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Hologram aberrations outside the binomial expansion**

By applying an asymptotic expansion we have considered the point-object aberrations in an axial hologram of Gabor-type outside the range of binomial expansion in the case of small and great magnifications. Two cases have been calculated and compared with the spherical aberration. A dependence of aberrations upon the hologram sizes has briefly given.

In the process of holographic imaging, even at the absence of lenses, there appear aberrations of the same kind as in the case of lens system [1].

The papers analysing the third order aberrations [2, 3] appeared almost at the same time (1965). They include the deviation of third order aberrations and a broad analysis of possible reduction of aberrations by an appropriate recording and reconstruction of the hologram. The later papers deal with non-paraxial imaging [4–7] and a possibility of reducing the imaging errors. Ofner [4] noticed a possibility of aberration compensation by introducing a prism to the system, while Lin and Collins, Jr., [8] discuss extensively the aberrations of holographic lens system indicating a possibility of applying the lenses of appropriately chosen aberrations to the compensation of unwanted aberrations. Their method of analysis is very general and comprises almost all the situations introducing imaging errors except for deformations and noise of the film. The influence of deformation of the photographic materials and their tolerances were considered by Matsamura [9].

It seems that the case of imaging in holographic microscope [3] is both interesting and not fully analysed so far. The aberrations cause a restricted applicability of Gabor microscope [10]. Although, it is possible to choose the conditions of hologram reconstruction in such a way that the third order aberrations disappear [1–3], the practical significance of these conditions is limited. The analyses given in [1–3] lead to the conclusion that this is possible for magnifications $M = 1$ and $M = 2$. The case $M = 1$ requires, that the wavelengths used in the reconstruction and recording, respectively, were equal each other, i.e. $\mu = \lambda_c/\lambda_0 = 1$, while $M = 2$ demands that $\mu = 2$. It may be seen that no aberration free magnification is possible except for $M = 2$. The case $M = 2$ is realizable, for two sources of frequencies differing by an octave but it does not seem to be of practical importance in the optical holography.

The imposing of an additional condition, i.e. the demand of achieving a resolving power of order of λ, requires a hologram of sizes comparable with the object-to-hologram distance [6]; it creates but then, however, such imaging conditions for which the estimation of aberrations based on binomial development are no longer sufficient. An analysis of imaging errors and their classification given by Meier [3] is based on expanding into power series of the phase term calculated with respect to the hologram centre

$$\varphi_q = \frac{2\pi}{\lambda_q} z_q \left(\sqrt{1+\frac{(\rho-\rho_q)^2}{z_q^2}} - \sqrt{1+\frac{(\rho_q)^2}{z_q^2}}\right), \quad (1)$$

where q ... o ... if it is attributed to object wave,
r ... if it is attributed to reference wave,
c ... if it is attributed to reconstructing wave
$\varrho^2 = x^2+y^2$
z ... coordinate of a Cartesian system associated with the hologram.

The series

$$\sqrt{1+\xi} \approx 1+\frac{1}{2}\xi-\frac{1}{8}\xi^2+ \ldots\ldots \quad (2)$$

is absolutely convergent for $|\xi| \leqslant 1$. In the general case the analysis based on such a development of

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wave phase o, r, c and the Gaussian spheres may be applied only to definite limiting apertures.

Under conditions, when $\xi = (\rho-\rho_q)/z_q^2$ is greater than 1, the series (2) becomes divergent. Different approaches aiming at estimation of imaging errors may be found in the literature. CHAMPAGNE [5] applies a binomial development with respect to ϱ/R_q, where $R_q^2 = \varrho^2+z_q^2$. The aberrational expressions obtained in the reversed reference system are rather troublesome while analysing. Other authors consider the problem from diffraction viewpoint [7] or use numerical ray-tracing [4, 6].

The proposed approach is based on the so-called asymptotic development [12, 13]. For $|\xi| \geqslant 1$

$$\sqrt{1+\xi} \approx \sqrt{\xi}\left(1+\frac{1}{2\xi}-\frac{1}{8\xi^2}+\frac{1}{16\xi^3}-\ldots\ldots\right). \quad (3)$$

Depending on the situation defined by the hologram and the object sizes, and the position of o, r, c sources the two square roots appearing in (1) may be developed according to (2) or (3). It is also possible that one of them is developed according to asymptotic expansion while the other according to traditional binomial one. Consider the case of a Gabor axial hologram (fig. 1) also discussed in [6].

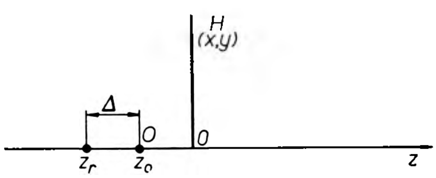

Fig. 1

The wavefront sources o, r, c lie on one straight line and therefore

$$\varrho_o = \varrho_r = \varrho_c = 0.$$

The source of the reference wave is close to the object and thus

$$\frac{\Delta}{z_o} = \frac{z_r-z_o}{z_o} \ll 1.$$

Hence the magnification in the recording stage

$$M_{\mathrm{rec}} = \frac{z_r}{\Delta} = 1+\frac{z_o}{\Delta}$$

is great and the spatial frequency range is highly reduced [2]; z_o and Δ should be matched to the resolving power of the photographic material.

The lateral magnification is given by the relation [2, 3]

$$M_{\mathrm{lat}\,R,V} = m\left(1\mp\frac{m^2}{\mu}\frac{z_o}{z_c}-\frac{z_o}{z_r}\right)^{-1}, \quad (4)$$

where $\mu = \lambda_c/\lambda_o$,

m — photographic magnification of hologram.

The upper sign is attributed to the wavefront

$$\Phi_R = \varphi_c-\varphi_o+\varphi_r,$$

while the lower one to the wavefront

$$\Phi_V = \varphi_c+\varphi_o-\varphi_r. \quad (5)$$

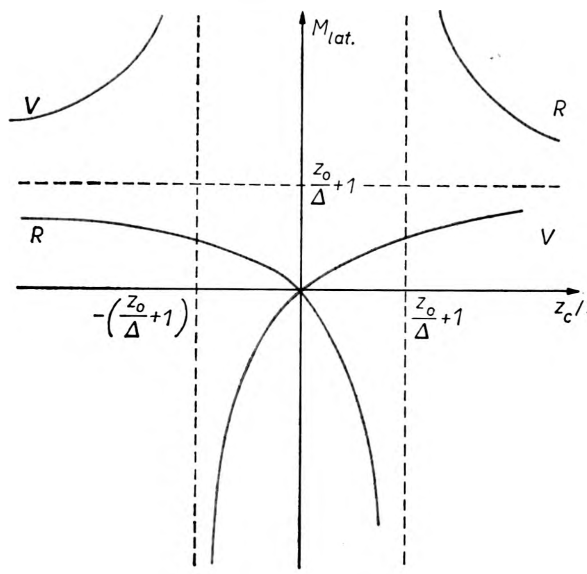

Fig. 2

Fig. 2 presents the dependence of the magnification upon the recording and reconstruction conditions determined by (4) for a Gabor axial hologram

$$M_{\mathrm{lat}\,R,V} = \left(\frac{1}{M_{\mathrm{rec}}}\mp\frac{z_o}{z_c}\frac{m^2}{\mu}\right)^{-1}. \quad (6)$$

The position of a point Gaussian-conjugate with the point O satisfies the relations

$$\frac{1}{z_{G_{R,V}}} = \frac{1}{z_c}\mp\frac{\mu}{m^2 z_o}\pm\frac{\mu}{m^2 z_r}, \quad (7)$$

where the upper signs concern the R wavefront while the lower ones — the V wavefront.

For the Gabor axial hologram they take the form

$$\frac{z_{GR}}{z_o} \approx \frac{z_c/z_o}{1-\frac{z_c}{z_o}\frac{\Delta}{z_o}},$$
$$\frac{z_{GV}}{z_o} = \frac{\frac{z_c}{z_o}\left(1+\frac{\Delta}{z_o}\right)}{\left(1+\frac{\Delta}{z_o}\right)+\frac{z_c}{z_o}\frac{\Delta}{z_o}}, \tag{8}$$
$$\mu = m = 1.$$

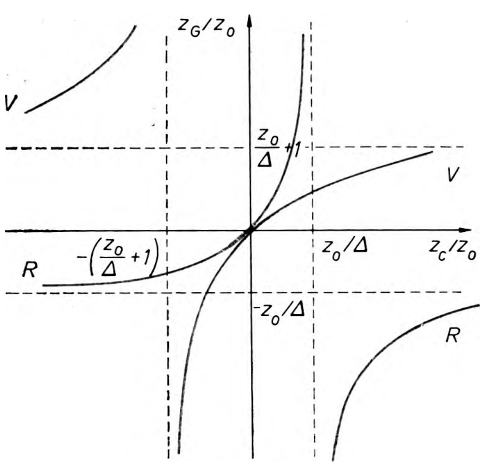

Fig. 3

Fig. 3 being a graphical illustration of these relations.

Let us assume that z_0, z_r and the dimensions of hologram are chosen so that the phases φ_0 and φ_r will be subject to an asymptotic development. The phase of the Gaussian references sphere Φ_G will be subject to this development if

$$\frac{z_{G_{R,V}}}{z_r} \leqslant 1.$$

These conditions reduce the values of z_c to

$$z_c \leqslant \frac{z_r}{1-\frac{z_r\Delta}{z_0^2}} \text{ for } R \text{ wave},$$

$$z_c \leqslant \frac{z_r}{1-\frac{\Delta}{z_0}} \text{ for } V \text{ wave}.$$

If we require additionally that also φ_c be subject to this development (small magnification) then the first of square roots in the expression (4) for the phase φ_q will be expanded asymptotically while the other will be equal to unity

$$\varphi_q = \frac{2\pi}{\lambda_q} z_q \left[\sqrt{\frac{(\rho-\rho_q)^2}{z_q^2}}\left(1+\frac{z_q^2}{2(\rho-\rho_q)^2} - \frac{z_q^4}{8(\rho-\rho_q)^4}+\dots\right)-1\right] =$$
$$= \frac{2\pi}{\lambda_q}\left[|\rho-\rho_q|\left(1+\frac{z_q^2}{2(\rho-\rho_q)^2} - \frac{z_q^4}{8(\rho-\rho_q)^4}+\dots\right)-z_q.\right]$$

In the face of (9) the phases of the R and V wavefront will have the form

$$\Phi_{R,V} = \varphi_c \mp \varphi_0 \pm \varphi_r = \frac{2\pi}{\lambda_c}\left\{\sqrt{x^2+y^2} + \frac{1}{2\sqrt{x^2+y^2}}\left(z_c^2 \mp \frac{\mu}{m} z_0^2 \pm \frac{\mu}{m} z_r^2\right) - \frac{1}{8(x^2+y^2)^{3/2}}\left(z_c^4 \mp \frac{\mu}{m^3} z_0^4 \pm \frac{\mu}{m^3} z_r^4\right) + \dots\dots - (z_c \mp \mu z_0 \pm \mu z_r)\right\}. \tag{10}$$

And similarly

$$\Phi_G = \frac{2\pi}{\lambda_c}\left(\sqrt{x^2+y^2} + \frac{z_G^2}{2\sqrt{x^2+y^2}} - \frac{z_G^4}{8(x^2+y^2)^{3/2}} + \dots\dots - z_G\right). \tag{11}$$

The sum of aberrations introduced during imaging by a hologram zone, for which the relations (10) and (11) are simultaneously valid, will be

$$\sum_i A_i = \Phi_{R,V} - \Phi_{G_{R,V}} =$$
$$= \frac{2\pi}{\lambda_c}\left\{\frac{1}{2\sqrt{x^2+y^2}}\left(z_c^2 \mp \frac{\mu}{m} z_0^2 \pm \frac{\mu}{m} z_r^2 - z_G^2\right) - \frac{1}{8(x^2+y^2)^{3/2}}\left(z_c^4 \mp \frac{\mu}{m^3} z_0^4 \pm \frac{\mu}{m^3} z_r^4 - z_G^4\right) + \dots - (z_c \mp \mu z_0 \pm \mu z_r - z_G)\right\}$$
$$= \frac{2\pi}{\lambda_c}\left\{P + \frac{1}{2\varrho} Q - \frac{1}{8\varrho^3} R + \dots\right\}. \tag{12}$$

The linear term P and coefficients Q and R at ϱ^n dependent on z_0, z_r and z_c may be treated as new aberration coefficients. The commonly accepted dependence of aberrations upon the hologram sizes [11] concerns only the aberrations introduced by the paraxial part of the holograms (spherical aberration $\sim\varrho^4$, coma $\sim\varrho^3$, astigmatism $\sim\varrho^2$, field curvature $\sim\varrho^2$, distortion $\sim\varrho$). Within the range of the asymptotic development we observe the proportionality of aberrations to ϱ^{-1}, $\varrho^{-3}, \ldots$, then, however, aberration coefficients take a new form.

The coefficients of "classical" aberrations are the functions of reciprocals of z_0, z_c, z_r and z_G, while the coefficients (12) creating somehow their inversions (they are proportional to the positive powers of z_0, z_c, z_r, z_G) will compensate the influence of ϱ^{-n}. When talking about the dependence of the hologram aberrations upon its sizes should be kept in mind that beside the explicite proportionality to ϱ^n, it may also be comprised implicity in the aberrational coefficients.

The examples in table 1 illustrate the behaviour of aberration for two cases described by relation (12). For the point O lying on the axis, of all the classical aberrations only spherical aberration remains. For the comparative reasons the coefficient S and values of spherical aberration have been placed at the initial part of the table 1. The aberrations are expressed in $z_o\lambda_c/2\pi$ units. Empty positions for the values $\varrho/z_o = 1$ and $\varrho/z_o = 1.1$ in the first example are due to the fact that in this range the phase φ_c cannot be subject to asymptotic development. The magnification corresponding to the conditions required by (12) is low being close to unity.

As may it be seen from table 1 the sums of "asymptotic" aberrations have a minimum and the aberrations of both wavefronts R and V differ few times from each other. At slightly differing magnifications, all close to unity, it is possible to choose a variant of lower aberrations.

The high magnifications at fixed Δ and $z_c/z_o > 0$ (comp. fig. 2) require that z_c/z_o be close to $z_c/\Delta+1$ for the R branch or $z_c \to \infty$ for the V branch. Under these circumstances the phase φ_c will be subject to

Table 1

$\Delta = 0.1z_o$, $z_r = 1.1z_o$, $z_c = 1.2z_c$						
Aberration	Coefficient	ϱ/z_o	Aberration R	$\sum_i A_{iR}$	Aberration V	$\sum_i A_{iV}$
$-\frac{1}{8}\varrho^4 S$	$S_R = -0.0792$ $S_V = 0.0390$	0.1	0.0000009		−0.0000004	
		0.2	0.0000158		−0.0000078	
		0.3	0.0000799		−0.0000393	
		0.5	0.000619		−0.000305	
		0.7	0.00238		−0.00117	
		0.9	0.00649		−0.00320	
		1.0	0.0099		−0.00487	
$-1\cdot P$	$O_R = -0.0470$ $O_V = 0.0180$	1				
		1.1	0.0470		−0.0180	
		1.2				
		1.3				
		1.5				
		1.7				
		2.0				
		2.3	0.0470		−0.0180	
$\frac{1}{2\varrho}Q$	$P_R = -0.1644$ $P_V = 0.0593$	1				
		1.1				
		1.2	−0.0685	0.0330	0.0247	−0.0106
		1.3	−0.0632	0.0267	0.0228	−0.0088
		1.5	−0.0548	0.0201	0.0198	−0.0070
		1.7	−0.0484	0.0178	0.0174	−0.0067
		2.0	−0.0411	0.0177	0.0148	−0.0069
		2.3	−0.0357	0.0190	0.0129	−0.0076
$-\frac{1}{8\varrho^3}R$	$Q_R = -0.7543$ $Q_V = +0.2390$	1				
		1.1				
		1.2	0.0545		−0.0173	
		1.3	0.0429		−0.0136	
		1.5	0.0279		−0.0088	
		1.7	0.0192		−0.0061	
		2.0	0.0118		−0.0037	
		2.3	0.0077		−0.0025	
			$M_{\text{lat }R} = -1.3470$		$M_{\text{lat }V} = 1.0820$	

Table 1 (cont.)

$\Delta = 0.05z_c$, $z_r = 1.05z_c$, $z_c = 1.1z_o$						
Aberration	Coefficient	ϱ/z_o	Aberration R	$\sum_i A_{iR}$	Aberration V	$\sum_i A_{iV}$
$-\frac{1}{8}\varrho^4 S$	$S_R = -0.0243$ $S_V = 0.0119$	0.1	0.0000003		−0.0000001	
		0.2	0.0000048		−0.0000023	
		0.3	0.0000245		−0.0000120	
		0.5	0.000190		−0.0000920	
		0.7	0.000729		−0.000357	
		0.9	0.00199		−0.000976	
		1.0	0.00303		−0.00149	
$-1\cdot P$	$Q_R = 0.0108$ $Q_V = 0.0921$	1.0	−0.0108		0.0921	
		1.2				
		1.3				
		1.5				
		1.7				
		2.0				
		2.3	−0.0108		0.0921	
$\frac{1}{2\varrho}Q$	$P_R = -0.0350$ $P_V = -0.1969$	1.1	−0.0159	−0.0139	−0.0895	0.0451
		1.2	−0.0146	−0.0156	−0.0820	0.0428
		1.3	−0.0135	−0.0166	−0.0757	0.0422
		1.5	−0.0117	−0.0175	−0.0656	0.0433
		1.7	−0.0103	−0.0176	−0.0579	0.0457
		2.0	−0.0088	−0.0175	−0.0492	0.0500
		2.3	−0.0076	−0.0170	−0.0428	0.0540
$-\frac{1}{8\varrho^3}R$	$Q_R = -0.1360$ $Q_V = -0.4528$	1.0	0.0128		0.0425	
		1.2	0.0098		0.0327	
		1.3	0.0077		0.0258	
		1.5	0.0050		0.0168	
		1.7	0.0035		0.0115	
		2.0	0.0021		0.0071	
		2.3	0.0014		0.0047	
			$M_{\text{lat }R} = -1.1608$		$M_{\text{lat }V} = 1.0453$	

the binomial development. Within the range where also Φ_G will be subject of such development the following relation is valid

$$\sum A_i = \Phi_{R,V} - \Phi_{G_{R,V}} =$$

$$= \frac{2\pi}{\lambda_c}\left\{\frac{1}{2}\frac{1}{\sqrt{x^2+y^2}}\frac{\mu}{m}(\mp z_0^2 \pm z_r^2) - \right.$$

$$-\frac{1}{8}\frac{1}{(x^2+y^2)^{3/2}}\frac{\mu}{m^3}(\mp z_0^4 \pm z_r^4) - \mu(\mp z_0 \pm z_r) +$$

$$+\frac{1}{2}(x^2+y^2)\left(\frac{1}{z_c}-\frac{1}{z_G}\right)+$$

$$\left. +\frac{1}{8}(x^2+y^2)^2\left(\frac{1}{z_c^3}-\frac{1}{z_G^3}\right)\ldots\right\} \tag{13}$$

insted of (12).

The examples illustrating the relation (13) are given in table 2. The form of this table does not differ essentially from that of table 1. Here, no aberration components are given but only their sums for both the kinds of expansion. Each pair of column for Φ_R and Φ_V corresponds to other recording and reconstruction conditions. As in table 1 there exists a possibility of choosing a variant of smaller aberrations.

The sum of aberrations should be a continuous function of ϱ/z_o. Thus in both asymptotic and binomial kinds of expansion, respectively, great number of terms should be taken into account. For ϱ/z_o tending to unity from the left hand side, it is no sufficient to restrict the attention to the third order spherical aberration only, while for ϱ/z_o tending to 1 from the right hand side it is necessary to establish the optimal number of terms for asymptotic expansion which would assure the same accuracy of approximation as in the region $\varrho/z_o < 1$. This work will be continued.

Table 2

Aberrations	ϱ/z_0	$\Delta = 0.1z_0,\ z_c = 12z_0$ for Φ_R	for Φ_V	$\Delta = 0.05z_0,\ z_c = 22z_0$ for Φ_R	for Φ_V
$\frac{1}{8}\varrho^4 S$	0.1	0.000003	−0.000003	0.000002	−0.0000016
	0.2	0.000050	−0.000049	0.000027	−0.0000271
	0.3	0.000251	−0.000246	0.000137	−0.000137
	0.5	0.00194	−0.00191	0.00106	−0.001058
	0.7	0.00744	−0.00732	0.00408	−0.00407
	0.9	0.0203	−0.0200	0.0112	−0.01111
	1.0	0.0300	−0.0305	0.0170	−0.01693
$\sum_i A_i$ asympt	1.0	−0.0481	0.0481	−0.0236	0.0236
	1.2	−0.0461	0.0461	−0.0222	0.0222
	1.3	−0.0456	0.0456	−0.0229	0.0229
	1.5	−0.0472	0.0472	−0.0238	0.0238
	1.7	−0.0500	0.0500	−0.0254	0.0254
	2.0	−0.0548	0.0548	−0.0281	0.0281
$\sum_i A_i$ binomial	1.1	0.0549	−0.0559	0.0287	−0.0289
	1.2	0.0653	−0.0666	0.0341	−0.0341
	1.3	0.0766	−0.0785	0.0399	−0.0405
	1.5	0.1019	−0.1053	0.0531	−0.0541
	1.7	0.1308	−0.1363	0.0680	−0.0696
	2.0	0.1806	−0.1912	0.0937	−0.1966
$\sum_i A_i$	1.1	0.0068	−0.0078	0.0051	−0.0053
	1.2	0.0192	−0.0205	0.0119	−0.0119
	1.3	0.0310	−0.0329	0.0170	−0.0176
	1.5	0.0547	−0.0581	0.0293	−0.0303
	1.7	0.0808	−0.0863	0.0426	−0.0442
	2.0	0.1258	−0.1364	0.0656	−0.0685
M_{lat}		131.6	5.74	476	10.75

Аберрации голограммы вне двоичного разложения в ряд

С применением асимптотического разложения в ряд рассмотрены аберрации точечного объекта в осевой голограмме Габора вне пределов двоичного разложения в ряд в случае малых и больших увеличений. Вычислены по два примера и, для примера, сопоставлены со сферической аберрацией. Кратко представлена зависимость аберрации от размеров голограмм.

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