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Solution of ray trajectory equation in a gas lens*

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An analytic method of solving the ray trajectory equation for a gas lens in cylindric coordinate system is presented. An approximate solution has been found for the integral equations by employing the method of successive approximations. An analytic form of the solution is very simple and allows to calculate quickly the main parameters of the gas lens.

1. Introduction

Gas lenses, as elements of the waveguides, are essentially nonuniform media characterized by refractive index gradient in gas, induced by changing such parameters like temperature or pressure. Thermal gas lens is composed of a brass tube of constant temperature T_w with a chrome-nickel wire reeled on it to heat the gas flowing through the tube. The gas of the room temperature $T_0 < T_w$ flows into the lens and is heated by its walls. Since the refractive index changes inversely proportionally to the temperature it reaches its greatest value on the lens axis. The flowing gas acts a converging lens and focusses the light rays passing through the tube.

The analysis of the light propagation in the gas lens may be carried out within the geometric optics approximation, since the changes of the refractive index along the distance comparable with the wavelength value are negligable with respect to the value of the refractive index itself.

In the uniform media the light propagates along the straight lines, while in the heterogeneous media it travels along the curved trajectories. Deformation of light ray trajectory in the nonuniform media may be easily determined by using numerical methods. However, for theoretical discussions of the ray trajectory properties the knowledge of an analytic (even approximate) solution of the ray equation is more desirable. In the case when the refractive index of the gas medium is a function of only one variable the analytic solution of the ray equation is reduced to solving a relatively simple differential equation of the second order [1]. If two variables are taken into account (it is assumed that the optical system is of rotational symmetry) the initial form of the differential equation is more complex and the rigorous analytic solution of this equation for the refractive index distribution existing in the gas lens is impossible. Sodha, Ghatak, Malik and Goval developed the theory of electromagnetic wave pro-

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pagation in the gas lens by using the geometric optics approximation [2, 3]. However, the analytic form of the solution obtained by them is rather complex; therefore an analytic method of solving the ray equation with the help of integral equations proposed in this work is based on iterative method.

2. Differential equation of ray trajectory in the cylindric coordinates

The light ray trajectory in the isotropic medium of arbitrary distribution of the refractive index is described by a vector ray equations [4]:

$$\frac{d}{ds}\left(n\frac{d\boldsymbol{r}}{ds}\right) = \operatorname{grad} n, \tag{1}$$

where $\boldsymbol{r}(x, y, z)$ — travelling radius of an arbitrary point of the light ray,
s — arc length of the light ray,
$d\boldsymbol{r}/ds$ — unity vector normal to the wave surface,
$n(x, y, z)$ — refractive index of the isotropic medium.

By passing from the ray eq. (1) in the Cartesian coordinates to the ray equation in the cylindric coordinates ϱ and z (under assumption that the gas medium is of rotational symmetry around the z axis, in other words that $\partial n/\partial\varphi = 0$ the following relations are obtained:

$$\begin{aligned} &\frac{d}{ds}\left(n\frac{d\varrho}{ds}\right) - n\varrho\left(\frac{d\varphi}{ds}\right)^2 = \frac{\partial n}{\partial\varrho}, \\ &n\frac{d\varrho}{ds}\frac{d\varphi}{ds} + \frac{d}{ds}\left(n\varrho\frac{d\varphi}{ds}\right) = 0, \\ &\frac{d}{ds}\left(n\frac{dz}{ds}\right) = \frac{\partial n}{dz}. \end{aligned} \tag{2}$$

By eliminating ds from those equations and taking account of the rays lying in the meridional plane the following differential equations are obtained:

$$\frac{d^2\varrho}{dz^2} = \frac{1}{n}\left[1+\left(\frac{d\varrho}{dz}\right)^2\right]\left(\frac{\partial n}{\partial\varrho} - \frac{d\varrho}{dz}\frac{\partial n}{\partial z}\right). \tag{3}$$

3. Analytic solution of the ray equation in the medium of refractive index depending on one variable

By considering the medium of refractive index being a one variable function a simple differential equation is obtained which may be solved analytically. In the case when the refractive index changes only with the distance ϱ from the lens axis the equation (3) has the form

$$\frac{d^2\varrho}{dz^2} = \frac{1}{n}\left[1+\left(\frac{d\varrho}{dz}\right)^2\right]\frac{dn}{d\varrho}, \tag{4}$$

and the solution is

$$z(\varrho) = z_0 \pm \int\limits_{\varrho_0}^{\varrho}\left[\frac{1+\left(\frac{d\varrho}{dz}\right)_0^2}{n^2(\varrho_0)}\, n^2(\varrho)-1\right]^{-1/2} d\varrho. \tag{5}$$

If the refractive index depends only on the lens length z then by solving the equation

$$\frac{d^2\varrho}{dz^2} + \frac{1}{n}\left[1+\left(\frac{d\varrho}{dz}\right)^2\right]\frac{dn}{dz}\frac{d\varrho}{dz} = 0, \tag{6}$$

we get

$$\varrho(z) = \varrho_0 \pm \int\limits_{z_0}^{z}\left[\frac{1+z_0}{n^2(z_0)}\, n^2(z)-1\right]^{-1/2} dz. \tag{7}$$

The values z_0, ϱ_0, $(d\varrho/dz)_0$ correspond to the initial conditions: z_0, ϱ_0 — point coordinates of the light ray at the input to the lens, $(d\varrho/d_z)_0$ — tangent of the ray inclination angle at the input. For the ray entering at $z_0 = 0$ parallelly to the axis $\left((d\varrho/dz)_0 = 0\right)$ the point coordinates of the ray change according to (5) as follows

$$z(\varrho) = \pm \int\limits_{\varrho_0}^{\varrho}\left[\frac{n^2(\varrho)}{n^2(\varrho_0)} -1\right]^{-1/2} d\varrho \tag{5a}$$

or, in the face of (7), according to the following formulae

$$\varrho(z) = \varrho_0 \pm \int\limits_{0}^{z}\left[\frac{n^2(z)}{n^2(0)} -1\right]^{-1/2} dz. \tag{7a}$$

If the light ray entering parallelly the lens at the height ϱ_0 is focussed, the minus sign is taken in formulae (5a) and (7a), while if it is defocussed the plus sign should be accepted.

4. Analytic solution of the ray equation in the gaseous medium of refractive index distribution $n(\rho, z) = n_0(z)(1-a_2(z)\rho^2)$

Since the gas lenses are of great focal lengths in relation to their transversal sizes, the quantity $(d\varrho/dz)^2$ is negligible in comparison to the number 1, and then

$$\frac{d^2\varrho}{dz^2} = \frac{1}{n}\left(\frac{\partial n}{\partial \varrho} - \frac{d\varrho}{dz}\frac{\partial n}{\partial z}\right). \tag{8}$$

By introducing the dimensionless quantities u and x

$$u = \frac{\varrho}{a}, \quad x = \frac{\alpha}{a^2 v_0} z,$$

where ϱ — distance from the tube axis,
a — tube radius,
v_0 — axial velocity of the gas flow,
z — length coordinate,
α — parameter defining the gas properties: $\alpha = \frac{k}{\tilde{\varrho} c_p}$ (k — coefficient of thermal conductivity, c_p — specific heat at constant pressure, $\tilde{\varrho}$ — average density of gas):

we get from (8)

$$\frac{d^2 u}{dx^2} = \frac{1}{n}\left[K \frac{\partial n(u, x)}{\partial u} - \frac{du}{dx} \frac{\partial n(u, x)}{\partial x}\right], \tag{9}$$

where $K = \left(\frac{v_0}{V(L)} \frac{L}{a}\right)^2$, $V(L) = \frac{\alpha L}{a^2}$, L — lens length. The following temperature distribution [3]:

$$T(u, x) = T_w - (T_w - T_0)(1 - u^2) e^{-4x} \tag{10}$$

(T_w — temperature of the tube walls, T_0 — temperature of the gas the tube entrance) is identical with the exact values of temperature in the gas lens and fulfills the energy balance equation and its limiting conditions in such a lens. By assuming the radial parabolic change of refractive index [2] in the form

$$n(u, x) = n_0(x)[1 - a^2 a_2(z) u^2] \tag{11}$$

and substituting the temperature distribution (10) to the expression relating the temperature and refractive index

$$n(u, x) = 1 + (n_p - 1) \frac{T_0}{T(u, x)} \tag{12}$$

the following formulae for $n_0(x)$ and $a_2(x)$ are obtained:

$$n_0(x) = 1 + (n_p - 1)\left[1 - \frac{T_w - T_0}{T_0} (1 - e^{-4x})\right], \tag{13}$$

$$a_2(x) = \frac{2(n_p - 1)(T_w - T_0)}{a^2 n_0(x) T_0} e^{-4x}, \tag{14}$$

where n_p — refractive index in gas for $T(0, 0) = T_0$. By eliminating $n_0(x)$ and $a_2(x)$ from (11) the distribution of the refractive index takes the form

$$n(u, x) = n_0 + p(1 - u^2) e^{-4x}, \tag{15}$$

where

$$n_0 = n_p - (n_p - 1)\frac{T_w - T_0}{T_0}, \tag{15a}$$

$$p = (n_p - 1)\frac{T_w - T_0}{T_0}. \tag{15b}$$

If a new variable $y = u/\sqrt{K}$ is introduced and if notation $m(y, x) = \ln n(y, x)$ is accepted, the eq. (9) is led to the form

$$\frac{d^2 y}{dx^2} = \frac{\partial m(y, x)}{\partial y} - \frac{dy}{dx}\frac{\partial m(y, x)}{\partial x}, \tag{16}$$

and the refractive index is

$$n(y, x) = n_0 + p(1 - Ky^2)e^{-4x}. \tag{17}$$

The partial derivatives of the function m appearing in (16) are equal to

$$\begin{aligned} \frac{\partial m(y, x)}{\partial y} &= -\frac{2Kpy}{n(y, x)}e^{-4x} = h_1(y, x), \\ \frac{\partial m(y, x)}{\partial x} &= -\frac{4p(1 - Ky^2)}{n(y, x)}e^{-4x} = h_2(y, x). \end{aligned} \tag{18}$$

The solution of (16) is sought in the form $y = y(x)$. By substituting this solution to the formulae (18) the variable y may be eliminated and in this way the partial derivatives of the function $m(y, x)$ become only the functions of one-variable x. Denoting by

$$\begin{aligned} g(x) &= -\frac{2Kpy(x)}{n[y(x), x]}e^{-4x}, \\ f(x) &= -\frac{4p[1 - Ky^2(x)]}{n[y(x), x]}e^{-4x}, \end{aligned} \tag{19}$$

the eq. (16) is reduced to a nonuniform linear differential equation of second order

$$\frac{d^2 y}{dx^2} + f(x)\frac{dy}{dx} - g(x) = 0. \tag{20}$$

By lowering the order of equation a nonuniform linear differential equation of first order is obtained

$$\frac{dz}{dx} + f(x)z - g(x) = 0, \tag{21}$$

where $z(x) = dy/dx$. The functions $f(x)$ and $g(x)$ are continuous within an interval of the variable x. The solution of this equation is the following

$$z(x) = \left(z_0 + \int\limits_{x_0}^{x} g(x')\, e^{\int\limits_{x_0}^{x'} f(x)dx} dx'\right) e^{-\int\limits_{x_0}^{x} f(x')dx'} \tag{22}$$

The solution of the differential equation of second order (20) with the initial condition $x = x_p$, $y = y_0$ is

$$y(x) = y_0 + z_0 \int\limits_{x_p}^{x} e^{-\int\limits_{x_0}^{x''} f(x')dx'} dx''$$
$$+ \int\limits_{x_p}^{x} \left(e^{-\int\limits_{x_0}^{x''} f(x')dx'} \int\limits_{x_0}^{x''} g(x')\, e^{\int\limits_{x_0}^{x'} f(x)dx} dx'\right) dx''. \tag{23}$$

Now, coming back to the function h_1 and h_2 (eq. (18)) depending upon x and y, which were temporarily assumed to depend only on x, we obtain an integral equation of the type

$$y(x) = y_0 + z_0 \int\limits_{x_p}^{x} e^{-\int\limits_{x_0}^{x''} h_2(y,x')dx'} dx''$$
$$+ \int\limits_{x_p}^{x} \left(e^{-\int\limits_{x_0}^{x'} h_2(y,x')dx'} \int\limits_{x_0}^{x''} h_1(y, x')\, e^{\int\limits_{x_0}^{x'} h_2(y,x)dx} dx'\right) dx''. \tag{24}$$

The solution is also an equation of ray trajectory $y = y(x)$, which is found by applying the method of successive approximations due to Picard. In the first step of iteration it is assumed that the functions $h_1(x, y)$ and $h_2(y, x)$ are almost equal to zero. Then from (24)

$$y(x) = y_0 + z_0(x - x_p).$$

By employing the initial conditions: $x = x_p$, $y = y_0$, $(dy/dx)_{x=x_p} = 0$ we obtain $y = y_0$. This is true since, as it is well known, in the case of constant refractive index of the medium the parallel ray incident at certain height y_0 moves in this medium along the straight line $y = y_0$. The solution obtained in the first approximation $y(x) = y_0$ is substituted into (18) in the second iterative step

$$\begin{aligned} h_2(y_0, x) &= -\frac{4p(1-Ky_0^2)}{n(y_0, x)} e^{-4x} = f(x), \\ h_1(y_0, x) &= -\frac{2Kpy_0}{n(y_0, x)} e^{-4x} = g(x), \end{aligned} \tag{25}$$

where $n(y_0, x) = n_0 + p(1-Ky_0^2)e^{-4x}$.

Since $n(y_0; x)$ differs only slightly from n_0 it is allowed to assume $n(y_0, x) = n_0$ in the nominator, getting

$$\begin{aligned} h_1(y_0, x) &= -\frac{4}{n_0} p(1-Ky_0^2)e^{-4x} = 4ae^{-4x}, \\ h_2(y_0, x) &= -\frac{2}{n_0} Kpy_0 e^{-4x} = -2be^{-4x}, \end{aligned} \tag{26}$$

where

$$\begin{aligned} a &= -\frac{p(1-Ky_0^2)}{n_0}, \\ b &= \frac{Kpy_0}{n_0}. \end{aligned} \tag{27}$$

By calculating the particular integrals in the formula (24) we get the solutions $y(x)$

$$\begin{aligned} y(x) = y_0 + \left(z_0 + \frac{b}{2a}\right) e^{-ae^{-4x_0}} \Bigg\{ (x-x_p) - \frac{1}{4} \Bigg[\left(\frac{ae^{-4x}}{1\cdot 1!} + \frac{a^2e^{-8x}}{2\cdot 2!} + \ldots \right) \\ - \left(\frac{ae^{-4x_p}}{1\cdot 1!} + \frac{a^2e^{-8x_p}}{2\cdot 2!} + \ldots \right) \Bigg] \Bigg\} - \frac{b}{2a}(x-x_p). \end{aligned} \tag{28}$$

The further iteration consists in resubstituting the known solution $y = y(x)$ to (18) and finding, according to (24), the solution for new functions h_1 and h_2, and so on. However, the next substitution leads to a very troublesome calculation of integrals and therefore the actual analytic calculations have been stopped at this point.

From the formula (22) the first derivative of $y(x)$ function is calculated

$$\frac{dy(x)}{dx} = \left(z_0 + \frac{b}{2a}\right) e^{-ae^{-4x_0}} e^{ae^{-4x}} - \frac{b}{2a}, \tag{29}$$

and the initial condition $x = x_p$, $(dy/dx) = 0$ and (29) yield

$$z_0 + \frac{b}{2a} = \frac{b}{2a} e^{-ae^{-4x_p}} e^{ae^{-4x_0}}. \tag{30}$$

By eliminating the value $z_0 + b/2a$ from the functions $y(x)$ and $dy(x)/dx$ we have

$$\begin{aligned} y(x) = y_0 - \frac{b}{2a}\left(1 - e^{-ae^{-4x_p}}\right)(x-x_p) \\ + \frac{b}{8a} e^{-ae^{-4x_p}} \left(\frac{ae^{-4x_p}}{1\cdot 1!} + \frac{a^2e^{-8x_p}}{2\cdot 2!} + \ldots \right) \\ - \frac{b}{8a} e^{-ae^{-4x_p}} \left(\frac{ae^{-4x}}{1\cdot 1!} + \frac{a^2e^{-8x}}{2\cdot 2!} + \ldots \right), \end{aligned} \tag{31}$$

$$\frac{dy(x)}{dx} = \frac{b}{2a}\left(e^{-ae^{-4x_p}} e^{ae^{-4x}} - 1\right). \tag{32}$$

Returning to the variables u and x and replacing the constants a and b with constants p an n_0 determining the refractive index we get relations that determine the position $u(x)$ of the point on the light ray, and the slope du/dx of this ray with respect to the axis

$$u(x) = u_0 - K\Big\{\frac{u_0}{2(u_0^2-1)}\big[1-e^{-\frac{p(u_0^2-1)}{n_0}e^{-4x_p}}\big](x-x_p)$$
$$-\frac{u_0}{8(u_0^2-1)}e^{-\frac{p(u_0^2-1)}{n_0}e^{-4x_p}}\Big[\frac{p(u_0^2-1)}{n_0}e^{-4x_p}+\frac{1}{4}\frac{p^2(u_0^2-1)^2}{n_0^2}e^{-8x_p}+\ldots\Big]$$
$$+\frac{u_0}{8(u_0^2-1)}e^{-\frac{p(u_0^2-1)}{n_0}e^{-4x_p}}\times\Big[\frac{p(u_0^2-1)}{n_0}e^{-4x}+\frac{1}{4}\frac{p^2(u_0^2-1)^2}{n_0^2}e^{-8x}+\ldots\Big]\Big\}, \tag{33}$$

$$\frac{du(x)}{dx} = \frac{Ku_0}{2(u_0^2-1)}\big[e^{-\frac{p(u_0^2-1)}{n_0}(e^{-4x_p}-e^{-4x})}-1\big]. \tag{34}$$

It is sufficient to use only two first terms of the series, since the next one is only of the order of 10^{-15}. The form of these expressions is very simple and the main parameters of the lens may be found quickly. This solution may be applied with success as a first estimate of the focal length and position of the principal surface.

5. Final remarks

The formulae (33) and (34) enable to define the properties of the gas lenses. If the directions of the light ray and that of the gas flow are consistent, then the initial conditions are the following: $x_p = 0$, $u = u(0)$, $(du/dx)_{x=0} = 0$. In the case when these directions are opposite: $x_p = x_L$, $u = u(0)$, $(du/dx)_{x=x_L} = 0$. The focal length and the principal surface shape are defined by the relations [5]:

$$\begin{aligned}
x_F^+ &= -\frac{u_+(0)}{(du/dx)^+_{x=x_L}},\\
x_H^+ &= \frac{V(L)}{v_0}+\frac{u_+(0)-u_+(x_L)}{(du/dx)^+_{x=x_L}},\\
x_F^- &= \frac{u_-(x_L)}{(du/dx)^-_{x=0}},\\
x_H^- &= \frac{u_-(x_L)-u_-(0)}{(du/dx)^-_{x=0}},
\end{aligned} \tag{35}$$

where the indices "+" and "−" denote the consistence of the light ray and gas flow directions, respectively.

The analytic expressions (33) and (34) are useful especially for calculation of focal lenght x_F (the distance of the principal surface from the lens input deviate from the exact values).

It should be remembered that when looking for analytic solution of the ray equation we have stopped the procedure after the second iterative step. If we did not of it and calculated the separate integrals according to (24), the values of parameters would be presumably much more accurate, compared with those calculated with the paraxial approximation [6] in the off axis region [7].

The values of the focal lengths may be estimated with sufficient accuracy for the distances $u(0)$ from the axes and contained within the interval $0 < u(0) < 0.7$, and for the normalized gas velocities within the interval $5 < v_0/V(L) < 10$. The values of n_0 and p for $n_p = 1.000273$, $T_w - T_0 = 50$ K, $T_0 = 293$ K have been found from (15a) and (15b). For the above values the principal parameters have been calculated. For instance, for $a = 0.3$ cm, $v_0/V(L) = 6.25$, $u(0) = 0.1$, and $L = 20$ cm, $-z_F^+/L = 3.2697$, while for non-axial approximation the value obtained is $z_F^+/L = 3.3761$ [7].

The form of the integral eq. (24) may be employed to find the light ray trajectory in a cylindric nonuniform medium of arbitraty distribution of refractive index, in the case when the transversal sizes of the lens are much less than the its longitudinal sizes.

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Решение уравнения траектории радиуса в газовой линзе

Предложен метод аналитического решения уравнения траектории радиуса в газовой линзе в цилиндрических координатах. Приближённое решение было найдено с помощью интегральных уравнений на основе метода последовательных приближений. Аналитическая форма решения очень проста и позволяет быстро рассчитать главные параметры газовой линзы.