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Determination of both the state and the degree of polarization for a mixture of nonuniform, incoherent light beams with the aid of Poincaré sphere

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It is assumed that the total light beam is composed of two (or more) mutually overlapping incoherent beams of light which are partially polarized being all in different states of polarization. The resultant state and degree of polarization for the sum of these beams may be determined with the aid of Poincaré sphere. In this paper, the "mixing" rules for polarization states have been formulated for the case of incoherent beams.

1. Introduction

In the paper, the Poincaré sphere is used in a slightly modified form as compared to the version introduced by SHURCLIFF [1], for instance, and the Stokes' vector in the form used, for instance, in paper [2]. The modification of the Poincaré sphere consists in the fact that the polarization state of light is determined as usual by the polar coordinates of the prolonged sphere radius but the intensity of the polarized part of the light need not be necessarily equal to the sphere radius. On the prolonged radius of the Poincaré sphere determining the state of polarization of the light beam two points are marked: one of them is marked at a distance from the sphere centre proportional to the intensity of the polarized part of the beam and another one – at a distance proportional to the total intensity.

The aim of this paper is to find a simple "vectorial" rule of determining the state of polarization and degree of polarization for a mixture of mutually incoherent light beams. For the sake of simplicity, we have restricted our considerations to the case of two beams, while the generalization of the method over a greater number of beams is rather obvious.

2. Theory

Suppose there are two overlapping and mutually incoherent light beams characterized by intensities I_1 and I_2, degrees of polarization p_1 and p_2 being in polarization states described by ellipticities ϑ_1 and ϑ_2 and azimuths α_1 and α_2, respectively. On the Poincaré sphere (see Fig. 1) a radius is drawn lying in the meridional plane of geographic longitude $2\alpha_1$ and creating with the meridional plane the angle $2\vartheta_1$. This corresponds to the Stokes' vector $[S_1]$ of the beam indexed by 1. Along the direction of this radius two points are distinguished: one marked by a circle with a cross inside denoting the distance I_1 from the sphere centre and the other marked by a filled in circle denoting the distance p_1I_1 from the centre. The intersection point of the radius and the Poincaré sphere is denoted by an empty circle. Also for the beam $[S_2]$ and for a hypothetical beam of resultant polarization state $[S]$ the due radii are drawn from the sphere centre. The angles occurring between these radii are: $[S_1][S_2] = c$, $[S][S_1] = c_1$ and $[S][S_2] = c_2$.

Let us define the Stokes' vector as in paper [2]:

$$[S] = \begin{bmatrix} I \\ pI\ \cos 2\vartheta \cos 2\alpha \\ pI\ \cos 2\vartheta \sin 2\alpha \\ pI\ \sin 2\vartheta \end{bmatrix} = \begin{bmatrix} I \\ pIM \\ pIC \\ pIS \end{bmatrix}. \tag{1}$$

Let us recall that $M^2+C^2+S^2 = 1$.

For the mutually incoherent beams the following equality holds:

$$\begin{bmatrix} I_1 \\ p_1I_1M_1 \\ p_1I_1C_1 \\ p_1I_1S_1 \end{bmatrix} + \begin{bmatrix} I_2 \\ p_2I_2M_2 \\ p_2I_2C_2 \\ p_2I_2S_2 \end{bmatrix} = \begin{bmatrix} I_1+I_2 \\ p_1I_1M_1+p_2I_2M_2 \\ p_1I_1C_1+p_2I_2C_2 \\ p_1I_1S_1+p_2I_2S_2 \end{bmatrix} = \begin{bmatrix} I \\ pIM \\ pIC \\ pIS \end{bmatrix}. \tag{2}$$

Let us recall again that $M_1M_2+C_1C_2+S_1S_2 = \cos c$, for $0 \leqslant c \leqslant 180$, where c is an angle between the radii of the Poincaré sphere corresponding to the states of polarization $[S_1]$ and $[S_2]$, respectively.

Thus, the intensity of the polarized part of the resultant beam is equal to

$$I_p = pI = \sqrt{(p_1I_1M_1+p_2I_2M_2)^2+(p_1I_1C_1+p_2I_2C_2)^2+(p_1I_1S_1+p_2I_2S_2)^2},$$

which after some simple rearrangement leads to the following result:

$$I_p = pI = \sqrt{p_1^2I_1^2+p_2^2I_2^2+2p_1p_2I_1I_2\cos c}. \tag{3}$$

The polarization degree of the nonuniform beam amounts to

$$p = \frac{I_p}{I_1+I_2} = \sqrt{\frac{p_1^2I_1^2+p_2^2I_2^2+2p_1p_2I_1I_2\cos c}{(I_1+I_2)^2}}, \tag{4}$$

while the total intensity is $I = I_1+I_2$.

Let us return to Figure 1 and make a cross-section of the Poincaré sphere (Fig. 2) by the plane containing the radii corresponding to $[S_1]$ and $[S_2]$. If the segments $0p_1I_1$ and $0p_2I_2$ of the sphere radii being considered as vectors are added in a vectorial way the resultant vector $0pI$ is obtained the length of which is

$$pI = \sqrt{p_1^2I_1^2+p_2^2I_2^2+2p_1p_2I_1I_2\cos c}. \tag{5}$$

Formulae (3) and (5) are identical which means that the applied procedure of vectorial addition of the intensities of the polarized parts of the beams of polarization

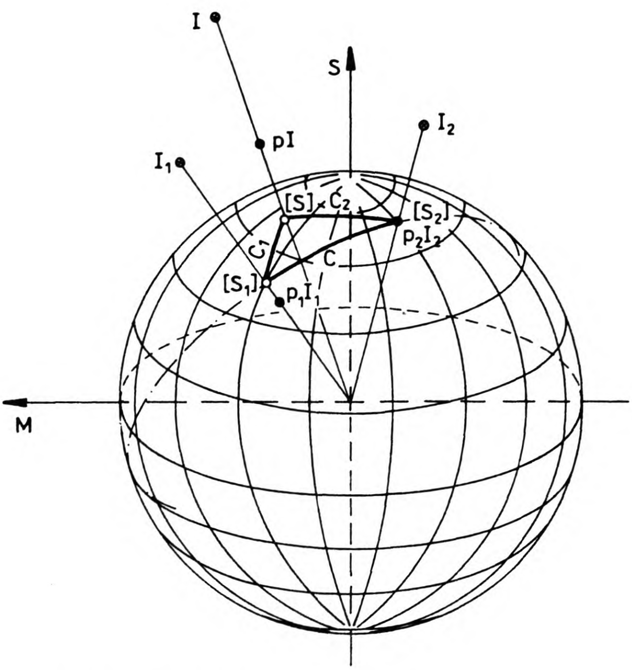

Fig. 1. On the Poincaré sphere three prolonged radii are presented: the first one passing through the polarization state of certain light beam $[S_1]$, the second one through the polarization state $[S_2]$ of another beam. Both beams are mutually incoherent. After mixing them together the third light beam of hypothetical polarization state $[S]$ is appeared. Point ⊗ denotes the beam intensity, • the intensity of the polarized part of the beam, o polarization state of the polarized part of the beam

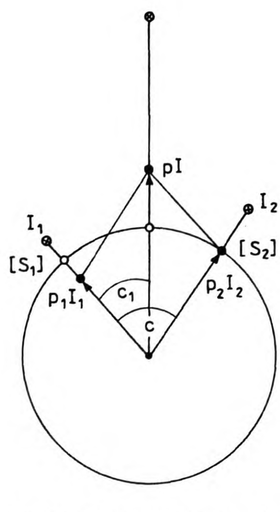

Fig. 2. Cross-section on the Poincaré sphere by a plane passing through two prolonged radii $[S_1]$ and $[S_2]$. The intensity of the polarization state of the polarized part of the beam $[S]$ composed of $[S_1]$ and $[S_2]$ is defined by the resultant vector of the intensities of the polarized parts. The intensity of beam $[S]$ is an algebraic sum of intensities of the beams $[S_1]$ and $[S_2]$

states $[S_1]$ and $[S_2]$ is justified. From elementary calculations, it follows that the resultant vector $[S]$ creates with the vector $[S_1]$ an angle

$$c_1 = \arctan\left(\frac{p_2 I_2 \sin c}{p_1 I_1 + p_2 I_2 \cos c}\right). \tag{6}$$

3. Conclusions

From the above discussion, the following conclusions may be formulated:

1. The Stokes' vector $[S]$ of the beam composed of two mutually incoherent beams described by the Stokes' vectors $[S_1]$ and $[S_2]$ lies between the latter on the Poincaré sphere, *i.e.*, on the great circle joining them, at the angular distance c_1 (formula (6)) from the point $[S_1]$.

2. The intensity of the polarized light of the beam composed of two (or more) beams is given as a modulus of the vectorial sum of the intensities of the polarized parts of the component beams.

3. The resultant intensity of the composed beam is an algebraic sum of the total intensities of the component beams.

4. The polarization degree of a composed beam (formula (4)) diminishes with the increase of the angle c from the initial value $p = \frac{p_1 I_1 + p_2 I_2}{I_1 + I_2}$ for $c = 0°$ to the value $p = \frac{p_1 I_1 - p_2 I_2}{I_1 + I_2}$ for $c = 180°$. In particular, a beam composed of two beams completely polarized ($p_1 = p_2 = 1$) but being in different polarization states is always partially polarized. The beam composed of two orthogonally polarized beams of equal intensities and equal polarization degrees is not polarized at all.

It is surprising how much the laws of "mixing" the polarization states of the incoherent beams are similar to the laws of colour mixing.

Point 1 of the above remarks corresponds to the statement that the colour created by mixing two other colours lies in the chromaticity diagram on the straight line joining the two points representing the components.

Point 4 corresponds to the statement that the colour created by mixing two saturated colours is not saturated while the white light appears if two complementary colours are mixed.

When analysing the spectrum the white light being the mixture of all wavelengths can be distinguished from the white light obtained by mixing the complementary colours. Till now, we have not managed to find the way of distinguishing the natural light being the mixture of all polarization states from the "natural" light (it may be more proper to say non-polarized light) created by mixing two beams of orthogonal states of polarization.

References

[1] SHURCLIFF W. A., *Polarized Light Production and Use*, Harvard University Press, Cambridge, Massachusetts 1962.

[2] RATAJCZYK F., WOŹNIAK W. A., KURZYNOWSKI P., Optik (in press).

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