$$

$$\alpha_+ = 0,$$

$$\alpha_- = -(1-m)a_i.$$

With the help of these relations the amplitude of the scattered wave for $z = -s$ will be

$$B(-s) = 2mA_0 \times$$

$$\times \frac{sh[(1-m)a_i s]}{(1-m)ch[(1-m)a_i s]+(1+m)sh[(1-m)a_i s]}. \tag{35}$$

If there exists the maximum scattering with respect to the thickness the following condition must be satisfied

$$\frac{dB(-s)}{ds} = 0.$$

This leads to the condition

$$\frac{(1-m)^2 a_i}{(1-m)ch[(1-m)a_i s]+(1+m)sh[(1-m)a_i s]} = 0, \tag{36}$$

which is fulfilled only for infinitely great thickness of the layer. Thus, the critical thickness does not exist.

For $m = 1$ the solution of equation (16) for c and d cannot be applied, because there appears dividing by zero. Hence, the equation (12) should be a starting point. The value of amplitude of scattered wave is, however, so small that the application of the perturbation calculus may generate great errors. Moreover, the smallness of scattering amplitude leads to the case of no practical meaning.

Влияние толщины слоя на эффективность связи плоских волноводов с периодически переменным коэффициентом преломления

Выведены уравнения связи для плоского волновода с периодически переменным коэффициентом преломления, причем была предположена слабая модуляция коэффициента преломления; принято также, что коэффициент преломления у окружающей среды равен среднему коэффициенту преломления волновода. Приведены решения уравнения связи для собственно фазовой и амплитудной модуляции. Проанализировано влияние толщины на значение связи.

References

[1] Kogelnik H., *Coupled Wave Theory for Thick Hologram Gratings*, BSTJ **48**, 2909 2947 (1969).

[2] Marcuse D., *Light Transmission Optics*, New York 1972.

[3] Marcuse D., *Theory of Dielectric Optical Waveguides*, Academic Press, New York 1974.

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