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Marechal's intensity degradation for Gaussian beams in the presence of higher order aberrations*

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The Marechal's intensity degradation formula has been derived for Gaussian beams in the presence of primary, secondary and tertiary aberrations. It was also shown how the odd and even aberrations of a different order should be balanced giving the best central intensity. For different Gaussian-parameter values the lowest intensity degradation were evaluated.

Introduction

The peak intensity degradation is relatively simple but very helpful criterion for an image quality assesment in optical systems. This criterion was firstly formulated by RAYLEIGH [1] as a quarter-wavelength rule and later more exactly by STREHL [2], and MARECHAL [3]. HOPKINS [4] has shown how to balance the primary and secondary aberrations to obtain the lowest peak intensity degradation. This, however, was done only for the case of uniform amplitude distribution. LOWENTHAL [5] has obtained Marechal's formula for a Gaussian beam in the presence of spherical aberrations of all orders and the first order aberrations of other types. Our purpose is to derive Marechal's expression for Gaussian beams in the presence of primary, secondary and tertiary aberrations and to find an optimal balance of aberrations of various types.

Marechal's formula for Gaussian aperture

From scalar diffraction theory [6] it follows that the intensity at the diffraction focus normalized by analogous quality for diffraction limited case (called Strehl ratio — S.R.) has the form

$$\text{S.R.} = \frac{\left|\int\limits_0^1\int\limits_0^{2\pi} A(\varrho)\exp[-ik\{v\varrho\cos\Theta+\tfrac{1}{2}u\varrho^2-\Phi(\varrho,\Theta)\}]\varrho d\varrho d\Theta\right|^2}{\left|\int\limits_0^1\int\limits_0^{2\pi} A(\varrho)\varrho d\varrho d\Theta\right|^2}, \tag{1}$$

where Θ — are the polar coordinates in the aperture; u, v — represents shifts of observation position in the longitudinal and transversal directions, respectively, measured from Gaussian image point; $A(\varrho)$ — describes illumination of the aperture; $\Phi(\varrho, \Theta)$ — is the wave aberration and k — means the wave number.

The Marechal's approximation of the S.R. is obtained after expanding the exponent function into series and droping all terms of order higher than second in $k\Phi$.

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The result is the following:

$$i = 1-\left(\frac{2\pi}{\lambda}\right)^2 \{\overline{\Phi^2}-(\overline{\Phi})^2\}. \tag{2}$$

In the case of Gaussian aperture ($A(\varrho) = A_0 \exp\left(-(\varrho/\gamma)^2\right)$, γ — Gaussian parameter) the expression for image degradation has form analogous to (2) [5]. Namely

$$i = 1-\left(\frac{2\pi}{\lambda}\right)^2 [\overline{\Delta^2}-(\overline{\Delta})^2] \tag{3}$$

or

$$i = 1-\left(\frac{2\pi}{\lambda}\right)^2 \overline{E}, \tag{3a}$$

where

$\tilde{E} = \overline{\Delta^2}-(\overline{\Delta})^2$,

$\overline{\Delta^n}$ — denotes n-th moment of the Δ^n function ($\Delta^n = \Phi^n \exp[-(\varrho/\gamma)^2]$). This moment is defined in the following way:

$$\overline{\Delta^n} = \frac{1}{\pi\gamma^2\left\{1-\exp\left[-\left(\frac{1}{\gamma}\right)^2\right]\right\}} \int_0^1 \int_0^{2\pi} \exp[-(\varrho/\gamma)^2]\Phi^n(\varrho, \Theta)\varrho \, d\varrho \, d\Theta. \tag{4}$$

It is obvious that relation (3) goes into (2) for uniform illumination.

Marechal's expression for Gaussian beam in the presence of primary, secondary and tertiary aberrations

To find an analytical form of the relation (3) we must have at first the wave aberration and then evaluate the integrals (4) for $n = 1,2$.

Complete wave aberration up to the fourth order has the following form [7]:

$$\begin{aligned}\Phi(\varrho, \Theta) = {} & W_{20}\varrho^2+W_{40}\varrho^4+W_{60}\varrho^6+W_{80}\varrho^8+ \\ & +(W_{11}\varrho+W_{31}\varrho^3+W_{51}\varrho^5+W_{71}\varrho^7)\cos\Theta+ \\ & +(W_{22}\varrho^2+W_{24}\varrho^4+W_{26}\varrho^6)\cos 2\Theta+ \\ & +(W_{33}\varrho^3+W_{35}\varrho^5)\cos 3\Theta+ \\ & +W_{44}\varrho^4\cos 4\Theta.\end{aligned} \tag{5}$$

Let us for convenience rewrite the formula (5) in the following form

$$\Phi(\varrho, \Theta) = \sum_{n=1}^{4}(a_n\varrho^{2n})+a\cos\Theta+b\cos 2\Theta+c\cos 3\Theta+d\cos 4\Theta$$

where:

$$\begin{aligned} a_1 &= W_{20}, a_2 = W_{40}, a_3 = W_{60}, a_4 = W_{80}, \text{ and} \\ a &= W_{11}\varrho + W_{31}\varrho^3 + W_{51}\varrho^5 + W_{71}\varrho^7, \\ b &= W_{22}\varrho^2 + W_{24}\varrho^4 + W_{26}\varrho^6, \\ c &= W_{33}\varrho^3 + W_{35}\varrho^5, \\ d &= W_{44}\varrho^4. \end{aligned} \tag{6}$$

Now the moment Δ is equal to

$$\Delta = \frac{1}{\pi\gamma^2\{1-\exp[-(1/\gamma)^2]\}} \int_0^1 \int_0^{2\pi} \exp-[(\varrho/\gamma)^2]\Big(\sum_{n=1}^{4} a_n\varrho^{2n} + a\cos\Theta + b\cos 2\Theta + c\cos 3\Theta + d\cos 4\Theta\Big)\varrho\, d\varrho\, d\Theta. \tag{7}$$

As the integration all terms depending on Θ gives zero we have

$$\Delta = \frac{2}{\gamma^2\{1-\exp[-(1/\gamma)^2]\}} \int_0^1 \exp-[(\varrho/\gamma)^2]\Big(\sum_{n=1}^{4} a_n\varrho^{2n}\Big)\varrho\, d\varrho. \tag{8}$$

From the same reason the second moment is

$$\Delta^2 = \frac{2}{\gamma^2\{1-\exp[-(1/\gamma)^2]\}} \int_0^1 \exp-[(\varrho/\gamma)^2]\Big[\sum_{m=2}^{8} b_m\varrho^{2m} + 1/2\,(a^2+b^2+c^2+d^2)\Big]\varrho\, d\varrho. \tag{9}$$

Coefficients b_n may be found with the help of the recurrence formula [8]

$$b_n a_0 n = \sum_{l=1}^{n} b_{n-l}a_l(3l-n),\ b_0 = a_0^2 = W_{10}^2. \tag{10}$$

Therefore we have

$$\begin{aligned} b_2 &= W_{20}^2, \\ b_3 &= 2W_{20}W_{40}, \\ b_4 &= 2W_{20}W_{60} + W_{40}^2, \\ b_5 &= 2(W_{20}W_{80} + W_{40}W_{60}), \\ b_6 &= 2W_{40}W_{80} + W_{60}^2, \\ b_7 &= 2W_{60}W_{80}, \\ b_8 &= W_{80}^2. \end{aligned} \tag{11}$$

The integration of (8) and (9) can be carried out analytically [5]. The results of integration are

$$\Delta = F(\gamma)\left[\frac{1}{2} W_{20}S_1(\gamma) + \frac{1}{3} W_{40}S_2(\gamma) + \frac{1}{4} W_{60}S_3(\gamma) + \frac{1}{5} W_{80}S_4(\gamma)\right], \tag{12}$$

and

$$\Delta^2 = F(\gamma)\Big[\frac{1}{4} W_{11}^2 S_1(\gamma) + \frac{1}{3}(W_{20}^2 + W_{31}W_{11} + \frac{1}{2}W_{22}^2)S_2(\gamma) +$$

$$+\frac{1}{4}(2W_{20}W_{40} + W_{11}W_{51} + \frac{1}{2}W_{31}^2 + W_{22}W_{24} + \frac{1}{2}W_{33}^2)S_3(\gamma) +$$

$$+\frac{1}{5}(2W_{20}W_{60} + W_{40}^2 + W_{11}W_{71} + W_{31}W_{51} + \frac{1}{2}W_{24}^2 +$$

$$+W_{22}W_{26} + W_{33}W_{35} + \frac{1}{2}W_{44}^2)S_4(\gamma) + \frac{1}{6}(2W_{20}W_{80} +$$

$$+2W_{40}W_{60} + \frac{1}{2}W_{51}^2 + W_{31}W_{71} + W_{24}W_{26} + \frac{1}{2}W_{35}^2)\times \tag{13}$$

$$\times S_5(\gamma) + \frac{1}{7}(2W_{40}W_{80} + W_{60}^2 + W_{51}W_{71} + \frac{1}{2}W_{26}^2)S_6(\gamma) +$$

$$+\frac{1}{8}(2W_{60}W_{80} + \frac{1}{2}W_{71}^2)S_7(\gamma) + \frac{1}{9}W_{80}^2 S_8(\gamma)\Big].$$

Functions $S_n(\gamma)$, $F(\gamma)$ are defined [5] in the following way:

$$F(\gamma) = \frac{1}{[\gamma^2(e^{1/\gamma^2}-1)]},$$

$$S_n(\gamma) = 1 + \frac{1/\gamma^2}{(n+2)} + \frac{(1/\gamma^2)^2}{(n+2)\,(n+3)} + \dots, \tag{14}$$

and for $\gamma > 2$ are practically equal to unity.

More details may be found in [5]. After inserting (12), (13) into (3), the final formula for Marechal's intensity degradation for Gaussian beams is the following

$$i = 1-(2\pi/\lambda)^2\, F(\gamma)\Big\{\frac{1}{4}W_{11}^2 S_1(\gamma) + \frac{1}{3}(W_{20}^2 + W_{11}W_{31} + \frac{1}{2}W_{22}^2)S_2(\gamma) +$$

$$+\frac{1}{4}(2W_{20}W_{40} + W_{11}W_{51} + \frac{1}{2}W_{31}^2 + W_{22}W_{24} + \frac{1}{2}W_{33}^2)S_3(\gamma) + \frac{1}{5}(2W_{20}W_{60} +$$

$$+W_{40}^2 + W_{11}W_{71} + W_{31}W_{51} + \frac{1}{2}W_{24}^2 + W_{22}W_{26} + W_{33}W_{35} +$$

$$+\frac{1}{2}W_{44}^2)S_4(\gamma) + \frac{1}{6}(2W_{20}W_{80} + 2W_{40}W_{60} + \frac{1}{2}W_{51}^2 + W_{31}W_{71} +$$

$$+W_{24}W_{26} + \frac{1}{2}W_{35}^2)S_5(\gamma) + \frac{1}{7}(2W_{40}W_{80} + W_{60}^2 + W_{51}W_{71} + \frac{1}{2}W_{26}^2)S_6(\gamma) +$$

$$+\frac{1}{8}(2W_{60}W_{80} + \frac{1}{2}W_{71}^2)S_7(\gamma) + \frac{1}{9}W_{80}^2 S_8(\gamma) - F(\gamma)\Big[\frac{1}{2}W_{20}S_1(\gamma) +$$

$$+\frac{1}{3}W_{40}S_2(\gamma) + \frac{1}{4}W_{60}S_3(\gamma) + \frac{1}{5}W_{80}S_4(\gamma)\Big]^2\Big\}. \tag{15}$$

Optimal balancing of the specified types of aberrations

Having in mind that Marechal's criterion is good for well corrected systems we restrict further considerations to primary and secondary aberrations. To find the lowest image intensity degradation we have to find the minimum value of the expression in { } bracket in eq. (15).

Spherical aberration

From (15) it is easy to see that in the case of spherical aberrations of 1-st and 2-nd orders the variance $\tilde{E}$ is

$$\tilde{E} = F(\gamma)\left\{\frac{1}{3}W_{20}^2S_2(\gamma)+\frac{1}{2}W_{20}W_{40}S_3(\gamma)+\frac{1}{5}(2W_{20}W_{60}+W_{40}^2)\times S_4(\gamma)+\right.$$
$$+\frac{1}{3}W_{40}W_{60}S_5(\gamma)+\frac{1}{7}W_{60}^2S_6(\gamma)-F(\gamma)\left[\frac{1}{4}W_{20}^2S_1^2+\right.$$
$$+\frac{1}{3}W_{20}W_{40}S_1(\gamma)S_2(\gamma)+\frac{1}{4}W_{20}W_{60}S_1(\gamma)S_3(\gamma)+\frac{1}{9}W_{40}^2S_2^2(\gamma)+ \quad (16)$$
$$\left.\left.+\frac{1}{6}W_{40}W_{60}S_2(\gamma)S_3(\gamma)+\frac{1}{16}W_{60}^2S_3^2(\gamma)\right]\right\}.$$

Of course the plane of the best focus is given by the condition

$$\frac{\partial\tilde{E}}{\partial W_{20}} = 0. \quad (17)$$

After differentiation of (16) we obtain the equation

$$\frac{2}{3}W_{20}S_2(\gamma)+\frac{1}{2}W_{40}S_3(\gamma)+\frac{2}{5}W_{60}S_4(\gamma)-F(\gamma)\times\left[\frac{1}{2}W_{20}S_1^2(\gamma)+\right.$$
$$\left.+\frac{1}{3}W_{40}S_1(\gamma)S_2(\gamma)+\frac{1}{4}W_{60}S_1(\gamma)S_3(\gamma)\right]=0. \quad (18)$$

Hence

$$\beta_{26} = \frac{\frac{1}{2}\beta_{46}S_3(\gamma)-\frac{1}{3}\beta_{46}S_1(\gamma)S_2(\gamma)F(\gamma)-\frac{2}{5}S_4(\gamma)+\frac{1}{4}S_1(\gamma)S_3(\gamma)F(\gamma)}{\frac{2}{3}S_2(\gamma)-\frac{1}{2}S_1^2(\gamma)F(\gamma)}, \quad (19)$$

where

$$\beta_{46} = -\frac{W_{40}}{W_{60}},$$

$$\beta_{26} = \frac{W_{20}}{W_{60}}.$$

In the case of uniform illumination ($\gamma \to \infty$) which gives lim $F(\infty) = 1$, and lim $S_n(\infty) = 1$ for all n, the relation (19) is reduced to the form

$$W_{20}\big|_{\gamma\to\infty} = \left(\beta_{46} - \frac{9}{10}\right) W_{60}. \tag{20}$$

The last result is the same as that obtained by HOPKINS [4] for uniform illumination.

Now inserting the optimal W_{20} from (19) into (16) we obtain the optimal variance $\tilde{E}$:

$$\tilde{E} = F(\gamma)\Big\{\beta_{26}^2\Big[\frac{1}{3}S_2(\gamma) - \frac{1}{4}S_1^2(\gamma)F(\gamma)\Big] + \beta_{26}\beta_{46}\Big[\frac{1}{3}S_1(\gamma)S_2(\gamma)F(\gamma) - \frac{1}{2}S_3(\gamma)\Big] + \beta_{46}^2\Big[\frac{1}{5}S_4(\gamma) - \frac{1}{9}S_2^2(\gamma)F(\gamma)\Big] + \beta_{26}\Big[\frac{2}{5}S_4(\gamma) - \frac{1}{4}S_1(\gamma)S_3(\gamma)F(\gamma)\Big] + \beta_{46}\Big[\frac{1}{6}S_2(\gamma)S_3(\gamma)F(\gamma) - \frac{1}{3}S_5(\gamma)\Big] + \frac{1}{7}S_6(\gamma) - \frac{1}{16}S_3^2(\gamma)F(\gamma)\Big\}W_{60}^2. \tag{21}$$

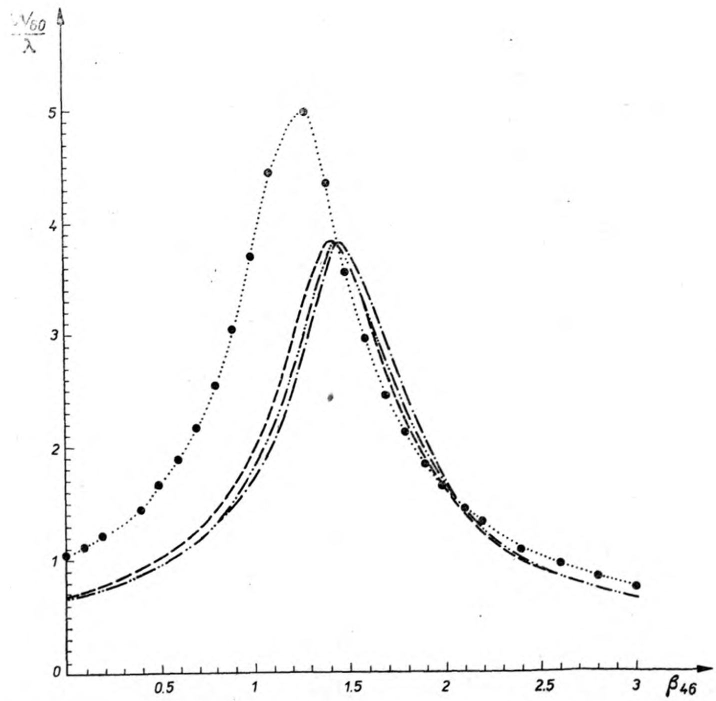

Fig. 1. The dependence of tolerance for spherical aberration W_{60} on the form of correction of β_{46} based on the Marechal's condition for different values of Gaussian parameters ($\gamma = 0.5$ .....; $\gamma = 1$ — — — —; $\gamma = 1.5$ — ·· —; $\gamma = 2$ —·—·—·—; $\gamma = \infty$—·—·—·—·)

The tolerance on intensity degradation may be written in one of two equivalent forms, namely

$$i \geqslant 0.8 \text{ or } \tilde{E} \leqslant \frac{\lambda^2}{180}. \tag{21a}$$

Therefore, the relations (16), (19), (21a) may be used to find the optimal balancing of the spherical aberrations in the case of Gaussian beams. This procedure has been examplified numerically. Basing on the Marechal's condition (21a) and taking different values of the parametr γ, we have found the dependence of tolerance for secondary spherical aberration W_{60} on the form of correction of β_{46} (fig. 1).

For different values of γ we have also found optimal parameters β_{46} and β_{26}. These results are presented in table 1.

Table 1

Optimal coefficients β_{46}, β_{26} and maximal (W_{60}/λ) for different values of γ

γ	β_{46}	β_{26}	$(W_{60}/\lambda)_{max}$
0.5	1.29	0.41	5.0
1.0	1.44	0.54	3.8
1.5	1.47	0.59	3.8
2.0	1.49	0.59	3.8
∞	1.50	0.60	3.8

Having optimal β_{46} we have calculated minimal peak intensity degradation versus W_{60}/λ. The obtained results are presented in fig. 2a. From these curves $(W_{60}/\lambda)_{max}$ has been determined for which $i = 0.8$ (see tab. 1). To asses the accuracy of Marechal's formula we have drawn curves analogous to those mentioned above but obtained from the relation (1). By comparing fig. 2a and 2b we can state that for $0.8 \leqslant i \leqslant 1$ the values of central intensity obtained from Strehl's and Marechal's formulae are practically the same.

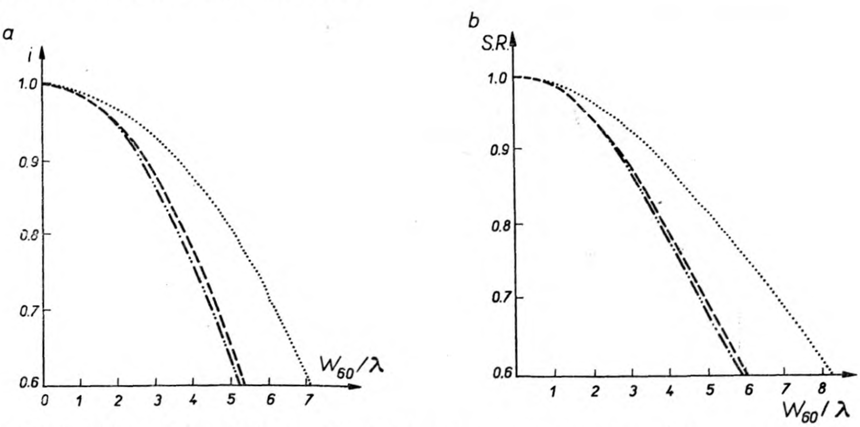

Fig. 2. Optimal image degradation vs. W_{60}/λ for different values of γ($\gamma = 0.5$ ......; $\gamma = 1$ — — —; $\gamma = 2$ and $\gamma = \infty$ —··—··—··—) obtained with the help of Marechal's (2a) and Strehl's (2b) formulae

Coma

In the case of coma the variance $\tilde{E}$ has the form

$$\tilde{E} = \frac{1}{2}\left[\frac{1}{2} W_{11}^2 S_1(\gamma) + \frac{2}{3} W_{11} W_{31} S_2(\gamma) + \frac{1}{4}(2W_{11}W_{51} + W_{31}^2) \times \right. \tag{22}$$

$$\left. \times S_3(\gamma) + \frac{2}{5} W_{31} W_{51} S_4(\gamma) + \frac{1}{6} W_{51}^2 S_5(\gamma)\right] F(\gamma).$$

The maximum in the diffraction pattern requires the fullfilment of the condition

$$\frac{\partial \tilde{E}}{\partial W_{11}} = 0. \tag{23}$$

From eqs. (22) and (23) we obtain

$$W_{11} = \frac{\left[\frac{2}{3}\beta_{35} S_2(\gamma) - \frac{1}{2} S_3(\gamma)\right] W_{51}}{S_1(\gamma)}, \tag{24}$$

where

$$\beta_{35} = -\frac{W_{31}}{W_{51}}.$$

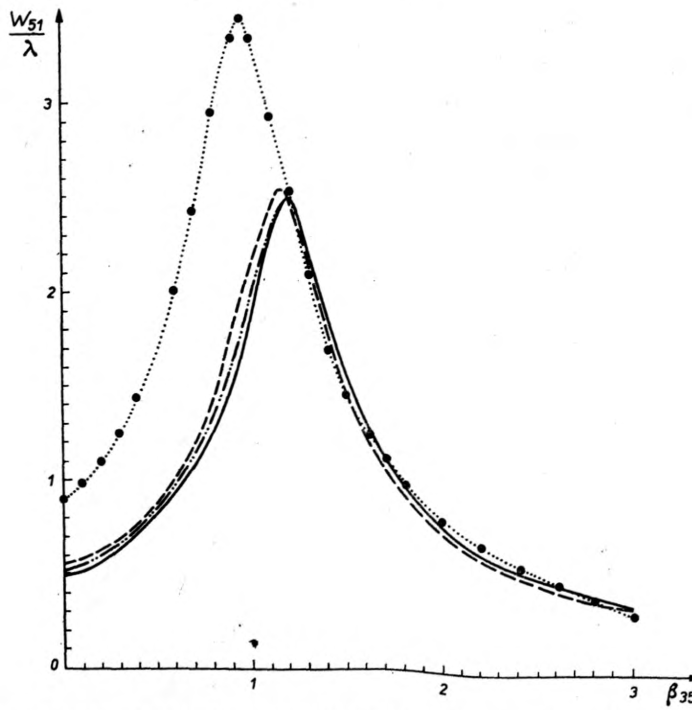

Fig. 3. Dependence of tolerance for secondary coma W_{51}/λ on the form of correction β_{35} for different values of γ ($\gamma = 0.5$ ......; $\gamma = 1$ — — — —; $\gamma = 1.5$ —··—··—··—··; $\gamma = 2$ and $\gamma = \infty$ —·—·—·—·—)

Again for a uniform illumination ($\gamma = \infty$) eq. (24) goes into Hopkins formula

$$W_{11} = \left(\frac{2}{3}\beta_{35} - \frac{1}{2}\right) W_{51}. \tag{25}$$

Setting the optimal values of W_{11} from eq. (24) into the expression (22) we obtain

$$\tilde{E} = \frac{1}{2} F(\gamma) \left\{ \left(\frac{2}{3}\beta_{35} S_2(\gamma) - \frac{1}{2} S_3(\gamma)\right) \left[\frac{1}{2}\left(\frac{2}{3}\beta_{35} S_2(\gamma) - \frac{1}{2} S_3(\gamma)\right) \frac{1}{S_1(\gamma)} - \frac{2}{3}\beta_{35}\frac{S_2(\gamma)}{S_1(\gamma)} + \left(\frac{1}{2S_1(\gamma)} + \frac{1}{4}\beta_{35}^2\right) S_3(\gamma)\right] - \frac{2}{5}\beta_{35} S_4(\gamma) + \frac{1}{6} S_5(\gamma) \right\} W_{51}^2. \tag{26}$$

The next thing to do is to find the relation between W_{51}/λ and β_{56} which satisfies Marechal's condition. The results are given in fig. 3.

Table 2

Optimal coefficients β_{35}, β_{15} and maximal (W_{51}/λ) for different values of γ

γ	β_{35}	β_{15}	$(W_{51}/\lambda)_{max}$
0.5	0.95	0.16	3.45
1.0	1.15	0.29	2.55
1.5	1.18	0.29	2.50
2.0	1.18	0.29	2.50
∞	1.20	0.30	2.50

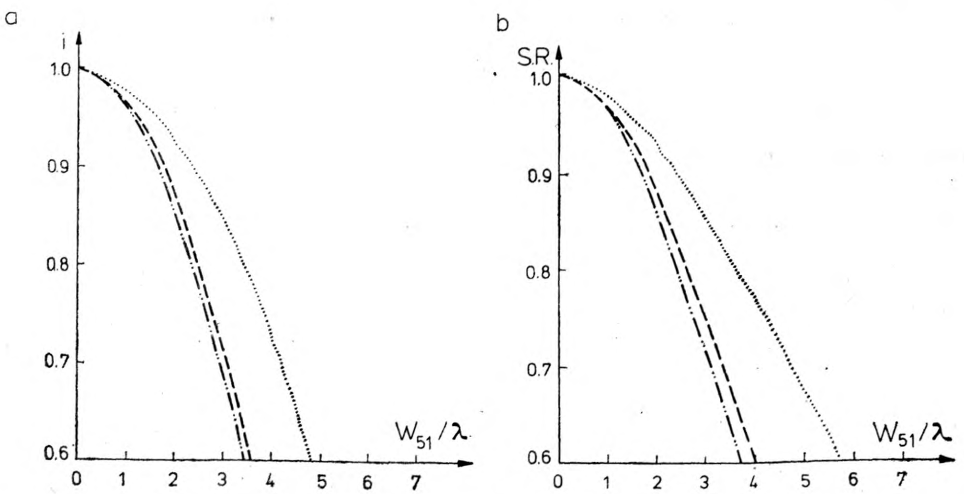

Fig. 4. The optimal intensity degradation vs. W_{51}/λ for different values of γ($\gamma = 0.5$ ....; $\gamma = 1$— — — — — ; $\gamma = 2$ and $\gamma = \infty$ —··—··—··—··—·—) obtained on the basis of Marechal's (4a) and Strehl's (4b) formulae

We have found optimal values of β_{35} and β_{15} (see tab. 2) and the corresponding optimal intensity degradation (fig. 4a). From the results presented in fig. 4 we see that for $0.8 \leqslant i \leqslant 1$ and all $\gamma \geqslant 0.5$ minimal central image degradation obtained from Marechal's approximation also for coma is good.

Conclusions

The expression for Marechal's intensity degradation for Gaussian beams in the presence of primary, secondary and tertiary both even and odd aberrations was derived. It has been shown that for Gaussian parameter greater than unity the optimal aberrational coefficients and, consequently, the optimal central image degradation are practically the same as that for uniform illumination.

For the parameter γ smaller than unity there exists the improvement of central intensity degradation. It has been also shown that for optimal central image degradation the Marechal's formula is a satisfactory approximation over the considered range of minimal intensity degradation.

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Снижение интенсивности Марешала для случая усеченных гауссовых пучков при наличии аберраций высших порядков

Получено выражение для падения интенсивности Марешала для аберраций третьего, пятого и седьмого порядков в случае гауссовых пучков. Показано, каким образом должны корректироваться парные и непарные аберрации высших порядков для получения наилучшей интенсивности Марешала. Для специфицированных параметров Гаусса определены минимальные снижения интенсивности, отвечающей центру линии поглощения.