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An analysis of the hologram aberration in the intermediate and far regions*

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The behaviour of two expansions into series are examined, i.e. that of the classical binomial expansion and the asymptotic expansion in the vicinity of $\varrho/z_q = 1$. It has been shown on an example of spherical aberration that the corresponding expressions for aberrations are too slowly convergent in this region. The dependences which bear the features of both the expansions and which allow to improve considerably the convergence have been proposed for both the intermediate and far regions. These dependences offer the advantage of easy physical interpretation: the aberrations of III order do exist in the region $\varrho/z_q > 1$ but the values of coefficients are correspondingly changed.

Introduction

In the paper [1] the application of the asymptotic expansion for the wavefront phases has been proposed in order to determine the hologram aberration outside the $\varrho/z_q \leqslant 1$ region**, which is valid for the MEIER approach [2]. In the example considered in [1] the attention has been restricted to several initial terms in the expansion which have been compared with the third-order spherical aberrations for the Gabor axial hologram. In the tables enclosed in [1] there were some vacancies near to $\varrho/z_o > 1$ region and drastic differences between spherical aberration increasing monotonously within the $\varrho/z_o < 1$ region and the aberration with $\varrho/z_o > 1$ region obtained from the asymptotic expansion.

Spherial aberrations

There exist no physical reason which would justify the above situation. The aberrations should be continuous functions within the whole variability region of ϱ. Therefore a special attention should be payed to the intermediate region where the both expansions:
binomial

$$\sqrt{1+\xi} \approx 1+\frac{1}{2}\xi-\frac{1}{8}\xi^2+\cdots \qquad |\xi| \leqslant 1, \tag{1}$$

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** Where: $\varrho = \sqrt{x^2+y^2}$ – coordinate in the hologram plane, z_q – distance of the source from the hologram plane.

and asymptotic

$$\sqrt{1+\xi} \approx \sqrt{\xi}\left(1+\frac{1}{2\xi}-\frac{1}{8\xi^2}+\frac{1}{16\xi^3}\cdots\right) \quad |\xi| \geqslant 1, \tag{2}$$

where $\xi = \varrho/z_q$, would be simultaneously taken into account.

Both these expansions are mutually contradictionary in a certain sense. The expansion (1) gives the more exact results the smaller are ξ (i.e. the closer to the axis is the point of interest the less number of terms are necessary to achieve the needed accuracy). In contrast to this, the expansion (2) allows to gain the requested accuracy for small number of terms if ξ is great enough. In the vicinity of $\xi \approx 1$ the convergence

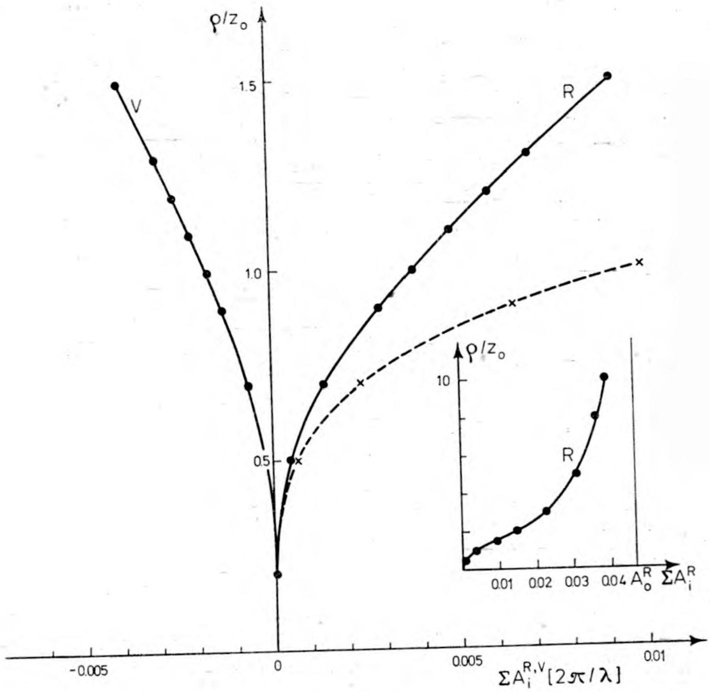

Fig. 1. Spherical aberration of wavefront Φ_R and Φ_V for $z_c = 1.2\,z_0$, $z_r = 1.1\,z_0$;
●——● total spherical aberration,
× – – × spherical aberration of third order of Φ_R

of both the expansions is very slow; whereby some oscillation about the sum value of the series appear with a slowly diminishing amplitude if the further terms are taken into account. These facts were the reason for a lack of continuity in the aberrations reported in [1]. The spherical aberration of the third order $-\frac{1}{8}\varrho^4 S$ describes less and less the real aberrations as the distance from the axis increases (see fig. 1).

To achieve a satisfying accuracy further terms must be taken into account, i.e.

$$A_n^{R,V} = a_n \varrho^{n+1}\left(\frac{1}{z_c^n} \mp \frac{1}{z_0^n} \pm \frac{1}{z_r^n} - \frac{1}{z_{G_{R,V}}^n}\right) \text{ for } \varrho \leqslant 1 \tag{3}$$

where $n \ldots 3, 5, 7 \ldots$

$$a_3 = -\frac{1}{8},\ a = -a_{n-2}\frac{n-2}{n+1} \text{ starting from } n = 5,$$

and

$$A_0^{R,V} = (z_c \mp z_o \pm z_r - z_{G_{R,V}}) \text{ for } \varrho \geqslant 1,$$
$$A_n^{R,V} = a_n \varrho^{1-n}(z_c^n \mp z_o^n \pm z_r^n - z_{G_{R,V}}^n), \tag{4}$$

where $n \ldots 2, 4, 6 \ldots,$

$$a_2 = \frac{1}{2},\ a_4 = -\frac{1}{8},\ a_n = -a_{n-2}\frac{n-3}{n} \text{ starting from } n = 6.$$

The upper signs refer to the index R while the lower ones to the index V.

Figures 1 and 2 together with the second column in the table illustrate the applicability of the formulae (3) and (4). The comparison of the exact

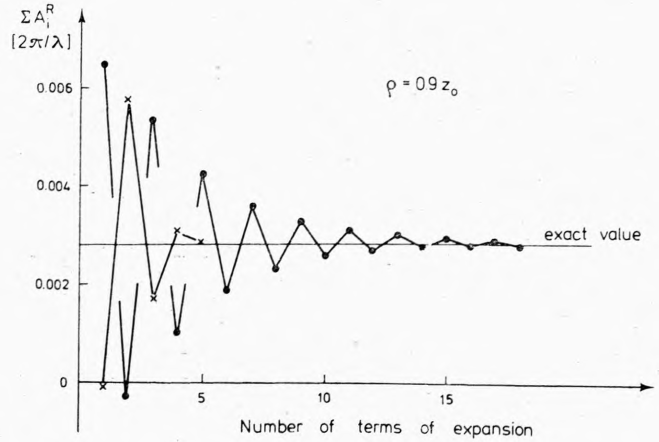

Fig. 2. Spherical aberration in the vicinity of $\varrho = 1$ — an illustration of the increase of convergence of relations (3) and (8)
●——● according to (3), ×——× according ot (8)

values obtained immediately from the differences in optical paths with the values obtained by summarizing the corresponding $A_n^{R,V}$ terms makes it clear that the convergence in the surrounding of $\varrho \approx 1$ is weak, which is illustrated by the number of terms of the expansion given in the parentheses beside in the table.

An improvement of convergence

Convergence may be improved as follows

$$\sqrt{1+\xi} = \sqrt{1+n+\xi-n} = \sqrt{1+n}\sqrt{1+\frac{\xi-n}{1+n}}$$

$$\approx \sqrt{1+n}\left(1+\frac{1}{2}\,\frac{\xi-n}{1+n}-\frac{1}{8}\left(\frac{\xi-n}{1+n}\right)^2+\frac{1}{16}\left(\frac{\xi-n}{1+n}\right)^3\ldots\right), \tag{5}$$

for $\left|\dfrac{\xi-n}{1+n}\right| \leqslant 1.$

The form of the relation (5) includes the features of both the asymptotic (2) and binomial (1) expansions. It is sufficient to notice that if $n+1 = \xi'$, then

$$\sqrt{1+\xi} = \sqrt{\xi'}\left(1+\frac{1}{2}\,\frac{(\xi-\xi')+1}{\xi'}-\frac{1}{8}\left(\frac{(\xi-\xi')+1}{\xi'}\right)^2\ldots\right).$$

By a suitable choice of n the convergence of the series may be considerably improved.

In our case

$$\xi = \left(\frac{\rho-\rho_q}{z_q}\right)^2, \text{ and } \left(\frac{\rho_q}{z_q}\right)^2.$$

The expression for the phase with respect to the hologram centre expanded into series according to (5), for instance, for $n = 1$, has the form

$$\varphi_q = \frac{2\pi}{\lambda} z_q \sqrt{2}\left\{\sqrt{1+\frac{\left(\frac{\rho-\rho_q}{z_q}\right)^2-1}{2}}-\sqrt{1+\frac{\left(\frac{\rho_q}{z_q}\right)^2-1}{2}}\right\}$$

$$= \frac{2\pi}{\lambda} z_q\sqrt{2}\left\{\left[1+\frac{1}{2}\,\frac{\left(\frac{\rho-\rho_q}{z_q}\right)^2-1}{2}-\frac{1}{8}\left(\frac{\left(\frac{\rho-\rho_q}{z_q}\right)^2-1}{2}\right)^2+\right.\right.$$

$$\left.+\frac{1}{16}\left(\frac{\left(\frac{\rho-\rho_q}{z_q}\right)^2-1}{2}\right)^3-\ldots\right]-\left[1+\frac{1}{2}\,\frac{\left(\frac{\rho_q}{z_q}\right)^2-1}{2}-\frac{1}{8}\left(\frac{\left(\frac{\rho_q}{z_q}\right)^2-1}{2}\right)^2+\right.$$

$$\left.\left.+\frac{1}{16}\left(\frac{\left(\frac{\rho_q}{z_q}\right)^2-1}{2}\right)^3-\ldots\right]\right\}, \tag{6}$$

or for small ρ_q

$$\varphi_q = \frac{2\pi}{\lambda} z_q \sqrt{2}\left(\sqrt{1+\frac{\left(\frac{\pi-\rho_q}{z_q}\right)^2-1}{2}}-\frac{1}{\sqrt{2}}\sqrt{1+\left(\frac{\rho_q}{z_q}\right)^2}\right)$$

$$= \frac{2\pi}{\lambda} z_q \sqrt{2}\left\{\left(1+\frac{1}{2}\frac{\left(\frac{\rho-\rho_q}{z_q}\right)^2-1}{2}-\frac{1}{8}\left(\frac{\left(\frac{\rho-\rho_q}{z_q}\right)^2-1}{2}\right)^2+\right.\right.$$

$$\left.+\frac{1}{16}\left(\frac{\left(\frac{\rho-\rho_q}{z_q}\right)^2-1}{2}\right)^3-\ldots\right)-$$

$$\left.-\frac{1}{\sqrt{2}}\left[1+\frac{1}{2}\left(\frac{\rho_q}{z_q}\right)^2-\frac{1}{8}\left(\frac{\rho_q}{z_q}\right)^4+\frac{1}{16}\left(\frac{\rho_q}{z_q}\right)^6-\ldots\right]\right\}. \tag{6a}$$

By expanding the binomials and grouping the similar terms we obtain

$$\varphi_q = \frac{2\pi}{\lambda}\sqrt{2}\left(\left(\frac{1}{4}+\frac{1}{16}+\frac{3}{128}+\ldots\right)\left(\frac{(\rho-\rho_q)^2}{z_q}-\frac{\rho_q^2}{z_p}\right)-\right.$$

$$-\left(\frac{1}{32}+\frac{3}{128}+\ldots\right)\left(\frac{(\rho-\pi_q)^4}{z_q^3}-\frac{\rho_q^4}{z_q^3}\right)+$$

$$\left.+\left(\frac{1}{128}+\ldots\right)\left(\frac{(\rho-\rho_q)^6}{z_q^3}-\frac{(\rho_q)^6}{z_q^5}\right)-\ldots\right), \tag{7}$$

and

$$\varphi_q = \frac{2\pi}{\lambda}\sqrt{2}\left(\left(1-\frac{1}{\sqrt{2}}-\frac{1}{4}-\frac{1}{32}-\frac{1}{128}-\ldots\right)z_q-\right.$$

$$-\left(\frac{\sqrt{2}-1}{4}\frac{\rho_q^2}{z_q}-\frac{2\sqrt{2}-1}{32}\frac{\rho_q^4}{z_q^3}+\frac{4\sqrt{2}}{128}\frac{\rho_q^6}{z_q^5}-\ldots\right)+$$

$$+\left(\frac{1}{4}+\frac{1}{16}+\frac{3}{128}+\ldots\right)\left(\frac{(\rho-\rho_q)^2}{z_q}-\frac{\rho_q^2}{z_q}\right)-$$

$$-\left(\frac{1}{32}+\frac{3}{128}+\ldots\right)\left(\frac{(\rho-\rho_q)^4}{z_q^3}-\frac{\rho_q^4}{z_q^3}\right)+$$

$$\left.+\left(\frac{1}{128}+\ldots\ldots\right)\left(\frac{(\rho-\rho_q)^6}{z_q^5}-\frac{\rho_q^6}{z_q^5}\right)-\ldots\right). \tag{7a}$$

By assuming the phases of the reconstructed wavefronts Φ_R and Φ_V calculated according to (7) and (7a), with the phase of a Gaussian reference sphere Φ_G we find that the expansion

$$\left(\frac{1}{4}+\frac{1}{16}+\frac{3}{128}+\ldots\right)\left(\frac{\rho^2-2\rho\rho_c}{z_c}\mp\frac{\rho^2-2\rho\rho_o}{z_o}\pm\right.$$
$$\left.\pm\frac{\rho^2-2\rho\rho_r}{z_r}-\frac{\rho^2-2\rho\rho_G}{z_G}\right.$$

is responsible for the Gaussian imaging, while the corresponding sums of terms $\rho-\rho_p$ in the fourth power with the coefficient $-\left(\frac{1}{32}+\frac{3}{128}+\ldots\right)$ are the sums of the thirds order aberrations. It may be said that the coefficients of third order aberrations increase $\sqrt{2}\,(1/32+3/128+\ldots)$ times in the vicinity of $\varrho\approx 1$. Besides, the relations (7) and (7a) differ from each other by the linear term depending on z_q and by a constant term depending on ϱ_q and z_q. The sequence of numbers containing the coefficient at z_q tends to zero, when the number of terms of expansion increases to ∞. If, however, we are forced to take account of several terms of the series it is reasonable to preserve this linear term too.

For the spherical aberration considered in the example, taking account of five terms of expansion, we obtain

$$\begin{aligned}\sum A_i^{R,V} = \frac{2\pi}{\lambda}\sqrt{2}\{&0.0005350\,(z_c\mp z_o\pm z_r-z_{G_{R,V}})\\ &-0.077\,880\,8\,S_3^{R,V}\varrho^4+0.026\,122\,0\,S_5^{R,V}\varrho^6\\ &-0.006\,713\,8\,S_7^{R,V}\varrho^8\\ &+0.000\,854\,4\,S_9^{R,V}\varrho^{10}\ldots\},\end{aligned} \tag{8}$$

where

$$S_n^{R,V}=\frac{1}{z_c^n}\mp\frac{1}{z_o^n}\pm\frac{1}{z_r^n}-\frac{1}{z^n_{G_{R,V}}}\,.$$

If we take account of another number of terms, also less than 5, we must calculate new numerical coefficient in (8). The calculation of the example for $z_c=1.2\,z_o$, $z_r=1.1\,z_o$ according to (8) is presented in table. It may be seen that the procedure based on the relation (5) gives the satisfactory results (see figs. 2 and 3) at the vicinity of $\varrho=1$. It is also seen, that e.g. for $\varrho=1.4$ another (new) value of n should be chosen to achieve an equally quick convergence.

Spherical aberration of the wavefront Φ_R for $z_c = 1.2\, z_o$, $z_r = 1.1\, z_o$

ϱ/z_o	Exact value							Absolute error of the results obtained from (8)
0.1	0.000 0011	0.000 000 99	(1)	values				
0.2	0.000 0150	0.000 015 0	(2)	obtained				
0.5	0.000 4617	0.000 461 4	(6)	on the base				
0.7	0.001 3944	0.001 394 4	(13)	of binomial				
0.9	0.002 8916	0.002 925 2	(17)	expansion	0.0028745	(5)		0.06 %
		0.002 863 8	(18)	(3)				
1	0.003 8112	0.004 979 3	(23)		0.0038057	(5)		0.15 %
		0.002 710 3	(24)					
		intermediate						
1.1	0.004 8136	region		binomial and	0.0048129	(5)		0.015 %
1.2	0.0058 749			asymptotic	0.0058726	(5)	values	0.04 %
1.3	0.0069 736			expansions		(5)	obtained	0.05 %
1.35	0.0075 308	0.019 901 8	(10)	are valid	0.0069379	(5)	on the base	
		0.012 499 8	(11)	simulateneously			of formula (8)	
1.4	0.0080 906	0.008 243 4	(23)		0.0078586	(5)		2.8 %
		0.008 012 9	(24)					
1.6	0.0103 221	0.010 064 9	(10)	values				
		0.010 484 8	(11)	obtained				
1.8	0.0124 813	0.012 451 6	(10)	on the base				
		0.012 496 3	(11)	of asymptotic				
2	0.0145 198	0.014 515 6	(10)	expansion				
		0.014 521 6	(11)	(4)				
3	0.0225 676	0.022 567 8	(7)					
10	0.0388 2	0.038 821 0	(3)					
$\to \infty$		$A_0^R = 0.046\,939$						

Note: in the parentheses located beside the aberration values expressed in $2\pi/\lambda$ units the numbers of term in the respective expansions are given.

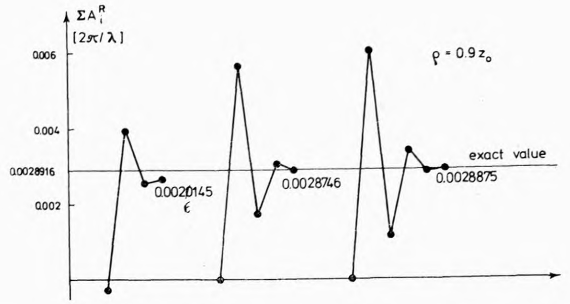

Fig. 3. Spherical aberration of the Φ_R for $z_c = 1.2\,z_o$ and $z_r = 1.1\,z_o$ computed from the relation (8) by taking into account of the four, five and six terms of the expansion

Conclusions

A quicker convergence of the asymptotic series may be achieved by applying a procedure analogical to that employed in (5)

$$\sqrt{1+\xi} = \sqrt{1-n+\xi+n} = \sqrt{\xi+n}\sqrt{1+\frac{n-1}{\xi+n}} \approx \tag{9}$$

$$\approx \sqrt{\xi+n}\left(1+\frac{1}{2}\frac{1-n}{\xi+n}-\frac{1}{8}\left(\frac{1-n}{\xi+n}\right)^2+\frac{1}{16}\left(\frac{n-1}{\xi+n}\right)^3-\ldots\right).$$

However, it does not seem to be convenient, since in the general case, when all the z_q lay outside the axis, the form of aberration sum is disadvantageous. This is because the distinction of these parts of the expansion which are responsible for the Gaussian imaging, spherical aberration, coma and so on, respectively, becomes difficult. Each of these parts is included in each of the asymptotic expansion terms. Due to the fact, that for great magnification (great z_c) the zone, where all the four phases φ_c, φ_0, φ_r, and Φ_G are simultaneously developable into asymptotic series is the far zone of the hologram, while the intermediate zone is very wide. Therefore, it may be sufficient to confine the attention to the formulae (7) and (7a) which are convenient in numerical calculation for properly choosen n and have a simple physical meaning. The purely asymptotic expansion will be, then, suitable for determination of spherical aberrations and shows how for very large ϱ the spherical aberration takes the constant value (fig. 1) determined by the expression $A_0^{R,v}$ from the relation (4).

References

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Анализ аберрации голограммы в промежуточной и далекой областях

Исследовано поведение обоих разложений в ряд классического биномиального а также асимптотического в кольцевой окрестности ϱ/z_q, близкой к 1. На примере сферической аберрации было выявлено, что соответствующие аберрационные выражения в этом диапазоне являются слишком медленно сходящимися. Для промежуточной и далёкой областей разработаны зависимости, носящие свойства обоих разложений, позволяющие значительно улучшить сходимости. Кроме того, эти зависимости имеют то преимущество, что они характеризуются лёгкой физической интерпретацией; аберрации III-его порядка продолжаются в области $\varrho/z_q > 1$ с изменёнными значениями коэффициентов.