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Light intensity distribution in the Fresnel diffraction region of a non-sinusoidal phase diffraction grating

KRZYSZTOF PATORSKI, PIOTR SZWAYKOWSKI

Warsaw Technical University, Institute of Design of Precise and Optical Instruments, ul. K. Chodkiewicza 8, 02-525 Warsaw, Poland.

One of the most spectacular diffraction phenomena obtainable with spatially coherent radiation is the so-called Talbot effect [1]. It refers to the imaging of the class of objects [2] without using any optical components. This is why the Talbot effect is frequently called the self-imaging phenomenon. Images formed in the Fresnel diffraction region are repeated along the illumination direction behind the object. Extensive theoretical studies of the Talbot effect have been published [3], and many practical applications proposed [4].

The gratings are the most important members of the class of objects suitable for self-imaging. In the majority of referenced works [4] the amplitude diffraction gratings have been employed. However, due to the great progress in the technology of phase gratings and their advantage over the amplitude ones with respect to their diffraction efficiency, the phase gratings replace the amplitude ones in many experimental arrangements. Moreover, the diffraction effects produced by optical phase gratings are of common interest to the specialists working in the fields of acoustooptics and electron microscopy. In the first case the optical grating is produced by an ultrasonic beam [5] and the second case relates to the correspondence between the scattering of electrons by a crystal lattice and the diffraction of light by a phase object [6]. In view of those facts further investigations of the Fresnel diffraction field properties of phase gratings are expected.

Until now the self-imaging phenomenon of phase gratings of sinusoidal profile has been studied. The case of optical gratings was discussed in [7–10], Fresnel diffraction patterns of ultrasonically produced gratings were analysed in [11–17], and the self-imaging phenomenon of a crystal lattice in electron microscopy systems was disucssed in [18, 19]. In this communication we shall derive the expression for irradiance distribution of the Fresnel diffraction field behind a nonsinusoidally modulated phase grating and report its potential practical applications.

The transmission function of nonsinusoidal phase grating illuminated by a plane spatially coherent wave is written in the form

$$T(x) \propto \exp\left\{i \sum_{m} v_m \cos\left[m\left(2\pi\frac{x}{d}+\Omega t\right)+\alpha_m\right]\right\}, \tag{1}$$

where v_m designates the Fourier coefficients (amplitude of the phase modulation of the m-th harmonic), d is the spatial period of the fundamental grating harmonic, Ω is the radial frequency of the fundamental, α_m designates the phase of the m-th harmonic (for the gratings with symmetrical groove shapes $\alpha_m = 0$), and t is the time parameter. Eq. (1) describes the case of progressive nonsinusoidal phase modulation (i.e., ultrasonically produced phase diffraction grating), the case of stationary optical nad crystal-lattice gratings is obtained by putting $\Omega = 0$. Each factor of eq. (1) can be written in the Jacobi series form

$$\exp\left\{iv_m\cos\left[m\left(2\pi\frac{x}{d}+\Omega t\right)+\alpha_m\right]\right\} = \sum_{n} i^n J_n(v_m)\exp\left\{in\left[m\left(2\pi\frac{x}{d}+\Omega t\right)+\alpha_m\right]\right\}. \tag{2}$$

Taking all the terms the expression (1) becomes

$$T(x, t) = \prod_{m=1}^{\infty} \sum_{n} i^n J_n(v_m)\exp\left\{in\left[m\left(2\pi\frac{x}{d}+\Omega t\right)+\alpha_m\right]\right\}. \tag{3}$$

In order to determine the amplitude in a plane $z > 0$ belonging to the Fresnel diffraction region it is necessary to multiply [20] each spectral component $\exp(imn2\pi x/d)$ by a factor $\exp[-i(mn)^2\pi\lambda z/d^2]$

$$U(x, z, t) = \sum_{n_1}\sum_{n_2}\dots\sum_{n_m}\dots\left[i^{\sum\limits_{m=1}^{\infty} n_m} J_{n_1}(v_1)\dots J_{n_m}(v_m)\dots\exp i\left\{-\sum_{m=1}^{\infty}\left[mn\left(2\pi\frac{x}{d}+\Omega t\right)+n_m\alpha_m\right]\right.\right.$$
$$\left.\left.-\left(\sum_{m=1}^{\infty} mn_m\right)^2\pi\frac{\lambda z}{d^2}\right]. \tag{4}$$

In order to shorten the notation the following symbols are introduced

$$A = 2\pi\frac{x}{d}+\Omega t, \tag{5}$$

$$B = \pi\frac{\lambda z}{d^2}. \tag{6}$$

The intensity distribution is given by the expression

$$I(x, z, t) = U(x, z, t)\,U^*(x, z, t) = \sum_{n_1}\sum_{p_1}\dots\sum_{n_m}\sum_{p_m}$$
$$\times\left\{i^{\sum\limits_{m=1}^{\infty}(n_m-p_m)} J_{n_1}(v_1)J_{p_1}(v_1)\dots\exp i\left\{\sum_{m=1}^{\infty}[(mn_m-mp_m)A+(n_m-p_m)\alpha_m]\right.\right.$$
$$\left.\left.-\left[\sum_{m=1}^{\infty}\sum_{l=1}^{\infty} mln_mn_1-\sum_{m=1}^{\infty}\sum_{l=1}^{\infty} mlp_mp_1\right]B\right\}\right\}. \tag{7}$$

Putting $n_m = g_m+p_m$ the last equation takes the form

$$I(x, z, t) = \sum_{g_1}\sum_{p_1}\dots\sum_{g_m}\sum_{g_p}\left\{i^{\sum\limits_{m=1} g_m} J_{g_1+p_1}(v_1)J_{p_1}(v_1)\dots\exp\left[i\sum_{m=1}^{\infty}(mg_mA+g_m\alpha_m)\right]\right.$$
$$\left.\times\exp\left[-i\sum_{m=1}^{\infty}\sum_{l=1}^{\infty} mlg_mg_lB\right]\exp\left[-i2\sum_{m-1}^{\infty}\sum_{l=1}^{\infty} mlg_lp_mB\right]\right\}. \tag{8}$$

Using m times the Graf's addition theorem [21] we finally obtain

$$I(x, z, t) = \sum_{g_1}\dots\sum_{g_m}\dots J_{g_1}\left[2v_1\sin\sum_{l=1}^{\infty} lg_l\pi\frac{\lambda z}{d^2}\right]$$
$$\dots J_{g_m}\left[2v_m\sin\sum_{l=1}^{\infty} mlg_l\pi\frac{\lambda z}{d^2}\right]\dots\left[\cos\sum_{m=1}^{\infty} mg_m\left(2\pi\frac{x}{d}+\Omega t\right)+g_m\alpha_m\right]. \tag{9}$$

Equation (9) describes the intensity distribution in the Fresnel diffraction field of a non-sinusoidal phase grating whose amplitude transmittance is given by eq. (1). In general, it can be used to determine numerically the intensity distribution at an arbitrary Fresnel diffraction plane. We shall discuss now some special cases being of practical importance.

1. $z = Nd^2/\lambda$, where N is an integer,

$$I\left(x, z = N\frac{d^2}{\lambda}, t\right) = \text{const}. \tag{10}$$

In the self-image planes, lying at the distances z equal the multiplicity of the Talbot distance d^2/λ, the uniform light intensity distribution is encountered.

2. $z = (N+1/2)d^2/\lambda, t$

$$I[x, z=(N+1/2)d^2/\lambda, t] = \sum_{g_1} \dots \sum_{g_m} \dots J_{g_1}\left[2v_1 \sin \sum_{l=1}^{\infty} l g_l \left(N+\frac{1}{2}\right)\pi\right]$$

$$\dots J_{g_m}\left[2v_m \sin \sum_{l=1}^{\infty} m\, l g_l \left(N+\frac{1}{2}\right)\pi\right] \dots \cos\left[\sum_{m=1}^{\infty} m g_m \left(2\pi \frac{x}{d} + \Omega t\right) + g_m \alpha_m\right]. \tag{11}$$

It is easy to notice that in this case all harmonics for m equal even numbers vanish.

$$I\left[x, z=\left(N+\frac{1}{2}\right) d^2/\lambda, t\right] = \sum_{g_1} \dots \sum_{g_{2m-1}} \dots J_{g_1}\left[2v_1 \sin \sum_{l=1}^{\infty} (2l-1) g_{2l-1} \left(N+\frac{1}{2}\right)\pi\right]$$

$$\dots J_{g_{2m-1}}\left[2v_{2m-1} \sin \sum_{l=1}^{\infty} (2m-1)(2l-1) g_{2l-1} \left(N+\frac{1}{2}\right)\pi\right]$$

$$\dots \cos\left\{\sum_{m=1}^{\infty}\left[(2m-1) g_{2m-1}\left(2\pi \frac{x}{d} + \Omega t\right) + g_{2m-1}\alpha_{2m-1}\right]\right\}. \tag{12}$$

In general, for $z = \left(N + \frac{m}{k}\right) d^2/\lambda$, where $m = 1, 2, 3, \dots$, k-th harmonics are absent.

3. The case of sinusoidal phase modulation. Putting $v_m = 0$ for m ≥ 2 we obtain the case of pure sinu soidal grating

$$I(x, z, t) = 1 + 2 \sum_{l=1}^{\infty} J_l \left[2v \sin\left(l\pi \frac{\lambda z}{d^2}\right)\right] \cos l\left(2\pi \frac{x}{d} + \Omega t\right), \tag{13}$$

discussed in [7–19].

4. The case of presence of the fundamental and its k-th harmonics. Putting $v_1 \neq 0$, $v_k \neq 0$, $v_m = 0$ for $m = 2, 3, \dots, k-1, k+1, \dots$ into eq. (9) we have

$$I(x, z, t) = \sum_{g_1} \sum_{g_k} J_{g_1}\left[2v_1 \sin(g_1 + k g_k)\pi \frac{\lambda z}{d^2}\right] J_{g_k}\left[2v_k \sin k(g_1 + k g_k)\pi \frac{\lambda z}{d_2}\right]$$

$$\times \cos\left[(g_1 + k g_k)\left(2\pi \frac{x}{d} + \Omega t\right) + g_1 \alpha_1 + g_k \alpha_k\right]. \tag{14}$$

In general, using eq. (9), we can calculate the Fresnel field intensity distribution curves in an arbitrary diffraction plane and compare those curves with the ones obtained experimentally. In this way the presence of harmonics in the non-sinusoidal phase modulation can be estimated quantitatively. This approach was proposed by Colbert and Zankel [14]; however, the referenced authors gave the expressions for intensity distribution in the particular diffraction planes only.

Equation (9) can be exploited for quantitative analysis of the harmonics and, therefore, for the determination of the shape of phase modulation. In many practical cases the harmonic distortion is small, and we can assume that the components of eq. (3) for $m > 3$ can be neglected. In such cases eq. (9) becomes

$$I'(x, z, t) = \sum_{g_1} \sum_{g_2} \sum_{g_3} J_{g_1}\left[2v_1 \sin(g_1 + 2g_2 + 3g_3)\pi \frac{\lambda z}{d^2}\right] J_{g_2}\left[2v_2 \sin 2(g_1 + 2g_2 + 3g_3)\pi \frac{\lambda z}{d^2}\right]$$

$$\times J_{g_3}\left[2v_3 \sin 3(g_1 + 2g_2 + 3g_3)\pi \frac{\lambda z}{d^2}\right] \cos\left[(g_1 + 2g_2 + 3g_3)\left(2\pi \frac{x}{d} + \Omega t\right) + (g_1 \alpha_1 + g_2 \alpha_2 + g_3 \alpha_3)\right]. \tag{15}$$

Assuming small values of the parameters v_1, v_2, and v_3 we can omit the terms described by the Bessel functions of the order higher than the first one. Under this assumption eq. (15) becomes

$$I'(x,z,t) = \sum_{g_1=-1}^{1}\sum_{g_2=-1}^{1}\sum_{g_3=-1}^{1} J_{g_1}\left[2v_1\sin(g_1+2g_2+3g_3)\pi\frac{\lambda z}{d^2}\right]$$
$$\times J_{g_2}\left[2v_2\sin 2(g_1+2g_2+3g_3)\pi\frac{\lambda z}{d^2}\right] J_{g_3}\left[2v_3\sin 3(g_1+2g_2+3g_3)\pi\frac{\lambda z}{d_2}\right]$$
$$\times\cos\left[(g_1+2g_2+3g_3)\left(2\pi\frac{x}{d}+\Omega t\right)+(g_1\alpha_1+g_2\alpha_2+g_3\alpha_3)\right]. \tag{16}$$

If the technique of selective detection of temporal harmonics of the intensity distribution is used, and the fundamental temporal harmonics is measured [17], the parameters, g_1, g_2, and g_3 can take the values:

$$g_1 = \pm 1,\ g_2 = 0,\ g_3 = 0;$$
$$g_1 = \pm 1,\ g_2 = \mp 1,\ g_3 = 0; \tag{17}$$
$$g_1 = 0,\ g_2 = \pm 1,\ g_3 \mp 1.$$

Assuming the symmetrical groove profile of the phase modulation we obtain from eq. (17)

$$I'(x,z,t) = 2J_1\left[2v_1\sin\pi\frac{\lambda z}{d^2}\right] J_0\left[2v_2\sin 2\pi\frac{\lambda z}{d^2}\right] J_0\left[2v_3\sin 3\pi\frac{\lambda z}{d^2}\right]\cos\left(2\pi\frac{x}{d}+\Omega t\right)$$
$$-2J_1\left[2v_1\sin\pi\frac{\lambda z}{d^2}\right] J_1\left[2v_2\sin 2\pi\frac{\lambda z}{d^2}\right] J_0\left[2v_3\sin 3\pi\frac{\lambda z}{d^2}\right]\cos\left(2\pi\frac{x}{d}+\Omega t\right)$$
$$-2J_0\left[2v_1\sin\pi\frac{\lambda z}{d^2}\right] J_1\left[2v_2\sin 2\pi\frac{\lambda z}{d^2}\right] J_1\left[2v_3\sin 3\pi\frac{\lambda z}{d^2}\right]\cos\left(2\pi\frac{x}{d}+\Omega t\right). \tag{18}$$

Now, we choose for the experimental observation three diffraction planes, for example: $z = d^2/2\lambda$, $z = d^2/3\lambda$, and $z = d^2/4\lambda$, and put the photodetector onto the optical axis, $x = 0$. The following three equations with three unknown are obtained

$$2J_1(\sqrt{3}v_1)J_0(\sqrt{3}v_2) - 2J_1(\sqrt{3}v_1)J_1(\sqrt{3}v_2) = I_1,$$
$$2J_1(2v_1)J_0(-2v_3) = I_2,$$
$$2J_1(\sqrt{2}v_1)J_0(2v_2)J_0(\sqrt{2}v_3) - 2J_1(\sqrt{2}v_1)J_1(2v_2)J_0(\sqrt{2}v_3) - 2J_0(\sqrt{2}v_1)J_1(2v_2)J_1(\sqrt{2}v_3) = I_3. \tag{19}$$

which can be solved by computer. In this way the constitutive harmonics of the non-sinusoidal phase grating can be found and the grating profile determined.

The selective detection approach to the time-dependent Fresnel field intensity distribution of the non--sinusoidal gratings is automatically applicable to the phase gratings generated by progressive ultrasonic fields. Due to the continuous movement of the sonic wave the time dependence of the point-detector scanned Fresnel diffraction field is obtained. Nevertheless, the approach proposed can be easily extended to other types of phase gratings. For example, in the case of stationary optical grating the lateral shift of its Fresnel diffraction field is easily realized by changing periodically the illumination angle of the plane light wave impinging onto the grating [3].

In conclusion we can say that the light intensity distributions in the Fresnel diffraction field of the phase gratings can be exploited as the source of information of the harmonic content of the grating modulation profile. From this point of view the Fresnel diffraction field approach can compete and/or be complementary with the well known farfield approach (the amplitudes of the harmonic components of phase modulation are determined from the diffraction efficiency data of higher diffraction orders).

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