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Impulse response of coherent optical system of image multiplication by spatial sampling filtration

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Impulse response and transfer function of coherent optical system for image multiplication by spatial sampling filtration are presented. A means for matching lens aperture with filter dimensions is found. In the experimental part a multiplication of 3-D objects is shown, and the influence of spatial sampling filter dimensions on images fidelity discussed.

Introduction

Image multiplication by sampling the spatial frequencies consists in placing a suitable mask at the frequency plane of the optical system [1–5]. The masks are commonly black screens with holes spaced periodically. Holes transmittance $t_c(x_f, y_f)$ is identical across the whole mask. Light field, which realises here the Fourier image of the object is diffracted on the mask and becomes periodic. The filtration is manifested distinctly in the irradiance distribution of the multiple images. The samplying of spatial frequency space must preserve the information content in the Fourier image and prevent the overlapping of the multiple images in the exit plane of the optical system. For the right choise of proper sampling distances in the filtering mask we can employ the WITTAKER–SHANNON sampling theorem [6, 7] in the frequency domain.

Image multiplication by sampling spatial frequencies can be obtained in an optical system which consists of two lenses L_1 and L_2. The object is placed at the focal distance f_1 before the lens L_1. The mask is placed in focal plane of the lens L_1, while the second lens L_2 behind the mask, is distanced by f_2 from the latter. Multiplication can be also performed in single lens optical system. The object is then placed at the distance $d_0 > f_1$ from the lens [3]. The last method is useful for multiplication of 3-D objects with smaller depth (see fig. 1). In the remainder we shall deal with a multiplying optical system shown in fig. 1.

Theoretical results have been confirmed by image multiplication of 3-D objects obtained experimentally.

Light field in the image plane x_i, y_i, obtained by spatial filtration using the spatial mask with sampling holes distributed periodically at distances $\vec{d} = (d_x, d_y)$ within a rectangular lattice is described as follow:

$$U_{im}(x_i, y_i) \propto \sum_m \sum_n \hat{S}_i(x_i \pm m\Delta_{x_i}, y_i \pm n\Delta_{y_i}) \hat{T}(m\Delta_{x_i}, n\Delta_{y_i}) \qquad (1)$$

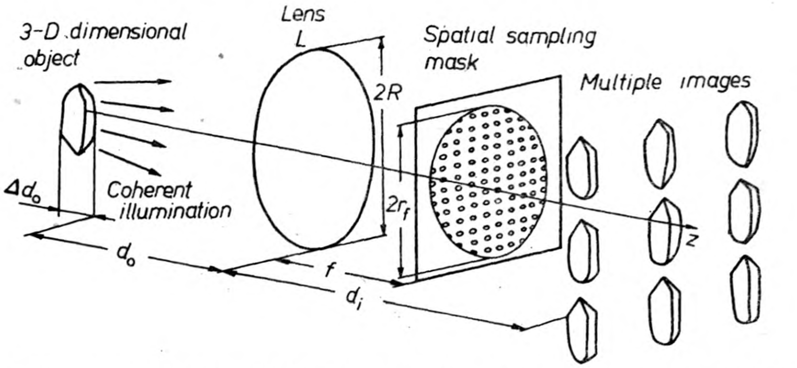


Fig. 1. Arrangement of an optical coherent system for image multiplications of 3-D object

where:

$\hat{S}_i(x_i \pm m\Delta_{x_i}, y_i \pm n\Delta_{y_i})$ — singular image of multiplied optical signal at every node of the rectangular lattice,

$\hat{T}(m\Delta_{x_i}, n\Delta_{y_i})$ — the Fourier transform of the sampling element transmittance in the sampling spatial filter,

$t_c(x_f, y_f) - x_f, y_f$ — coordinates in the Fourier plane of the optical system.

$|\hat{T}(m\Delta_{x_i}, n\Delta_{y_i})|^2$ is here the weighting factor of irradiance of the multiple images. The images lie at the nodes the Δ_{x_i}, Δ_{y_i} of the lattice, which is an inverse one to the lattice of samples in the spatial filter. Filter transmittance is here

$$t_f(x_f, y_f) = t_c(x_f, y_f \otimes \sum_m \sum_n \delta(x_f \pm m\Delta_{x_f}, y_f \pm n\Delta_{y_f}), \qquad (2)$$

where $\otimes$ — convolution symbol.

Impulse response

In linear approximation impulse response of multiplying optical system can be given by

$$\hat{h}(x_i, y_i, x_0, y_0) = \sum_k \hat{T}_k(m_k \Delta_{x_i}, n_k \Delta_{y_i}) \hat{h}_k(x_{ik}, y'_{ik}, x_0, y_0) \qquad (3)$$

where k denotes summation of all the multiple images, $\hat{h}_k$ is impulse response of the system for k-th image. When the samples in samplying spatial filter are approximated by δ, then the weighting factor $\hat{T}(m\Delta_{x_i}, n\Delta_{y_i}) \equiv 1$, ($x_0, y_0$ — input plane, x_i, y_i — output plane).

In this case, for a diffraction limited system including a lens and a spatial filter eq. (3) in geometrical optics approximation takes the form [9]:

$$\begin{aligned}\hat{h}(x_i, y_i, x_0, y_0) &= \sum_k \hat{h}_k(x_{ik}, y_{ik}, x_0, y_0) \\ &= M\delta(x_i+Mx_0, y_i+My_0)\otimes \\ &\otimes \sum_k \delta(x_i-m\Delta_{x_i}, y_i-n\Delta_{y_i}) \\ &= M\sum_m\sum_n \delta(x_i\pm m\Delta_{x_i}+Mx_0, y_i\pm n\Delta_{y_i}+My_0) \qquad (4)\end{aligned}$$

where:

$x_{ik} = (x_i \pm m_k \Delta_{x_i}), y_{ik} = (y_i\pm n_k\Delta_{y_i})$,

$M = d_i/d_0$ is images magnification,

$(\Delta_{x_i}, \Delta_{y_i}) = \bar{\Delta}_i$ is translation vector of the lattice of multiple images.

For physically realised optical system we must include the finite extent of the lens and the filter. $P_L(x, y)$ and $P_F(x_f, y_f)$ being the respective pupil functions of the lens and filter, where x, y denote the coordinates at the thin lens plane. The transmittance of the spatial filter may now be written as:

$$t_f(x_f, y_f) \propto \sum_m\sum_n \delta(x_f\pm m\Lambda_{x_f}, y_f\pm n\Lambda y_x). \qquad (5)$$

Diffraction effects caused by finite dimensions of used lens and spatial filter will disclose in every k-th image, they affect the partial impulse response $\hat{h}_k$. Our purpose is to find $\hat{h}_k$ for optical system with limited bounds.

The image is assumed to be at the distance d_i behind the lens in the Fresnel diffraction zone. In paraxial approximation, when assuming that the lens equation $1/d_0+1/d_i-1/f = 0$ is valid we can [9] represent the impulse response $\hat{h}_k$

$$\hat{h}_k(x_{ik}, y_{ik}, x_0, y_0) = \frac{1}{\lambda^2 d_i d_0}\iint\limits_{-\infty}^{+\infty} P_L(x, y)\tilde{P}_F\left(x\frac{d_i-f}{d_i}, y\frac{d_i-f}{d_i}\right)\times$$

$$\times\exp\left\{-i\frac{2\pi}{\lambda d_i}\cdot[(x_{ik}+Mx_0)x+(y_{ik}+My_0)y]\right\}dxdy. \qquad (6)$$

The finite extent of the filter can be represented mathematically by projecting the pupil function of the filter onto the lens, which yields an effective pupil function, denotes by

$$\tilde{P}_F\left(x\frac{d_i-f}{d_i}, y\frac{d_i-f}{d_i}\right),$$

where:

$$x_f = x\frac{d_i-f}{d_i},$$

$$y_f = y\frac{d_i-f}{d_i}.$$

Denoting $\tilde{x}$ by $\frac{x}{\lambda d_i}$ and $\tilde{y}$ by $\frac{y}{\lambda d_i}$ the equation (6) takes the form

$$h_k(x_{ik}, y_{ik}, x_0, y_0) \propto M \iint\limits_{-\infty}^{+\infty} P_L(\lambda d_i\tilde{x}, \lambda d_i\tilde{y})\tilde{P}_F\big((\lambda d_i-f)\tilde{x}, \lambda(d_i-f)\tilde{y}\big)\times$$
$$\times\exp\{-i2\pi[(x_{ik}+Mx_0)\tilde{x}+(y_{ik}+My_0)\tilde{y}]\}\,d\tilde{x}\,d\tilde{y}. \quad (7)$$

In the geometrical optics approximation [9], for $\lambda\to 0$, either $P_L(x, y)$ or

$$\tilde{P}_f\left(x\frac{d_i-f}{\lambda d_i}, y\frac{d_i-f}{d_i}\right)$$

is equal to unity for all the values of x, y. Then the partial impulse response becomes

$$\hat{h}_k(x_{ik}, y_{ik}, x_0, y_0) = \delta(x_{ik}+Mx_0, y_{ik}+My_0),$$

according to eq. (4). We have assumed that the coherent multiplying system is isoplanar:

$$\hat{h}_k(x_{ik}, y_{ik}, x_0, y_0) = \hat{h}_k(x_{ik}-x_0, y_{ik}-y_0)$$

for every $k = 1, 2, \ldots$.

By inserting $\tilde{x}_0 = -Mx_0$, $\tilde{y}_0 = -My_0$ into eq. (7) and denoting $\hat{\hat{h}}_k = M^{-1}\cdot\hat{h}_k$ we may write:

$$\hat{\hat{h}}_k(x_{ik}, y_{ik}) \propto \iint\limits_{-\infty}^{+\infty} P_L(\lambda d_i\tilde{x}, \lambda d_i\tilde{y})\tilde{P}_F\big(\lambda(d_i-f)\tilde{x}, \lambda(d_i-f)\tilde{y}\big)\times$$
$$\times\exp\{-2\pi i[(x_{ik}-\tilde{x}_0)\tilde{x}+(y_{ik}-\tilde{y}_0)\tilde{y}]\}\,d\tilde{x}\,d\tilde{y}. \quad (8)$$

Thus, impulse response $\hat{\hat{h}}_k(x_{ik}, y_{ik})$ is a Fourier transform of the effective aperture $P_{\text{eff}}(\tilde{x}, \tilde{y})$ of the multiplying optical system:

$$\hat{\hat{h}}_k(x_{ik}, y_{ik}) = \mathscr{F}\{P_{\text{eff}}(x, y)\}$$
$$= \mathscr{F}\{P_L(\lambda d_i\tilde{x}, \lambda d_i y)\tilde{P}_F\big(\lambda(d_i-f)\tilde{x}, \lambda(d_i-f)\tilde{y}\big)\}, \quad (9)$$

where $\mathscr{F}\{\ \}$ — denotes Fourier transformation operator.

From eq. (9) we have also

$$\hat{\hat{h}}_k(x_{ik}, y_{ik}) = \mathscr{F}\{P_L\}\otimes\mathscr{F}\{P_F\}. \quad (10)$$

Using the eqs. (9) and (10) we can define the coherent transfer function as

$$\hat{H}_k(f_x, f_y) = \mathscr{F}\{\hat{h}_k\} = \mathscr{F}^{-1}\{\mathscr{F}\{P_L\}\otimes\mathscr{F}\{\tilde{P}_F\}\} = P_L\cdot\tilde{P}_F, \quad (11)$$

i.e. as a product of a lens pupil function P_L and filter pupil functions $\tilde{P}_F$ projected onto the lens plane x, y; f_x, f_y being spatial frequencies. Assuming circular lens apertures of the both lens and filter with the respective radii R and $\tilde{r}_F$ chosen so that the projection of $\tilde{r}_F$ onto the lens plane x, y be equal R we have:

$$P_L(x, y) = \tilde{P}_F(x, y) = \operatorname{circ}\frac{r}{R}, \tag{12}$$

where

$$\operatorname{circ}\frac{x^2+y^2}{R}\begin{cases}1 \text{ for } r^2 = x^2+y^2 \leqslant R^2,\\ 0 \text{ for } r^2 = x^2+y^2 > R.\end{cases}$$

Under this assumptions impulse response $\hat{\tilde{h}}_k(x_{ik}, y_{ik})$ $k = 1, 2, 3, \ldots,$ takes the form

$$\tilde{h}_k(x_{ik}, y_{ik}) \propto \frac{R^2 J_1(2\pi\varrho_{ik}R)}{\varrho_{ik}^2}, \tag{13}$$

where:
$\varrho_{ik}^2 = x_{ik}^2+y_{ik}^2$,
$k = 1, 2, \ldots$.

Further we want look for such dimensions of spatial filter for which the spatial frequencies cutoff does not restrict the spectrum transmitted through the lens. It means that frequency cutoff $f_{0F} = \tilde{r}_F/2\lambda d_i$ of the filter must be equal to that of the lens $f_{0L} = R/2\lambda d_i$ [9], where

$$\tilde{r}_F = \frac{d_i}{d_i-f} r_F, \tag{14}$$

and $\tilde{r}_F$ is the projection of sampling filter radius r_F on the lens plane x, y.

Assuming that $f_{0F} = f_{0L}$ and $f_{0F} = r_F/2\lambda(d_i-f)$ we obtain finally

$$r_F = R\left(1-\frac{f}{d_i}\right). \tag{15}$$

The relationship (15) gives us an indication how to match the spatial filter to the lens dimensions, and vice versa.

Experimental

The possibility of image multiplication a single lens L_1 arrangement has been previously shown [3]. A 2-D object, coherently illuminated is placed at the distance $d_0 > f_1$ before the lens the sampling mask being placed in the Fourier plane of the objective lens L_1 at the focal distance f_1. Such optical arrangement is useful for 3-D object multiplication (fig. 1). Let the depth of 3-D luminous object along the z-axis be Δd_0. The Fourier spectrum of the object realized in the focal plane of the lens is:

$$\hat{U}_f(x_f, y_f) \propto \int_{d_0}^{d_0+\Delta d_0} \mathscr{F}\{\hat{U}_0(x_0, y_0, z)\} dz_0$$

$$= (i\lambda f)^{-1} \int_{d_0}^{d_0+\Delta d_0} \hat{F}(x_f, y_f, z_0) \exp\left[i \frac{\pi}{\lambda f}(x_f^2+y_f^2)\left(1-\frac{z_0}{f}\right)\right] dz_0,$$

where

$\hat{F}(x_f, y_f, z_0) = \mathscr{F}\{\hat{U}_0(x_0, y_0, z_0)\}$ – Fourier transform of the object light field,

$\mathscr{F}\{\ \}$ – Fourier operator, and z_0 parameter.

$\hat{U}_0(x_0, y_0, z_0)$ is defined for every $z_0 \in [(d_0+\Delta d_0)-d_0]$. The function $\hat{U}_0(x_0, y_0, z_0)$ represents the 3-D optical signal

$$\hat{S}(x_0, y_0, z_0) = \int_{d_0}^{d_0+\Delta d_0} \hat{U}(x_0, y_0, z_0) dz_0 \tag{17}$$

given at the input of the optical system, at the distance d_0 to the lens L_1. For every $z_0 \in [(d_0+\Delta d_0)-d_0]$ we have

$$\hat{F}(x_f, y_f, z_0) \equiv \hat{F}\left(\frac{x}{\lambda f}, \frac{y}{\lambda f}, z_0\right) = \hat{\hat{F}}_0(f_x, f_y, z_0). \tag{18}$$

At the Fourier plane x_f, y_f the objective lens we place a sampling mask with transmittance

$$t_f(x_f, y_f) \propto \sum_m \sum_n \delta(x_f \pm m\Delta x_f, y_f \pm n\Delta y_f). \tag{19}$$

The light field $\hat{U}_f(x_f, y_f)$ immediately behind the filter is:

$$\hat{U}_{f+}(x_f, y_f) = \hat{U}_{f-}(x_f, y_f) t_f(x_f, y_f) \propto$$

$$\propto (i\lambda f)^{-1} \sum_m \sum_n \int_{d_0}^{d_0+\Delta d_0} \hat{F}(x_f, y_f, z_0)\, \delta(x_f \pm m\Delta x_f, y_f \pm n\Delta y_f) \times \tag{20}$$

$$\times \exp\left[i \frac{\pi}{\lambda f}(x_f^2+y_f^2)\left(1-\frac{z_0}{f}\right)\right] dz_0.$$

In geometrical optics approximation the light field $\hat{U}_{im}(x_i, y_i, d_i)$ in the image x_i, y_i at the distance d_i behind the lens is given by

$$\hat{U}_{im}(x_i, y_i, d_i) = \sum_m \sum_n \hat{U}_{im}(x_i \pm m\Delta_{x_i}, y_i \pm n\Delta_{y_i}) \tag{21}$$

$$= \sum_m \sum_n \hat{U}_0(Mx_0 - x_i \pm m\Delta_{x_i}, My_0 - y_i \pm n\Delta_{y_i}),$$

where M is magnification.

Optical signal $\hat{S}(x_0, y_0, z_0)$ given at the input, and defined by eq. (17) has been multiplicated at the output of the optical system:

$$\hat{S}_{im}(x_i, y_i, n_i) = \sum_m \sum_n \hat{S}(x_i \pm m\Delta_{x_i}, y_i \pm n\Delta_{y_i}). \tag{22}$$

Spatial distances between multiple images $\Delta_{x_i} = \frac{\lambda f}{\Delta x_f} M$, $\Delta_{y_i} = \frac{\lambda f}{\Delta y_f} M$ depend on the depth of the object Δd_0 or the image Δd_i. The sampling constants Δx_f, Δy_f of the filter are matched to the object dimensions in the way stated in previous papers [3–5]. The matching condition follow from the sampling theorem in the spatial frequency domain

$$\Lambda_{x_f} \leqslant \frac{\lambda f}{D_{x_0}}, \quad \Lambda_{y_f} \leqslant \frac{\lambda f}{D_{y_0}}, \tag{23}$$

where D_{x_0}, D_{y_0} are dimensions of a rectangle circumscribing the greatest crossection of the 3-D object (fig. 2). Sampling of the Fourier spectrum of 3-D object results in multiplication in all images planes of the 3-D image. Impulse response and coherent transfer function for multiplication remain valid.

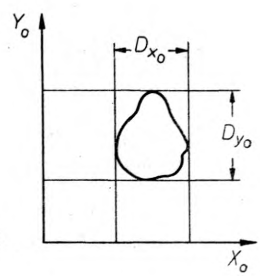


Fig. 2. The greatest crossection of the 3-D object, seen from the lens

A schematic diagram of a model experiment is shown in fig. 3. At the input of the optical system two transparents are placed at the respective distances d_{01} and d_{02} from the lens. In our experiment this distance was intentionally large: $\Delta d_{012} = 250$ mm.

Fig. 4 presents the Fourier image of the composite object at focal plane of the lens L_1. The same image after sampling by the inserted filter is shown in fig. 5. Multiplicated images of the object in the sharp focus image plane of the nearer transparent, and in the sharp image plane of the second transparent are shown in figs. 6a and 6b, respectively. Multiple images of both objects are mutually shifted, being separated at the input by the distance δx_0, δy_0.

Fig. 7 shows multiple images of a 3-D objects. In this case the dimensions given by radius $r_F = 6$ mm of the sampling spatial filter were matched (according to the eq. (15)) to the lens aperture radius $R = 17$ mm and

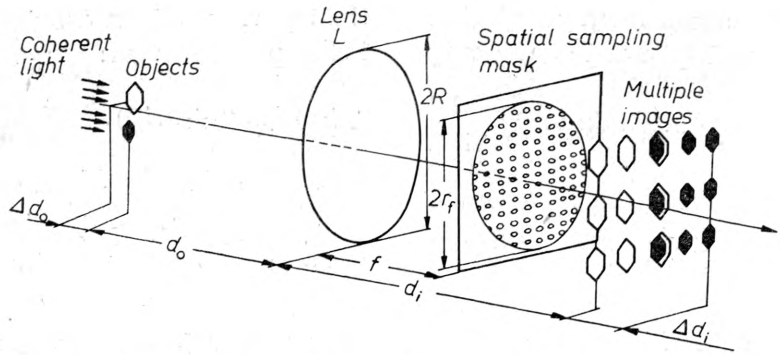


Fig. 3. Schematic diagram of a model experiment: 3-D object consists of two transparent, placed at distances d_{01} and d_{02} from the lens (coherent illumination). In reality the first pattern, placed nearer the lens L_1 is a small transparent regular hexagon. It is distinctly in fig. 6a. The second pattern, placed after first one is a hexagonal line inscribed into a square. It is seen distinctly in fig. 6b

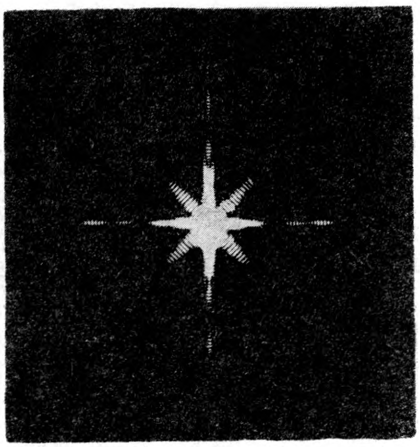


Fig. 4. Fourier image of this composite object at focal plane of the lens L

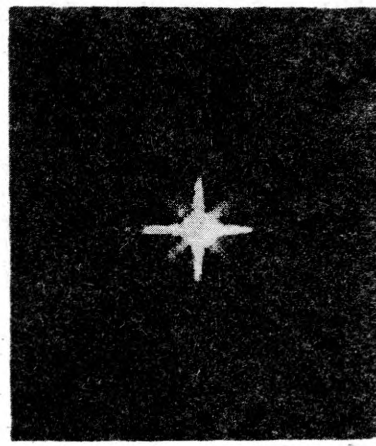


Fig. 5. The same as in fig. 4 after sampling by the inserted filter

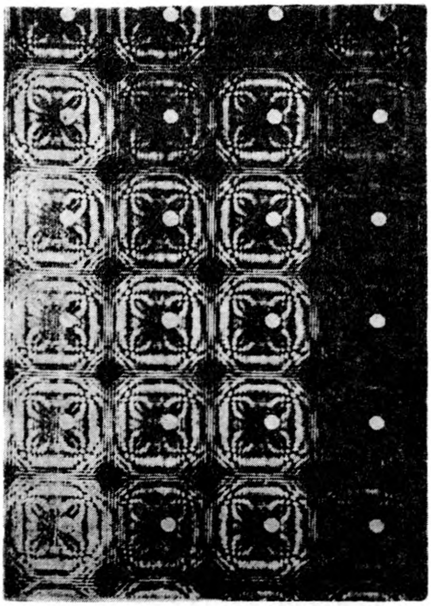
focal distance $f_1 = 360$ mm. The image was formed at the distance $d_i = 540$ mm.

The same object multiplicated by the use of a filter which does not fulfill the condition (15) is shown in fig. 8. The filter aperture radius $r_F = 3$ mm was too small compared to the above mentioned optical setup. Consequently, we can observed decreased fidelity of images and bluring of the details.

The filter used in the experiments had circular sampling holes of 10 μm radius spaced by the distances $\Lambda = \Lambda_{x_f} = \Lambda_{y_f} = 50$ μm. The weighting factor (see eqs. (1) and (2))

$$|\hat{T}_k(x_i, y_i)|^2 = \frac{r_0^2 J_1^2(2\pi \varrho_k r_0)}{\varrho_k^2} \tag{24}$$

a



b

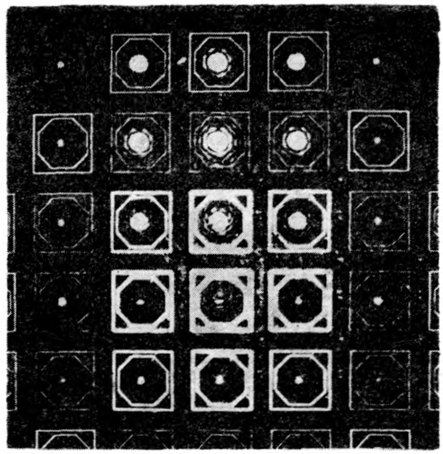


Fig. 6. a) Multiplicated image of the object in the sharp focus plane of the nearer transparent; b) the same as in fig. 6a in the sharpe focus image plane of the second transparent

Fig. 7. Multiple images of a 3-D object. Dimensions of the sampling filter were matched according to the eq. (15)

Fig. 8. The same as in fig. 7 3-D objects is multiplied by the use of a filter which does not fulfill the eq. (15)

determines the observed intensity distribution of the images, where

$$\varrho_k^2 = (m_k \Delta_{x_i})^2 + (n_k \Delta_{y_i})^2$$

is identic for

$$z_i \in [(d_i + \Delta d_i) - d_i].$$

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References

[1] Som S., J. Opt. Soc. Am. **60**, 1928 (1970).
[2] Vienot J. C. et al., Appl. Opt. **12**, 950 (1973).
[3] Kalestyński A., Appl. Opt. **14**, 2343 (1975), J. Opt. Soc. Am. **65**, 1443 (1975).
[4] Kalestyński A., Smolińska B., Appl. Opt. **16**, 24 (1977).
[5] Kalestyński A., Smolińska B., Appl. Opt. **16**, 2261 (1977).
[6] Shannon C. E., Proc. Inst. Radio Eng. **37**, 10 (1949).
[7] Linden D. A., Proc. Inst. Radio Eng. **47**, 1219 (1959).
[8] Parrent G. B., Thomson B. J., USA Patent 3320852.
[9] Goodman J., *Introduction to Fourier Optics*, McGraw-Hill Co., New York 1968.

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Импульсная характеристика когерентной оптической системы, мультипликативно увеличивающей изображение благодаря пробной фильтрации пространственных частот

Работа посвящена вопросу нахождения импульсной характеристики оптической системы когерентного света, обладающей свойством мультипликативного увеличения изображения благодаря пробному фильтрованию пространственных частот спектра фурье оптического сигнала, возбужденного на входе. Найдено условие согласования размеров оптических коленьев и фильтра, так чтобы усечение полосы пространственных частот фильтром не ограничивало частот, передаваемых оптическими коленьями. Опытная часть посвящена мультипликативному увеличению изображений трехмерных объектов с помощью названной системы; в ней эмпирически проиллюстрированы также результаты теоретических рассуждений.