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Object field transformation by a mirror with a focussing error*

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In the paper the light field transformation in the paraboloidal mirror is considered. The influence of the focussing error on the image creation is discussed. The dependence of the field distribution in the image upon the field distribution in the object as well as the dependence of the image field upon the Fourier-transform of the generalized pupil function have been shown. The application of the paraboloidal concave mirror to produce the image holograms has been suggested.

1. Introduction

In an arbitrary optical system the effect of light beam transformation is observed when passing from the object to image spaces [1]. Such transformation of the light beams occurs always in the imaging process or 'n the hologram recording consisting essentially in recording of the respectively transformed light fields [2] — to name the most typical cases.

In the present paper the transformation of the object field in the image light field by a mirror system is discussed and the influence of the focussing error upon the imaging process examined. The transformed field of this kind may be exploited for producing the image holograms characterized by the fact that the object informations are recorded locally. This means that the wave emitted by the given point is transformed with the help of an optical system (which was paraboloidal concave mirror in the case considered) and recorded at the point or in a small region of the recording plane. Due to this the reconstructed images appear in the plane of hologram or in its vicinity.

The assumption of the relatively small spread of reconstructed images, allows to reconstruct the wavefront with the help of an extended source of the white light. Under this condition the image confussion spot is proportional to the distance between the reconstructed images and the hologram [3].

The figure 1 presents a simple mirror system for making the image holograms with the help of a plane reference wave. If the distance from the object to the mirror vertex is $z_1 = 2f$ (where f is the mirror focal length) then the distance of the image plane from the mirror vertex amount also to $z_2 = 2f$, and the transversal magnification in the image is $M = 1$.

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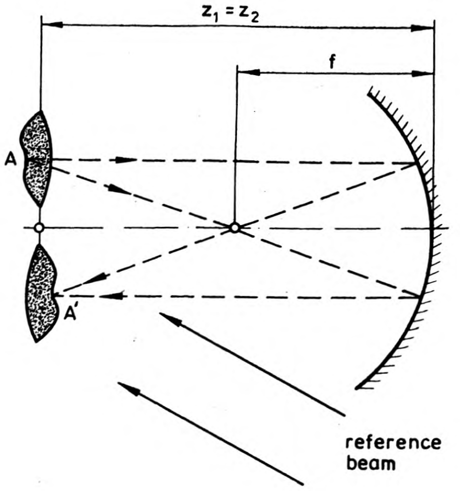

Fig. 1. Creation of the image hologram with the help of a concave spherical mirror

2. Complex amplitude distribution in the image plane

Let us consider a plane object of amplitude transmittance $U_1(x_1, y_1)$ located at the distance z_1 from the vertex of concave paraboloidal mirror. Let the coherent plane wave fall upon the object in the direction perpendicular to the axis of the optical system. After passing the object the perturbed light distribution [2] is described by the Fresnel diffraction integral

$$U(x, y) = \frac{\exp\left[i\frac{k}{2z_1}(x^2+y^2)\right]}{i\lambda z_1}\int_{-\infty}^{+\infty}\int U_1(x_1, y_1)$$

$$\times \exp\left[i\frac{k}{2z_1}(x_1^2+y_1^2)\right]\exp\left[-i\frac{k}{z_1}(xx_1+yy_1)\right]dx_1dy_1. \tag{1}$$

In the above expression the constant phase factor has been neglected, $k = 2\pi/\lambda$ is the wave number.

Due to the light reflection from the mirror surface the field is transformed in such a way that the complex amplitude distribution takes the form

$$U'(x, y) = U(x, y)P(x, y)\exp\left[-i\frac{k}{2f}(x^2+y^2)\right]. \tag{2}$$

The pupil function in the exit pupil plane has the cylindrical form

$$P(x, y) = \mathrm{cyl}\left(\frac{\sqrt{x^2+y^2}}{D}\right) \tag{3a}$$

and takes the following values

$$\operatorname{cyl}\left(\frac{\sqrt{x^2+y^2}}{D}\right) = \begin{cases} 1; & 0 \leqslant \sqrt{x^2+y^2} < \dfrac{D}{2} \\ \dfrac{1}{2}; & \sqrt{x^2+y^2} = \dfrac{D}{2} \\ 0; & \sqrt{x^2+y^2} > \dfrac{D}{2} \end{cases} \tag{3b}$$

where D is the diameter of the exit pupil of the system.

After substituting (1) to formula (2) we obtain

$$U'(x, y) = \frac{\exp\left[i\dfrac{k}{2}\left(\dfrac{1}{z_1}-\dfrac{1}{f}\right)(x^2+y^2)\right]}{i\lambda z_1} P(x, y) \times \iint_{-\infty}^{\infty} U_1(x_1, y_1)\exp\left[i\frac{k}{2z_1}(x_1^2+y_1^2)\right] \exp\left[-i\frac{k}{z_1}(x_1x+y_1y)\right] dx_1 dy_1. \tag{4}$$

By using again the Huygens-Fresnel principle in the image space of the system we will find the complex amplitude distribution in the observation plane distant by z_2 from the mirror vertex:

$$U_2(x_2, y_2) = -\frac{\exp\left[i\dfrac{k}{2z_2}(x_2^2+y_2^2)\right]}{\lambda^2 z_1 z_2} \iint_{-\infty}^{\infty} U_1(x_1, y_1) \times\exp\left[i\frac{k}{2z_1}(x_1^2+y_1^2)\right] dx_1 dy_1 \iint_{-\infty}^{\infty} P(x, y) \times\exp\left[i\frac{k}{2}\left(\frac{1}{z_1}+\frac{1}{z_2}-\frac{1}{f}\right)(x^2+y^2)\right] \times\exp\left[-i\frac{k}{z_2}(xx_2+yy_2)\right]\exp\left[-i\frac{k}{z_1}(xx_1+yy_1)\right] dxdy. \tag{5}$$

The second term of the expression (5) described by the surface integral in the aperture diaphragm plane is the Fourier-transform of the generalized pupil function

$$P(x, y)\exp\left[i\frac{k}{2F}(x^2+y^2)\right], \tag{6}$$

for the spatial frequencies

$$\xi = \frac{1}{\lambda}\left(\frac{x_1}{z_1}+\frac{x_2}{z_2}\right),$$
$$\eta = \frac{1}{\lambda}\left(\frac{y_1}{z_1}+\frac{y_2}{z_2}\right). \tag{7}$$

The generalized pupil function takes account of the phase error in the exit pupil plane, whereby the parameter

$$\frac{1}{F} = \frac{1}{z_1}+\frac{1}{z_2}-\frac{1}{f} \tag{8a}$$

is the measure of the deviation from the ideal imaging and determines the defocussing error of the light beams creating the image of the examined object. Thus if Δ denotes a shift of the observation plane with respect to the Gaussian image plane, then in accordance to the ideal imaging law

$$\frac{1}{f} = \frac{1}{z_1}+\frac{1}{z_2+\Delta} \tag{8b}$$

we obtain

$$\frac{1}{F} = \frac{\Delta}{z_2(z_2+\Delta)}. \tag{9}$$

By taking advantage of the conditions (3b) the Fourier transform of the generalized function may be put in the form

$$\iint_{-\infty}^{\infty} P(x, y)\exp\left[i\frac{k}{2F}(x^2+y^2)\right]\exp[-i2\pi(x\xi+y\eta)]dxdy$$
$$= \exp\left[-i\frac{k}{2}\lambda^2F(\xi^2+\eta^2)\right]\otimes\frac{J_1\left(2\pi\sqrt{\xi^2+\eta^2}\right)}{\sqrt{\xi^2+\eta^2}}, \tag{10}$$

up to a constant factor. Here J_1 is the Bessel function of the first kind and the first order. After substituting the above expression to the eq. (5) we obtain the following form of the complex amplitude of the optical field in the observation plane

$$U_2(x_2, y_2) = -\frac{\exp\left[i\frac{k}{2z_2}(x_2^2+y_2^2)\right]}{\lambda^2 z_1 z_2}\iint_{-\infty}^{\infty} U_1(x_1, y_1)$$
$$\times\exp\left[i\frac{k}{2z_1}(x_1^2+y_1^2)\right]\exp\left[-i\frac{k}{2}\lambda^2F(\xi^2+\eta^2)\right]$$
$$\otimes\frac{J_1\left(2\pi\sqrt{\xi^2+\eta^2}\right)}{\sqrt{\xi^2+\eta^2}}dx_1dy_1, \tag{11a}$$

which is proportional to Fourier transform of convolution of two functions: the modified object complex amplitude and the sombrero function:

$$U_2(x_2, y_2) = -\frac{\exp\left[i\frac{k}{2z_2}\left(1-\frac{F}{z_2}\right)(x_2^2+y_2^2)\right]}{\lambda^2 z_1 z_2}$$

$$\iint_{-\infty}^{\infty} U_1(x_1, y_1)\exp\left[i\frac{k}{2z_1}\left(1-\frac{F}{z_1}\right)(x_1^2+y_1^2)\right] \otimes \frac{J_1\left(2\pi\sqrt{\xi^2+\eta^2}\right)}{\sqrt{\xi^2+\eta^2}}$$

$$\times\exp\left[-i\frac{kF}{z_1 z_2}(x_1x_2+y_1y_2)\right]dx_1dy_1. \tag{11b}$$

If the object transmittance is located at the focal plane of the mirror (i.e. under the assumption that $z_1 = f$) we obtain $z_2 = F$ from eq. (8). Then, the field distribution in the image focal plane fulfils the condition $F = f$, and the phase curvatures in the expression (11b) disappear. This is because we have

$$U_2(x_2, y_2) = -\frac{1}{\lambda^2 f^2}\iint_{-\infty}^{\infty} U_1(x_1, y_1) \otimes \frac{J_1\left(2\pi\sqrt{\xi^2+\eta^2}\right)}{\sqrt{\xi^2+\eta^2}}$$

$$\exp\left[-i\frac{k}{f}(x_1x_2+y_1y_2)\right]dx_1dy_1. \tag{11c}$$

Hence an obvious conclusion follows that the complex amplitude distribution in the image focal plane of the mirror is a product of two functions: Fourier-transform of the field distribution in the object focal plane and the pupil function of the form

$$P(x_2, y_2) = \mathrm{cyl}\left(\sqrt{x_2^2+y_2^2}\right).$$

3. The effect of the defocussing error on the shape of complex amplitude in the image

In a particular case when the object transmittance plane is placed at the distance $z_1 = 2f$ from the mirror vertex the focussing of the paraxial beams occurs also at the distance $z_2+\Delta = 2f$, thus fulfilling the condition of ideal imaging (8b). Nevertheless, the prevailing quantity of light energy of the optical field reflected from the paraboloidal mirror is focussed outside the Gaussian image plane. Therefore, the phase curvature depends upon the focussing error in the following way:

$$\exp\left[i\frac{k}{2z_2}\left(1-\frac{F}{z_2}\right)(x_2^2+y_2^2)\right] = \exp\left[-i\frac{k}{2\Delta}(x_2^2+y_2^2)\right],$$

$$\exp\left[i\frac{k}{2z_1}\left(1-\frac{F}{z_1}\right)(x_1^2+y_1^2)\right] = \exp\left[-i\frac{k}{2\Delta}\left(1-\frac{\Delta}{f}\right)(x_1^2+y_1^2)\right],$$

and the optical field distribution defined by the formula (11b) takes the form

$$U_2(x_2, y_2) = -\frac{\exp\left[-i\frac{k}{2\Delta}(x_2^2+y_2^2)\right]}{4f^2\lambda^2}\iint_{-\infty}^{\infty} U_1(x_1, y_1)$$

$$\times\exp\left[-i\frac{k}{2\Delta}\left(1-\frac{\Delta}{f}\right)(x_1^2+y_1^2)\right]$$

$$\otimes\frac{J_1\left(2\pi\sqrt{\xi^2+\eta^2}\right)}{\sqrt{\xi^2+\eta^2}}\exp\left[-i2\pi(x_1u+y_1v)\right]dx_1dy_1. \quad (12a)$$

This means that the complex amplitude of the field in the observation plane located at the distance $z_2 = 2f-\Delta$ from the mirror vertex is proportional to the Fourier transform of the convolution of two functions: modified object function and sombrero-function. This transform depends on the focussing error of the light beams at the observation plane being distant by Δ from the Gaussian image plane. The Fourier spectrum frequencies are here as follows

$$u = \frac{x_2}{\lambda\Delta}, \quad v = \frac{y_2}{\lambda\Delta}. \quad (13)$$

Finally, after performing the Fourier transform operation the expression (12a) takes the form

$$U_2(x_2, y_2) = A\,\mathrm{cyl}\left(\sqrt{x_2^2+y_2^2}\right)\exp\left[-i\frac{k}{2\Delta}(x_2^2+y_2^2)\right]$$

$$\exp\left[i\frac{k}{2}\lambda^2\frac{\Delta}{\left(1-\frac{\Delta}{f}\right)}(u^2+v^2)\right]\otimes\mathscr{F}\left[U_1(x_1, y_1)\right], \quad (12b)$$

where $\mathscr{F}$ denotes the Fourier transform, and the constant

$$A = \frac{i}{4f\lambda}\sqrt{\frac{2(2f-\Delta)}{f-\Delta}}$$

takes account of the constant factor appearing due to integrating of the expression (10).

The figure 2 illustrates the beam focussing in the observation plane located at the distance Δ with respect to Gaussian image plane. Under assumption that the transversal sizes of the diffraction image are equal to the spread caused by the Fraunhofer diffraction in the Gaussian image plane we obtain

$$(\delta x)_d = \frac{2\lambda z_2}{D}. \quad (14)$$

But the transversal sizes of the image spread function in the Gaussian plane evoked by the defocussing error Δ are determined as follows (see fig. 2)

$$(\delta x)_\Delta = \frac{D\Delta}{z_2}. \tag{15}$$

Therefore the image spot area corresponding to an object point is not focussed around the Gaussian plane point ([1, 4]) but is recorded around the Gaussian image within the region of sizes determined by the formula

$$\delta = [(\delta x)_d^2 + (\delta x)_\Delta^2]^{1/2}. \tag{16}$$

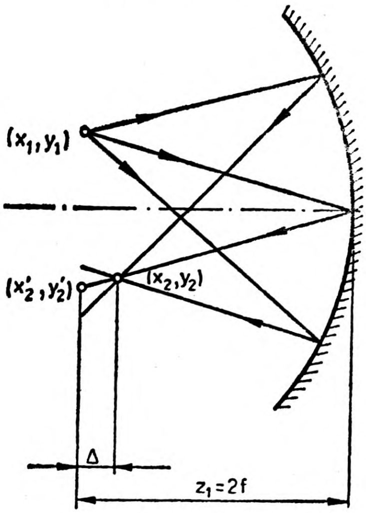

Fig. 2. The influence of the aperture of the concave mirror on the light beam focussing. Point (x_2', y_2') is the Gaussian image of the point (x_1, y_1)

4. Final conclusions

The above considerations concerning the light beam transformations in a mirror may be exploited to examination of the image light field in this kind of systems [5]. When analysing the expressions (5) and (12) a distinct influence of the focussing error on the complex amplitude of the field in the image space of the system is easily noticeable. On the one hand, we have a dependence of the complex image amplitude upon the Fourier-transform of the generalized pupil function, while on the other one, we see the relation of this amplitude to the Fourier-transform of convolution of the modified distribution of the object field defined by the focussing error and the sombrero-function.

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Трансформация предметного поля в элементарной зеркальной системе с аберрацией перефокусировки

Обсуждены преображения светового поля в зеркальной параболоидной системе. Рассмотрено влияние аберрации перефокусировки на создание изображения. Показана зависимость распределения поля изображении от распределения поля на предмете, а также зависимость поля изображения от преобразования Фурье обобщённой зрачковой функции. Предложено также использование параболоидного вогнутого зеркала для создания голограмм сфокуссированных изображений.