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# On tolerancing of the refractive index gradient in optical systems *

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In the paper the effect of the refractive index gradient in elements of an optical system on the Strehl definition has been determined. The final formulae have been obtained under assumption that the rays run parallely to the optical axis of the particular elements of the system. This assumption restricts the applicability of the obtained results. However, for certain class of objectives the derived formulae may offer a basis for tolerancing the gradient.

## Introduction

The material of which the optical elements are made is almost absolutely uniform; the typical material errors being bubbles, striae, birefringence and slowly-varying nonuniformities of the refractive index. The first three of the mentioned errors may be relatively easily detected and measured. Their influence on the imaging quality has found a wide representation in the literature, for instance, [1–3]. However, the measurement of the slow changes of the refractive index is relatively difficult and expensive.

The literature concerning the effect of this type of errors on the imaging quality includes three papers [4–6]. The influence of the gradient in the reflecting prisms and systems of such prisms on their focussing properties has been determined in the papers [4] and [5], while the effect of the gradient in lenses and refracting prisms on the Strehl definition being examined in [6]. In the present paper the results published in [6] have been generalized to include the systems perturbed by aberrations consisting of many lenses. The final formulae have been derived under the assumption that the rays travel — at least approximately — parallelly to the optical axis. This assumption restricts the applicability of the results to some types of objectives. It seems that a general solution of the problem is not possible. Some expectation may be connected with the method presented in the paper [7]. This method is, however, very complicated and the final formulae would be probably so complex that their application to practical tolerancing problems would be very problematic.

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## Theory

The following notation for the coordinates in the plane of the discussed lens have been assumed:

$x', y'$ — rectangular coordinate,

$x, y$ — dimensionless rectangular coordinates,

$x = x'/a,\ y = y'/a,$

$a$ — the radius of the cross-section of the aperture beam in the lens considered.

Let us introduce additionally the polar coordinates

$$r' = (x'^2+y'^2)^{1/2}, \quad \varphi = \tan^{-1}(y'/x'),$$

$$r = (x^2+y^2)^{1/2} = r'/a, \quad \varphi = \tan^{-1} = (y/x) = \varphi'.$$

Let us assume that the optical system consists of $N$ lenses and that in each of them there appears a refractive index gradient $G_i (i = 1, \ldots, N)$. The aberrations induced by the gradient in the $i$-th lens are [6]:

$$\begin{aligned} V_{G_i}(x, y) &= G_{x_i} x D_i(x, y) a_i + G_{y_i} y D_i(x, y) a_i \\ &= (G_{x_i}\cos\varphi + G_{y_i}\sin\varphi) r a_i D_i(r), \end{aligned} \tag{1}$$

where:

$G_{x_i}, G_{y_i}$ — are the gradient components in the plane perpendicular to the axis of $i$-th lens (fig. 1),

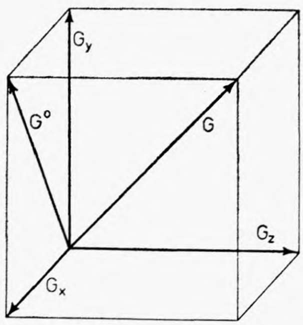


Fig. 1. The decomposition of the refractive index gradient $G$ into Cartesian components:
$G_x, G_y$ — in the lens plane, $G_z$ — in the direction optical axis, $G^0$ — the component of $G$ vector in the lens plane

$D_i(x, y)$ — the function of thickness for $i$-th lens [6]

$$D_i(x, y) = D_i(r) = D_{o_i}(1 - a_i r^2), \tag{2}$$

$$a_i = \frac{D_{o_i} - D_{b_i}}{D_{o_i}},$$

$D_{o_i} = D_i(0)$ — thickness of the $i$-th lens in its middle,

$D_{b_i} = D_i(1)$ — thickness of the $i$-th lens at the edge (fig. 2).

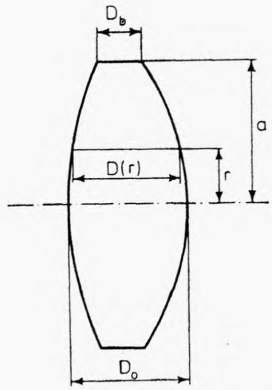


Fig. 2. The picture explaining the notation used in the text:
$D(r)$ – thickness function of the lens, $D_b$ – thickness at the lens edge, $D_o$ – thickness in the lens middle

The total aberrations $V_G(x, y)$ introduced by the gradient in all the lenses are respective sums of aberrations coming from the particular lenses

$$V_G(x, y) = \sum_i V_{G_i}(x, y) = \sum_i V_{G_i}(r, \varphi)$$
$$= \sum_i (G_{x_i}\cos\varphi + G_{y_i}\sin\varphi) D_{o_i}(r - a_i r^3) a_i.$$

The joint aberrations $W(x, y)$ of the system are sums of aberrations coming from the gradient and the own aberrations $\Delta(x, y)$ of the system

$$W(x, y) = V_G(x, y) + \Delta(x, y)$$
$$= \Delta(x, y) + \sum_i (G_{x_i}\cos\varphi + G_{y_i}\sin\varphi) D_{o_i}(r - a_i r^3) a_i.$$

To evaluate the effect of refractive index gradient on the imaging quality we shall use the Strehl definition I, which for small aberrations may be described by an approximate formula [8]:

$$I = 1 - k^2(\langle W^2\rangle - \langle W\rangle^2), \tag{4}$$

where:

$$\langle W\rangle = \int_0^1 \int_0^{2\pi} W(r, \varphi) r\, dr\, d\varphi,$$

$k = 2\pi/\lambda$, $\lambda$ – wavelength of the light in vacuum.

The aberrations coming from the gradient cause an inclination of the reference sphere. To take account of this effect $W(x, y)$ should be completed by adding the terms

$$Kr\sin\varphi + Lr\cos\varphi .$$

We obtain

$$W'(x, y) = W(x, y) + Kr\sin\varphi + Lr\cos\varphi\,. \tag{5}$$

In order to determine the optimal values of $K$ and $L$ the formula (5) should be inserted into (4) and $K$ and $L$ calculated from the conditions

$$\frac{\partial I}{\partial K} = \frac{\partial I}{\partial L} = 0\,. \tag{6}$$

Denote $K$ and $L$, fulfilling this conditions by $K_{\text{opt}}$ and $L_{\text{opt}}$. Thus, the aberrations coming from the gradient are defined by the formula

$$V_{G\text{opt}}(x, y) = V(x, y) + K_{\text{opt}} r\sin\varphi + L_{\text{opt}} r\cos\varphi\,. \tag{7}$$

The formula for the Strehl definition takes the final form

$$I_{\text{opt}} = I_{\Delta} - k^2[\langle V^2_{G\text{opt}}\rangle + 2\langle \Delta V_{G\text{opt}}\rangle], \tag{8}$$

where

$I_{\Delta} = 1 - k^2(\langle \Delta^2\rangle - \langle \Delta\rangle^2)$ – Strehl definitions in the optical system with its own aberrations.

The method presented in this paper is both general and possible to application in practice, if the own aberrations of the system are known. Only in special cases (for instance, for axial-symmetric aberrations) we have

$$\langle \Delta V_G\rangle = 0, \quad \text{thus} \quad \langle \Delta V_{G\text{opt}}\rangle = 0,$$

and then the formula (8) takes the simple form

$$I = I_{\Delta} - k^2\langle V^2_{G\text{opt}}\rangle\,. \tag{9}$$

In this case the parameters $K_{\text{opt}}$ and $L_{\text{opt}}$ determined from the condition (6) are equal to

$$K_{\text{opt}} = \sum_i G_{x_i} a_i D_{o_i}(\tfrac{2}{3}a_i - 1),$$

$$L_{\text{opt}} = \sum_i G_{y_i} a_i D_{o_i}(\tfrac{2}{3}a_i - 1)\,. \tag{10}$$

By inserting (10) and (5) into (8) we obtain

$$I_{\text{opt}} = I_{\Delta} - \frac{k^2}{72}\Big[\Big(\sum_i G_{x_i} a_i (D_{o_i} - D_{b_i})\Big)^2 + \Big(\sum_i G_{y_i} a_i (D_{o_i} - D_{b_i})\Big)^2\Big]\,. \tag{11}$$

Denoting by $G_i^o$ the gradient component in the plane of the $i$-th lens and by $\psi_i$ the angle of $G_i^o$ vector with the $x$-axis we obtain

$$G_{x_i} = G_i^o \cos\psi_i, \quad G_{y_i} = G_i^o \sin\psi_i\,.$$

The formula (11) after transformations takes a simpler form

$$I_{\text{opt}} = I_{\Delta} - \frac{k^2}{72}\Big[\sum_i \big(G_i^o a_i (D_{o_i} - D_{b_i})\big)^2 +$$

$$+2 \sum_{\substack{l,k \\ l \neq k}} +\cos(\psi_l - \psi_k) G_l^o G_k^o a_l a_k (D_{o_l} - D_{b_l})(D_{o_k} - D_{b_k})\Big],$$

$$l, k = 1, \ldots, N.$$

From the formula (12) it follows that the gradient effect on the imaging quality depends strongly on the directions of the vectors $G_i^o$. Since, in general, their direction is not known, we shall find such one for which the Strehl definition reaches its minimum.

It is easy to notice that this occurs when

$\psi_i = \psi_o$ – for positive lens,

$\psi_i = \psi_o + 180^0$ – for negative lens.

This means that the component of the gradient vector $G_i^o$ should be oriented in the same way in all the lenses, i.e. parallely in the positive lenses and antiparallely in the negative lenses. In this case the Strehl definition formula takes the form

$$I_{\text{opt}}^{\min} = I_{\Delta} - \frac{1}{72} k^2 \Big[\sum_i G_i^o a_i (D_{o_i} - D_{b_i})\Big]^2. \tag{13}$$

Thus, due to the gradient action the Strehl definition never reaches any value lower than that defined by the formula (13). The formula (13) gives a one-sided estimation of the deformation degree and may be a basis for tolerancing the gradient in some types of optical systems. The formula (13) is a generalization of formulas given in [6] for single aberration-free optical elements.

In the present paper it has been assumed that the rays run parallely to the axis of the optical system. In the real optical systems the rays subtend certain angles with the optical axis. Therefore, a direct application of method proposed in this paper to the real systems is connected with some errors. This error may be established as follows:

Let us denote by $V_o$ the aberrations generated by the gradient $\boldsymbol{G}$ for the ray running parallely to the axis and by $V_\gamma$ the aberrations for the ray subtending an angle $\gamma$ with the optical axis of the lens (fig. 3). The relative error $\Delta$ will be equal to

$$\Delta = \frac{V_o - V_\gamma}{V_\gamma}.$$

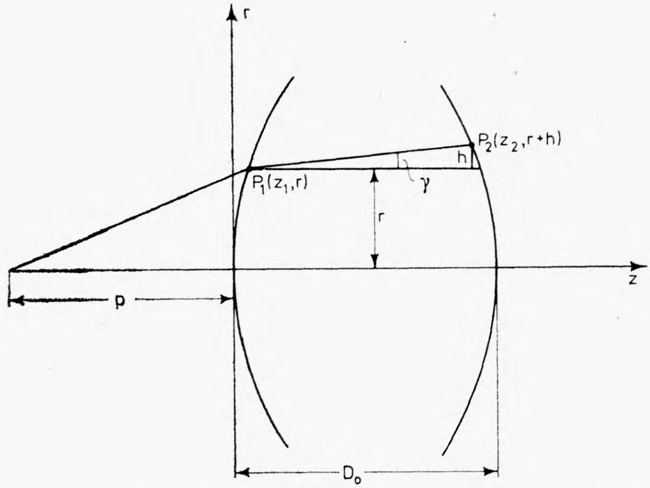


Fig. 3. An auxiliary lay-out to determine the error in the method presented

After detailed calculations we obtain

$$\Delta = \left(D_o + \frac{r^2}{2R_2} - \frac{r^2}{2R_1}\right)\left\{\left(D_o + \frac{(r+h)^2}{2R_2} - \frac{r^2}{2R_1}\right) \times \right.$$

$$\left. \times \left[1 + \frac{h}{r}\left(D_o + \frac{(r+h)^2}{2R_2} + \frac{r^2}{2R_1}\right)\right]\left[\left(D_o + \frac{(r+h)^2}{2R_2} - \frac{r^2}{2R_1}\right) - h\right]^{1/2}\right\}^{-1} . \quad (14)$$

This formula may be drastically simplified for $h \ll R_1, R_2$. Then we obtain:

$$\Delta = \left(1 + \frac{z_1 + z_2}{r} \tan\gamma\right)^{-1} - 1 . \quad (15)$$

The formula (15) should not be applied to estimate the error for small values of $\gamma$ angle, because

$$\lim_{\gamma \to 0} r = 0 .$$

In this case the formula (15) after respective rearrangements takes the form

$$\Delta = \left(1 + \frac{D_o}{n_o p}\right)^{-1} - 1 , \quad (16)$$

where: $n_o$ — refractive index in the lens centre,
$p$ — object distance.

Thus, formula (15) may be used to estimate the error for the rays angulary distant from the optical axis, while the formula (16) preserves its validity for the paraxial rays.

## An example and final conclusions

Let us consider an collimation objective 400/6 (fig. 4). The wave aberrations for this objective being axially-symmetrical and thus we can use the formula (13). Let us assume for the sake of simplicity that

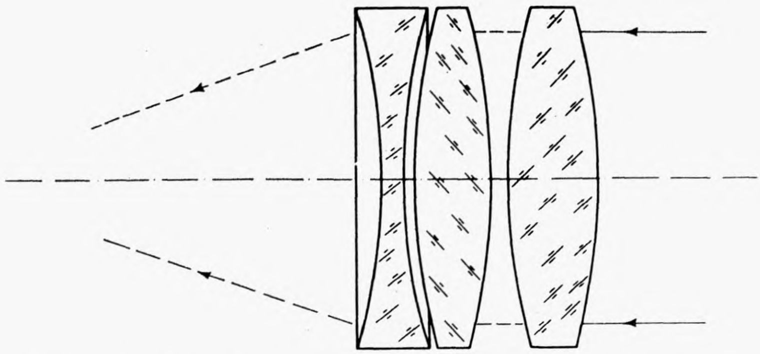

Fig. 4. A schematic picture of the collimator objective 400/6

$G_i^o = G^o$, i.e. that the gradient components in the plane perpendicular to the lens axis are the same in all the directions. Then the Strehl definition will be expressed by the formula:

$$I = I_\Delta - \frac{(kG^o)^2}{72}\Big[\sum_i a_i(D_{o_i} - D_{b_i})\Big]^2 .$$

The expression in square parenthesis in the above formula is calculated from the system design data. In our case it amounts to 3.8 cm². Therefore

$$\Delta I = I_\Delta - I = \frac{1}{72} k^2 G^{o2} 3.8 \text{ cm}^2 ,$$

where $\Delta I$ — drop in Strehl definition caused by the gradient action.

From the detailed calculations it follows that the error not exceed 4 % in this case.

The dependence of $\Delta I$ upon the value of $G^o$ for the wave-length $\lambda = 632.8$ nm has been shown in the table. From this table it follows that the values of gradient occurring in practice [9] may considerably worsen the imaging quality. Hence, it may be concluded that for high quality objectives the glass should be tested for refractive index gradient value. As the gradient component parallel to the optical

$\Delta I$ vs. the value of $G^o$ at $\lambda = 632.8$ nm for the collimator objective 400/6

| $G^o$ [$10^6$ cm$^{-1}$] | $\Delta I$ |
|---|---|
| 1 | 0.0005 |
| 2 | 0.0021 |
| 5 | 0.0130 |
| 10 | 0.0520 |
| 15 | 0.1171 |

axis does not affect practically the imaging quality the optical elements should be cut out of the glass block in such a way that the optical axis coincides with the gradient vector direction.

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## Допустимость градиента коэффициента преломления в оптических системах

В работе определено влияние градиента коэффициента преломления в элементах оптической системы на светлоту Стреля. Конечные формулы были получены при предположении, что лучи проходят параллельно оптической оси отдельных элементов системы. Эти предположения ограничивают пределы применяемости полученных результатов. Однако, для некоторого класса объективов, полученные формулы могут являться основой допустимости градиента.