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The *ABCD*-matrix for holographic gratings

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We derive the *ABCD*-matrix for the diffraction by a holographic grating. We show that this matrix can be decomposed into a part describing pure diffraction by a plane grating and a part describing pure reflection by an effective concave mirror. The possibilities of production of the gratings by interference of deformed wavefronts are included.

The theory of holographic concave gratings was developed by Cordelle et al. [1], Namioka and co-workers [2], Velzel [3], and others. The *ABCD*-matrix is described in [4]. The theory of using the *ABCD*-matrix for transformation of Gaussian beams was given by Kogelnik [5]. Examples of application of Gaussian beams are given in [6, 7, 9, 13].

For the incorporation of concave gratings in optical systems a first paraxial calculation is appropriate. In this connection the formulation by means of the *ABCD*-matrix is an useful tool. For plane uncorrected gratings the *ABCD*-matrix was provided by Kanstad and Wang [8]. In this paper the beam transformation matrix is derived for an in-plane recorded and in-plane used concave grating.

It has been suggested earlier ([12]), that for production of gratings, instead of spherical waves the interference of deformed wavefronts should be used. In this paper we derive the ABCD-matrices for the case of spherical waves, and the simple changes occurring in the matrices due to deformed wavefronts are explained.

The symbols used for recording and reconstruction of the gratings correspond to those in [1]. The definitions are given for symbols in fig. 1. The record of the grating is made on the grating surface G by interference of the coherent light coming from the two point sources C and D situated at the distances l_C and l_D from 0. The angles of $C0$ and $D0$ with the X-axis are γ and δ. These angles are greater than zero if they point to positive Y-axis direction. The use or reconstruction of the grating is made by a point source at A (distance l_A, angle α) and the image appears in B (distance l_B, angle β). The points A, B, C, and D are positioned in the X–Y plane. The surface G may be a sphere or a paraboloid, because we need only the equation of this surface up to the second order in Y and Z coordinates, e.g.

$$X = (Y^2+Z^2)/2R \tag{1}$$

with R being the radius of curvature of the surface in 0.

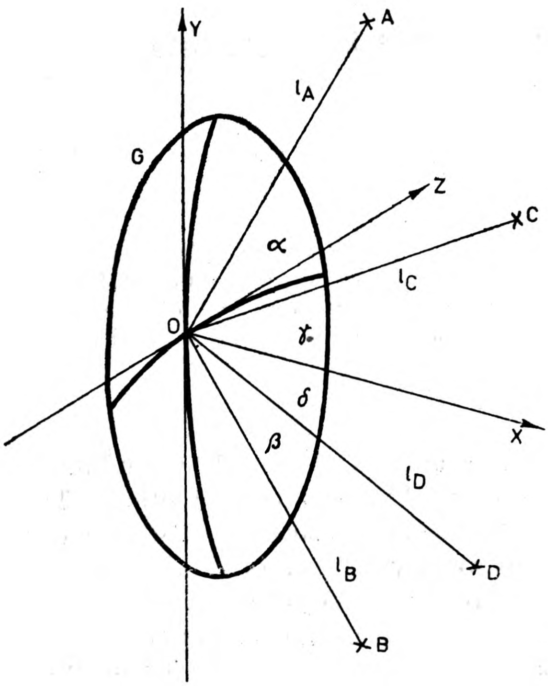

Fig. 1. Record and reconstruction of a holographic grating

Now, we consider the tangential focussing (see [1]), i.e. the focussing in the X–Y plane obtained by generalization of the method applied in [8]. Figure 2 shows the X–Y plane.

The central ray of the Gaussian beam arrives at 0 at the angle a and the spot size is w. The diffracted beam has the spot size w'. Now, the following question arises: Supposing that the incoming beam has a convergence (or divergence) Δa at its edge, how this is transformed into the convergence $\Delta\beta$ of the outgoing beam?

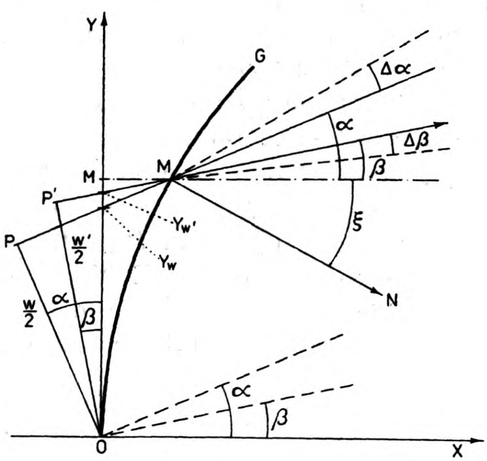

Fig. 2. Geometry of beam transformation

First we approximate $Y_M \approx Y_w \approx Y_{w'}$ in fig. 2, which means that $w/R \ll 1$, because in the contrary case the symmetry of the beam with respect to the positive and negative parts of the Y-axis is lost:

$$Y_M = \frac{w}{2}\left(\frac{1}{\cos\alpha} + \frac{w}{4R}\frac{\tan\alpha}{\cos^2\alpha} + \ldots\right) \approx \frac{w}{2\cos\alpha} = Y_w. \tag{2}$$

This is shown by a short calculation marked by $0, P, Y_M, Y_w$ and by M in the fig. 2.

The transformation of $\Delta\alpha$ into $\Delta\beta$ is determined by the diffraction equation at the point M related to the surface

$$\sin(\alpha+\Delta\alpha+\xi)+\sin(\beta-\Delta\beta+\xi) = \frac{k\lambda}{g(Y_M)}, \tag{3}$$

where λ is the wavelength under investigation, k is the order of the spectrum and $g(Y_w)$ is the local grating constant at M, ξ is simply derived by differentiating (1):

$$\tan\xi \approx \xi = \frac{Y_M}{R} = \frac{w}{2R\cos\alpha} = \frac{w'}{2R\cos\beta}, \tag{4}$$

$\Delta\alpha$, $\Delta\beta$ and ξ are assumed to be small. We use the addition theorem and expand the sines to get

$$\sin\alpha+(\Delta\alpha+\xi)\cos\alpha+\sin\beta+(\xi-\Delta\beta)\cos = \frac{k\lambda}{g(Y_M)} \tag{5}$$

$\sin\alpha$ and $\sin\beta$ are connected by the grating constant at the origin 0. By using (4) we obtain

$$\Delta\beta = \frac{\cos\alpha}{\cos\beta}\Delta\alpha + \frac{(\cos\alpha+\cos\beta)w}{2R\cos\alpha\cos\beta} + \frac{k\lambda}{\cos\beta}\left[\frac{1}{g(0)} - \frac{1}{g(Y_M)}\right]. \tag{6}$$

The calculation of $k\lambda\left[\frac{1}{g(0)} - \frac{1}{g(Y_M)}\right]$ is simplified by noting in [1] that in the well known expansion of the optical path function

$$\Delta = \overline{MA}+\overline{MB}-\frac{k\lambda}{\lambda_0}(\overline{MC}-\overline{MD}) \tag{7}$$

the term $-\frac{1}{\lambda_0}(\overline{MC}-\overline{MD})$ enumerates the grooves depending on M(λ_0 is the wavelength of production of the grating). The grooves per length unit are

$$\frac{1}{g(M)} = \frac{1}{g(Y_M)} = -\frac{d}{ds}\left(\frac{\overline{MC}-\overline{MD}}{\lambda_0}\right)$$
$$= -\frac{d}{dY_M}\left(\frac{\overline{MC}-\overline{MD}}{\lambda_0}\right)\frac{dY_M}{ds}, \tag{8}$$

where ds describes the line element along the curve given by G in the X–Y plane. It holds approximately

$$S \approx Y_M + \frac{1}{6}\frac{Y_M^3}{R^2} \approx Y_M \quad (Y_M/R \ll 1) \tag{9}$$

or $dY_M/ds = 1$. In this manner we take from [1] the expression

$$\frac{d}{dY_M}(\overline{MC} - \overline{MD}) = -\sin\gamma + \sin\delta + \left(\frac{\cos^2\gamma}{l_C} - \frac{\cos\gamma}{R} - \frac{\cos^2\delta}{l_D} + \frac{\cos\delta}{R}\right) Y_M \tag{10}$$

up to the second order. The term $\sin\gamma - \sin\delta$ is also connected with $1/g(0)$. Finally, by using (2) we obtain

$$\Delta\beta = \frac{\cos\alpha}{\cos\beta}\,\Delta\alpha + \frac{\cos\beta}{\cos\alpha}\,\frac{1}{R_2}\,w \tag{11}$$

with the effective radius for the tangential focussing R_2

$$R_2 = \frac{2R\cos^2\beta}{\cos\alpha + \cos\beta}\left\{1 + \frac{k\lambda}{\lambda_0}\,\frac{R}{\cos\alpha + \cos\beta} \times \left(\frac{\cos^2\gamma}{l_C} - \frac{\cos\gamma}{R}\,\frac{\cos^2\delta}{l_D} + \frac{\cos\delta}{R}\right)\right\}^{-1}. \tag{12}$$

Taking account of (4) we obtain

$$\frac{w'}{2} = \frac{\cos\beta}{\cos\alpha}\,\frac{w}{2}, \tag{13}$$

and the $ABCD$-matrix is

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \dfrac{\cos\beta}{\cos\alpha} & 0 \\ \dfrac{\cos\beta}{\cos\alpha}\,\dfrac{2}{R_2} & \dfrac{\cos\alpha}{\cos\beta} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \dfrac{2}{R_2} & 1 \end{pmatrix} \begin{pmatrix} \dfrac{\cos\beta}{\cos\alpha} & 0 \\ 0 & \dfrac{\cos\alpha}{\cos\beta} \end{pmatrix}. \tag{14}$$

The symbols A, B, C and D should not be confused with the point A, B, C and D in fig. 1. The eq. (14) shows that the $ABCD$-matrix of the transformation by a grating can be constructed by applying first the transformation to a plane grating (see [8]) and then the transformation to a concave mirror with an effective radius R_2.

In the special case of autocollimation ($\alpha = \beta$) R_2 can be calculated by the following simple argument: The tangential focal distance l_{B2} is obtained by equating to zero the coefficient of Y_M^2 in the expansion of (7):

$$0 = -\left(\frac{\cos^2\beta}{l_{B2}} + \frac{\cos^2\alpha}{l_A}\right) + \frac{\cos\alpha + \cos\beta}{R} + \frac{k\lambda}{\lambda_0}\left(\frac{\cos^2\gamma}{l_C} - \frac{\cos\gamma}{R} - \frac{\cos^2\delta}{l_D} + \frac{\cos\delta}{R}\right). \tag{15}$$

If $\alpha = \beta$, we remark in eq. (15) the combination $1/l_{B2}+1/l_A$ (which is in correspondence with the mirror imaging equation) equal to $2/R_2$. So defined R_2 is consistent with that determined in (12).

The paraxial ray transformation for all possible α and β in the radial case (focussing in Z-direction) can be calculated by the latter argument. The coefficient of Z_M^2 in (z), according to [1], is

$$0 = -\left(\frac{1}{l_A} + \frac{1}{l_{B1}}\right) + \frac{\cos\alpha + \cos\beta}{R} + \frac{k\lambda}{\lambda_0}\left(\frac{1}{l_C} - \frac{\cos\gamma}{R} - \frac{1}{l_D} + \frac{\cos\delta}{R}\right). \tag{16}$$

With $1/l_A+1/l_{B1} = 2/R_1$ we obtain the corresponding matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2/R_1 & 1 \end{pmatrix} \tag{17}$$

with

$$R_1 = \frac{2R}{\cos\alpha + \cos\beta}\left\{1 + \frac{k\lambda R}{\lambda_0(\cos\alpha + \cos\beta)}\left(\frac{1}{l_C} - \frac{\cos\gamma}{R} - \frac{1}{l_D} + \frac{\cos\delta}{R}\right)\right\}^{-1}. \tag{18}$$

The beam transformations resulting from (14) and (17) correspond to a toroidal mirror, which changes its focal distances from one wavelength to another. However, among the holographic concave gratings there are also well corrected examples, where the both focal distances do not essentially change. Then the mode conversion losses due to mismatch of the focal distance [10] are small.

In our calculations the terms Y^3 in the light path function are neglected. (This means Y^2 in the grating equation.) The beam waist and curvature radius of a diffracted Gaussian beam can be calculated up to the terms of order Y^2. This we obtain also by a wave-optical argument similar to that used in [11].

Now, we consider the possibility of deformed wavefronts. Such deformations can be realized by optical surfaces between the light sources C or D and the grating, for example by cylindrical or toroidal lenses or mirrors. The interference of such deformed waves with plane waves yields elliptical deformed Fresnel zones, as, for instance, used in optical processing of radar signals [15]. The deformation of the spherical waves from C is explained on fig. 3. In a coordinate system with $\tilde{X}$ along the distance l_C (see fig. 1), $\tilde{Y}$, respectively, rotated and $\tilde{Z}$ parallel to Z, the usual expansion of $\overline{CM}$, with $(\tilde{X}, \tilde{Y}, \tilde{Z}) = M$ near to 0, yields (without deforming elements) the rotational symmetric expansion

$$\overline{CM} = \overline{C0} - \tilde{X} + \mu(\tilde{Y}^2 + \tilde{Z}^2) + \ldots \tag{19}$$

which (after the appropriate rotation $\tilde{X}, \tilde{Y}, \tilde{Z} \rightarrow X, Y, Z$) gives the usual contribution to $\Delta = \overline{AM} + \overline{BM} - k\lambda(\overline{CM} - \overline{DM})/\lambda_0$. Therefore, at $\tilde{Y}^2$ and $\tilde{Z}^2$ equal factors in (19) imply the same distance l_C in (12) and (18).

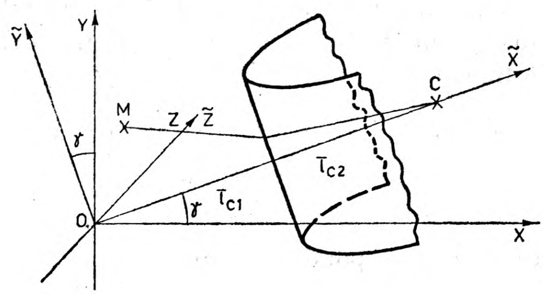

Fig. 3. Deformation by a parabolic cylinder

If the factors at $\tilde{Y}^2$ and $\tilde{Z}^2$ are unequally, then the effective l_C in (12) is unequally to l_C in (18). We given a short example shown already in fig. 3:

We divide the distance $\bar{l}_C$ into $\bar{l}_{C1}$ and $\bar{l}_{C2}$ $(l_C = \bar{l}_{C1} + \bar{l}_{C2})$. At the distance $\bar{l}_{C1}$ from 0 we locate a simple optical surface, say, a parabolic cylinder, whose symmetry-axis in the X–Y plane is perpendicular to $\tilde{X}$. The cylinder is filled with a medium of the refractive index n, and described by the equation

$$\tilde{X} = l_{C1} + F_{02}\tilde{Z}^2, \tag{20}$$

where F_{02} is one of the general surface expansion coefficients used in more extented calculations up to the third order. Generally, we obtain $\overline{CM}$ by variational elimination of the coordinates of the cylinder surface via Fermats principle (comparable with [14]). The result until the second order in $\tilde{X}$, $\tilde{Y}$ and $\tilde{Z}$ can also be very simply derived from the usual imaging equation for a spherical surface

$$\begin{aligned}\overline{CM} \approx\ & \bar{l}_{C1} + \bar{l}_{C2} - \tilde{X} + \frac{n}{2(\bar{l}_{C2} + n\bar{l}_{C1})}\tilde{Y}^2 \\ & + \frac{1}{2\bar{l}_{C1}}\frac{2(1-n)F_{02} + n/\bar{l}_{C2}}{2(1-n)F_{02} + 1/\bar{l}_{C1} + n/\bar{l}_{C2}}\tilde{Z}^2.\end{aligned} \tag{21}$$

After the rotation $\tilde{X}, \tilde{Y}, \tilde{Z} \rightarrow X, Y, Z$ we obtain $\overline{CM}$ expressed by X, Y, Z, taking account of (1). Now, Δ is available in (7). If we denote the factor of $\tilde{Y}^2$ in (21) by $1/2l_{C2}$ we find that the matrix (14) and eq. (12) are given by the substitution $l_C \to l_{C2}$. The denotation of the factor at $\tilde{Z}^2$ in (21) by $1/2l_{C1}$ dues to the matrix (17) and the formula (18) if the substitution $l_C \to l_{C1}$ is performed.

If we extent the argument also to $\overline{DM}$ and to toroidal gratings ($R \to \tilde{R}_1$ resp. $\tilde{R}_2$ in the two matrices with $\tilde{R}_1$ and $\tilde{R}_2$ the main curvature radii of the grating) we see that deformation in second order yields two independent gratings for the saggital case and for the meridional case. If we like a combination of a grating with suitable saggital properties with another grating with suitable meridional properties we can combine this properties by appropriate deformations of wavefronts or by a toroidal grating surface.

The main application of the given matrices we see in the inclusion of gratings in lens- and mirror-systems when attempting the of first simple optimizations of polychromators, monochromators, resonators [16] and other dispersing devices.

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Матрицы *ABCD* для голографических сеток

Выведены матрицы *ABCD* для дифракции на голографических сетках. Показано, что эти матрицы можно разделить на часть, описывающую чистую дифракцию на плоской сетке, а также на часть, описывающую чистое отражение эффективным вогнутым зеркалом. Указано также на возможность выполнения сеток при использовании интерференции деформированных фронтов волны.