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DERIVATION OF EQUATION OF MOTION IN CHANNELS WITH SIDE WEIRS

Using the differential form of the Bernoulli equation and the principle of conservation of momentum, an analysis of the one-dimensional description of flows of Newtonian fluids in prismatic channels with side weirs was carried out. A new form of the equation of motion – with a correction in the mass decrement term and an added momentum-variation term – has been derived from the principle of conservation of momentum. The modified dimensionless equation of motion is applicable to the computation of flows over side weirs, i.e. the determination of the free-surface profile on the weir and the volumetric flow rate.

DENOTATIONS

A – surface area of flow section, m^2,
b – width of flow section, m,
Fr_0 – Froude number in channel at beginning of weir ($l = 0$),
g – acceleration of gravity, m/s^2,
G – body force, N,
H – elevation of free-surface in channel (filling), m,
H_p – channel filling at beginning of overflow chamber ($l = 0$), m,
i – bottom slope of channel (overflow chamber),
J – hydraulic gradient,
k – ratio of coordinates of vectors $\vec{U}$ and $\vec{\upsilon}$ ($k = U/\upsilon$),
K_0 – shape number of channel at beginning of weir ($l = 0$),
$\vec{\imath}$ – unit vector having direction of local velocity vector $\vec{v}$, parallel to mean velocity vector $\vec{\upsilon}$,
l – axial coordinate parallel to channel bottom, m,
l_p – length of overflow edge, m,
L_0 – relative length of overflow edge ($L_0 = l_p/H_p$),
m – mass, kg,
N – force of hydrostatic thrust, N,
O_z – wetted perimeter of flow section, m,
p – elevation of overflow crest above channel bottom, m,
P_0 – relative height of overflow edge ($P_0 = p/H_p$),
q – dimensionless volumetric rate of flow in overflow chamber ($q = Q(l)/Q_d$),

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q_b – unit volume flow over side weir, m^3(s m),
q_r – coefficient of separation of flows on weir ($q_r = Q/Q_d$),
Q – volumetric rate of flow over weir, m^3/s,
$Q(l)$ – volumetric rate of flow in overlow chamber in section with abscissa l, m^3/s,
t – time, s,
T – friction force, N,
U – coordinate of longitudinal component $\vec{U}$ of velocity $\vec{\upsilon}_b$ of side stream along direction of velocity vector $\vec{\upsilon}$, m/s,
v – local velocity stream filament, m/s,
υ – mean velocity of main stream in channel, m/s,
υ_b – mean velocity of side-discharge stream, m/s,
V – volume, m^3,
z – elevation above datum level, m,
α – kinetic energy (Coriolis) coefficient,
β – momentum (Boussinesq) coefficient,
β_b – side-discharge stream momentum coefficient,
ζ – dimensionless ordinate of free-surface elevation in channel ($\zeta = H/H_p$),
η – experimental coefficient ($\eta = 2\beta - k\beta_b$),
Θ – angle of inclination of channel invert, °,
μ – weir discharge coefficient,
ξ – dimensionless abscissa of length ($\xi = l/l_p$),
Π – momentum of liquid mass, kg/s,
ρ – liquid density, kg/m^3,
τ – shear stress on channel wall, Pa.

1. INTRODUCTION

The problem of the calculation of water flows over conventional side has been investigated since the end of the last century but as yet no satisfactory, full analytical solution of the problem has been found. For practical purposes the formulas established for front weirs (e.g. the Poleni formula [1]) were adapted and later simple empirical formulas, based on experiments covering usually a narrow range of variation in the geometric and hydraulic parameters of side weirs (e.g. the Engels, Colemann and Kotowski formulas [1]) or derived theoretically (the Hager and Schaffernak formulas [2]), were used. Further development of methods of calculating flows over side weirs consisted in combining the description of the water–surface profile along the weir by means of differential equations of motion, with formulas (of the type mentioned above) for the volumetric flow over the weir (De MARCHI [3], PIETROV [4], FRAZER [5], VISCHER [6], SMITH [7]–[9], ISHIKAWA [10] and HAGER [11], [112]).

Most of the researchers dealing with free flows in open channels base their theoretical analyses on differential equations of motion derived (assuming that the specific energy is invariable) from the Bernoulli theorem, while few others use a differential equation of motion derived from the principle of conservation of momentum. Regardless of the approach, the following initial assumptions are usually made for prismatic channels having conventional side weirs:

• The volumetric rate of complete side overflow can be described by the following formula:

$$\frac{dQ}{dl} = \frac{2}{3}\mu\sqrt{2g}(H - p)^{3/2}. \tag{1}$$

• Friction-induced hydraulic slope J in any cross-section of the channel is the same as for the uniform flow at a depth H, equal to the actual depth of the flow in a given cross-section, and it can be calculated for a nonuniform flow by using the Chezy group formulas (acc. to VISHER [6], SMITH [7], HAGER [11]) or the Manning –Stricler formulas (acc. to ISHIKAWA [10], NOUGARO [13]) or it can be assumed as equal to the slope of the channel bottom ($J = i$), but it is usually neglected (acc. to De MARCHI [3], PIETROV [4] and KURGANOV [14]).

• (Coriolis) kinetic energy coefficient α and (Boussinesq) momentum coefficient β can be treated as constant along the weir, usually $\alpha = \beta = 1.0$ is assumed (acc. to De MARCHI [3], DALMAYRAC [15], NOUGARO [13], ISHIKAWA [10]).

• The last of the above assumptions is physically incorrect since, by definition, $\alpha \neq \beta$ and $\alpha > \beta > 1$ as:

$$\alpha = \frac{\int_A v^3 dA}{\upsilon^3 A}, \tag{2}$$

$$\beta = \frac{\int_A v^2 dA}{\upsilon^2 A}. \tag{3}$$

According to BIGGIERO and PINESE [16], $\beta = 1.05$ should be assumed for unilateral side weirs and $\beta = 1.10$ for bilateral side weirs. VITI [17] estimated, on the basis of the results of BUFFONI's studies [18], the value of the Coriolis coefficient at $\alpha \approx 2.0$. Whereas EL-KHASHAB and SMITH [8] showed that coefficients $\alpha \in <1.1; 3.0>$ and $\beta \in <1.0; 1.3>$ vary along conventional side weirs in trapezial channels, as did KOTOWSKI [2], [19]: $\alpha \in <1.1; 3.4>$ and $\beta \in <1.0; 1.6>$ for rectangular and cylindrical channels having nonconventional side weirs (with throttlig pipe).

Equality between the coordinate of the longitudinal component U of the velocity vector $\vec{\upsilon}_b$ of the separating itself lateral stream and the coordinate of the mean velocity (υ) vector at any cross-section of the overflow chamber (figure 1) is assumed when deriving an equation of motion from the Bernoulli equation. Thus it is assumed the flow over the side weir does not affect the total energy of a unit of the liquid mass remaining in the channel. Total energy head E_c in the flow section of the channel is given by the Bernoulli equation in the following differential form:

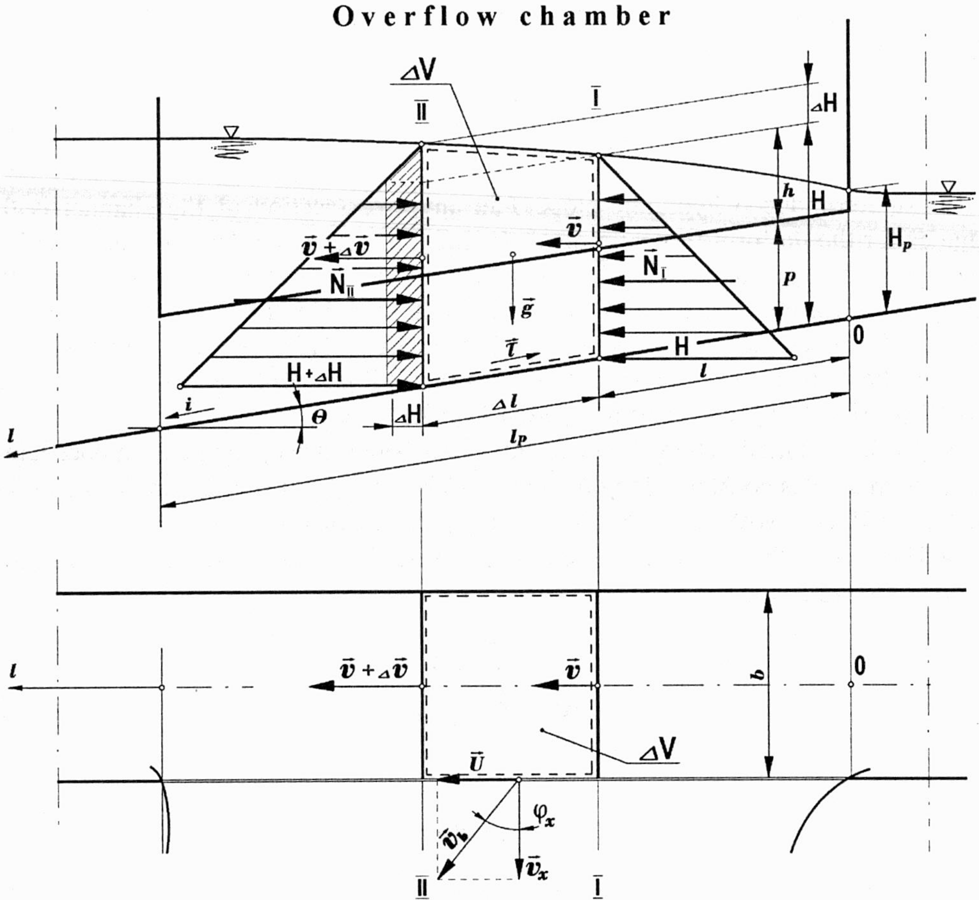

Fig. 1. Application of the principle of conservation of momentum to flows in channel with side weir

$$\frac{dE_c}{dl} = \frac{dz}{dl} + \frac{dH}{dl} + \frac{d}{dl}\left(\frac{\alpha Q^2}{2gA^2}\right), \tag{4}$$

where: $Q = Q(l)$ is the volumetric rate of flow in the overflow chamber.

The slopes of the particular lines relative to the datum level stand for: $dE_c/dl \equiv J$ – the hydraulic gradient; $dz/dl \equiv i$ – the slope of the channel bottom; $dH/dl \equiv i_z$ – the free-surface slope. The last term of equation (4) is written as follows (α = idem):

$$\frac{d}{dl}\left(\frac{\alpha Q^2}{2gA^2}\right) = \frac{\alpha}{2g}\left(\frac{2Q}{A^2}\frac{dQ}{dl} - \frac{2Q^2}{A^3}\frac{dA}{dl}\right) = \frac{\alpha Q}{gA^2}\frac{dQ}{dl} - \frac{\alpha Q^2}{gA^3}\left(\frac{\partial A}{\partial l} + \frac{\partial A}{\partial H}\frac{dH}{dl}\right). \tag{5}$$

For prismatic channels ($\partial A/\partial l = 0$, $\partial A/\partial H = b$), by introducing the above relations into equation (4), we get (after ordering):

$$\frac{dH}{dl}=\frac{i-J-\dfrac{\alpha Q}{gA^2}\dfrac{dQ}{dl}}{1-\dfrac{\alpha Q^2 b}{gA^3}}. \tag{6}$$

From the principle of conservation of momentum one can derive an equation of motion in which the coordinates U and υ can be related freely: $U < \upsilon$ (acc. to KURGANOV and FEDOROV [14]), $U = \upsilon$ (acc. to HAGER [11]), and $U > \upsilon$ (acc. to EL-KHASHAB and SMITH [8]). Thus, if the energy is taken into account, this method is more general. The following forces act on the liquid volume contained between two channel cross-sections: the liquid weight $G\sin\Theta$, the hydrostatic thrusts N_1 and N_2 and the friction force T. Thus the initial equation for the side overflow (acc. to [8]) can be written as follows:

$$\beta\rho\left(Q+\Delta Q\right)\left(\upsilon+\Delta\upsilon\right)-\rho\frac{dQ}{dl}\Delta lU-\beta\rho Q\upsilon=G\sin\Theta\ +N_1-N_2-T. \tag{7}$$

For an infinitesimal length of the interval Δl on the weir we get:

$$G\sin\Theta=\rho gAi\Delta l\,,$$

$$N_1-N_2=-\rho gA\Delta H, \tag{8}$$

$$T=-\rho gAJ\Delta l.$$

Using these relations and the equation of continuity of motion we get (after appropriate transformations) for prismatic channels with a side weir (β = idem) the following equation:

$$\frac{dH}{dl}=\frac{i-J-\dfrac{\left(2\beta\upsilon-U\right)}{g\,A}\dfrac{dQ}{dl}}{1-\beta\dfrac{Q^2 b}{g\,A^3}}. \tag{9}$$

Expression (9) is a differential equation for the overflow swelling or swelling or depression curve in the coordinates H and l. Since the axis l is assumed to run along the invert line, dH/dl is the water-surface slope relative to the channel invert. Equation (9), derived from the principle of conservation of momentum, enables a wider analysis of theoretically possible lateral outflow liquid-surface profiles than equation (6) derived from the differential form of the Bernoulli equation.

2. DERIVATION OF EQUATION OF MOTION

It follows from the principle of conservation of momentum in Newtonian continuous-medium mechanics that a change in momentum over time is equal to the sum of the body forces $\vec{G}$ and the surface forces $\vec{F}$:

$$\int_V \rho \frac{d\vec{\upsilon}}{dt} dV = \int_V \rho \vec{G} dV + \int_A \vec{F} dA. \tag{10}$$

Thus momentum change over time $dt \to 0$ is equal to the sum of forces acting on the infinitesimal control liquid volume ΔV contained between cross-sections I and II and the side edge of the weir. First, momentum will be balanced for this volume of the liquid and then the sum of forces acting on liquid will be calculated. On this basis, using the volume flow continuity and balance equation, an equation of motion of a liquid in a channel with flow a side weir will be derived (figure 1).

Momentum balance. The momentum brought in a liquid mass flowing through an element having the area dA in the time $dt \to 0$ is:

$$d(m\vec{v}) = (\rho v dA dt)\vec{v}, \tag{11}$$

where $\vec{v}$ denotes a local velocity vector perpendicular to the area dA cut out from the flow area A.

Thus the total momentum brought in the liquid mass flowing through a flow section having the area A in the time dt is:

$$d\vec{\Pi}_{\mathrm{I}} = \rho dt \int_A v\vec{v} dA = \vec{\imath} \rho dt \int_A v^2 dA, \tag{12}$$

where $\vec{\imath}$ means a unit vector, whose direction is the same as that of the local velocity vector $\vec{v}$, parallel to the mean velocity vector $\vec{\upsilon}$.

Momentum (12), expressed by the mean liquid flow velocity, can be written as:

$$d\vec{\Pi}_{\mathrm{I}} = \beta(\rho \upsilon A dt)\vec{\upsilon} = \beta(\rho \upsilon A dt)\upsilon \vec{\imath} = \beta \rho A \upsilon^2 dt \vec{\imath}, \tag{13}$$

where β is a dimensionless (corrective) coefficient of momentum.

It follows from equations (12) and (13) that:

$$\vec{\imath} \rho dt \int_A v^2 dA = \beta \vec{\imath} \rho A \upsilon^2 dt, \tag{14}$$

and hence the momentum coefficient β, also known as the Boussinesq coefficient, can be expressed by formula (3). It should be noted that the solid of velocities varies along the path l, similarly as the mean velocity does. This means that the coefficient β is constant only for uniform flow. Here it varies from section to section. This observation is based on the results of experiments carried out by the author [19] on nonconventional side weirs in rectangular and cylindrical channels.

The total momentum $d\vec{\Pi}_{\mathrm{I}}$ introduced into the control space ΔV with a liquid flowing through a flow section of the area A in the time $dt \to 0$ is given by the following value:

$$d\vec{\Pi}_{\mathrm{I}} = \beta\rho A\upsilon^2 dt\vec{\imath} = \beta\rho Q\upsilon dt\vec{\imath}, \tag{15}$$

where $Q = Q(l)$ is a volumetric flow rate in section A: $Q = A\upsilon$.

The momentum of the liquid leaving the interior of the control space through a flow section of the area $A + \Delta A$ in time $dt \to 0$ is:

$$\begin{aligned} d\vec{\Pi}_{\mathrm{II}} &= (\beta + \Delta\beta)\rho\,(A + \Delta A)(\upsilon + \Delta\upsilon)^2\, dt\,\vec{\imath} \\ &= (\beta + \Delta\beta)\rho\,(Q + \Delta Q)(\upsilon + \Delta\upsilon)dt\vec{\imath}. \end{aligned} \tag{16}$$

Equation (16) written for the section II is similar to equation (15) except that the quantities having the values β, A, υ, Q have been replaced with the quantities having the values $\beta + \Delta\beta$, $A + \Delta A$, $\upsilon + \Delta\upsilon$, $Q + \Delta Q$, differing in their increments (generally, positive or negative) from the former in the interval Δl.

At the same time ($dt \to 0$) a mass of liquid flows out from the control space, flowing over the weir length Δl at the mean velocity υ_b. The velocity changes its value and direction along the overflow edge. The momentum of the liquid mass flowing over the length Δl of the overflow edge is:

$$d\vec{\Pi}_b = \beta_b\,\rho\Delta l q_b dt\vec{\upsilon}_b, \tag{17}$$

where:

β_b – a lateral stream momentum (correction) coefficient,

q_b – a unit volume flow over the side weir (per length of the overflow crest).

The coefficient β_b relates here to the velocity $\vec{\upsilon}_b$ and it can be calculated using formula (3) and integrating along the elevation h of the layer of liquid above the weir. Thus it refers to the profile of the velocity $\vec{\upsilon}_b$ at the place indicated by the abscissa l, and not to a solid of velocities as in the sections I and II (figure 1).

It follows from formulae (15), (16) and (17) that the following change in momentum will occur in the time dt:

$$d\vec{\Pi}_{\mathrm{II}} + d\vec{\Pi}_b - d\vec{\Pi}_{\mathrm{I}} = d\vec{\Pi}, \tag{18}$$

which, after ordering and neglecting the terms containing products of two quantities having infinitesimal values, can be written as:

$$d\vec{\Pi} = \rho\,(\beta Q\Delta\upsilon\vec{\imath} + \beta\Delta Q\upsilon\vec{\imath} + \Delta\beta Q\upsilon\vec{\imath} + \beta_b q_b\Delta l\vec{\upsilon}_b)\,dt. \tag{19}$$

Surface and body forces. The unit surface forces here are:

- the hydrostatic pressure, increasing linearly towards the inside of the liquid, acting perpendicularly to the areas of the flow sections I and II;
- the shear stress $\vec{\tau}$ acting on the wetted surface of the channel's walls and the invert between the sections I and II.

The vector of a unit body force is the gravitational acceleration vector $\vec{g}$. The total force, called thrust, produced by the hydrostatic pressure in the sections I and II can be calculated on the basis of diagrams of thrust exerted on the walls of the area A and $A + \Delta A$. The numerical value of this force is equal to the weight of the diagram thrust solid. As it follows from figure 1, this force for the weir swelling curve is:

$$\Delta\vec{N} = -\left(\Delta HA + \frac{1}{2}\Delta H^2 b\right)\rho g \vec{\imath}, \tag{20}$$

where the unit vector $\vec{\imath}$ has the same direction as that of the mean velocity $\vec{\upsilon}$, and the minus sign is due to the fact that the depth of the liquid in the chamber in the section II is the sum of the elevatiom H in the section I and the increment (generally, positive or negative) ΔH. If the surface of liquid lowers along the weir (the depression curve), the direction of $\Delta\vec{N}$ agrees with $\vec{\imath}$, and the coordinate $-(\Delta HA + \Delta H^2 b/2)\rho g$ of the force $\Delta\vec{N}$ is positive.

The body force acting on the liquid mass considered in the central area assumes the following value (figure 1):

$$\Delta\vec{G} = \left(A\Delta l\cos\Theta + \frac{1}{2}\Delta H\Delta l\cos\Theta b\right)\rho\vec{g}. \tag{21}$$

The shear stresses $\vec{\tau}$ amount to the following resisting force:

$$\Delta\vec{T} = \left(O_z\Delta l\cos\Theta + \frac{1}{2}\Delta H\Delta l\cos\Theta\right)\vec{\tau}, \tag{22}$$

where O_z is the wetted perimeter in the section I. Thus the resultant of the surface and body forces can be expressed as follows:

$$\Delta\vec{W} = -A\Delta H\rho g\vec{\imath} + A\Delta l\cos\Theta\rho\vec{g} - O_z\Delta l\cos\Theta\vec{\tau}, \tag{23}$$

where the summands incorporating the products $\Delta l\Delta H$ and ΔH^2 are neglected as minor (in comparison with the minor first-order terms) second-order terms.

Equation of motion. The vector of momentum variation in the time dt, calculated using formula (19), is equal to resultant vector (23) of the forces $\Delta\vec{W}$ acting on the infinitesimal liquid volume ΔV considered. This means that the projection of these vectors onto the axis l satisfies the following equation:

$$\begin{aligned}\rho\left[(\beta Q\Delta\upsilon + \beta\Delta Q\upsilon + \Delta\beta Q\upsilon)\cos\Theta + \beta_b q_b\Delta lU\cos\Theta\right]\\ = -A\Delta H\rho g\cos\Theta + A\Delta l\rho g\sin\Theta\cos\Theta - O_z\Delta l\tau\cos\Theta,\end{aligned} \tag{24}$$

where U is a coordinate of the longitudinal component $\vec{U}$ of the velocity vector $\vec{\upsilon}_b$ of a lateral stream along direction of the mean velocity vector $\vec{\upsilon}$ of the main liquid stream in the channel.

Dividing both sides of equation (24) by the product $A\rho g\ \Delta l\cos\Theta$ we arrive at:

$$\frac{1}{gA}\left(\frac{\beta Q\Delta\upsilon+\beta\Delta Q\upsilon+\Delta\beta Q\upsilon}{\Delta l}+\beta_b q_b U\right)=-\frac{\Delta H}{\Delta l}+\sin\Theta-\frac{O_z\tau}{\rho gA}, \tag{25}$$

where:

$\sin\Theta\equiv i$ – bottom slope of the channel,

$O_z\tau/\rho gA = J$ – a hydraulic gradient (the magnitude of the energy loss due to frictional resistance per unit length) at $\tau\equiv|\vec{\tau}|$.

By introducing the above notations and performing lim operations at $\Delta l\to 0$ we obtain:

$$\frac{1}{gA}\left(\beta Q\frac{d\upsilon}{dl}+\beta\upsilon\frac{dQ}{dl}+\upsilon Q\frac{d\beta}{dl}+\beta_b q_b U\right)=-\frac{dH}{dl}+i-J. \tag{26}$$

It follows from the equation of the continuity of motion that $\upsilon = Q/A$, and hence:

$$\frac{d\upsilon}{dl}=\frac{d}{dl}\left(\frac{Q}{A}\right)=\frac{1}{A}\frac{dQ}{dl}-\frac{Q}{A^2}\frac{dA}{dl}, \tag{27}$$

where $dA/dl = bdH/dl$, i.e. the area increment ΔA along the path Δl occurs as a result of an increase in the height by ΔH, while the width of the overflow chamber is b (to be exact, $\Delta A = b\Delta H$, and after a limit transition we get $dA/dl = b\ dH/dl$). Thus:

$$\frac{d\upsilon}{dl}=\frac{1}{A}\frac{dQ}{dl}-b\frac{Q}{A^2}\frac{dH}{dl}, \tag{28}$$

and after inserting it into (26) and ordering we have:

$$\frac{dH}{dl}=\frac{i-J-\dfrac{1}{gA}\left(\dfrac{2\beta Q}{A}\dfrac{dQ}{dl}+\dfrac{Q^2}{A}\dfrac{d\beta}{dl}+\beta_b q_b U\right)}{1-\beta\dfrac{Q^2 b}{gA^3}}. \tag{29}$$

The volume flow over the length Δl of the side crest is:

$$\Delta Q=\frac{2}{3}\mu\Delta l\cos\Theta\sqrt{2g}(H-p)^{3/2}=\frac{2}{3}\mu\Delta l\cos\Theta\sqrt{2g}\,h^{3/2}, \tag{30}$$

where:

μ – a weir discharge coefficient (generally dependent on l),

h – the height of the layer of overflowing liquid measured on the channel axis relative to the elevation p of the overflow edge: $h = H - p$.

If we assume that $\cos\Theta = 1$ (e.g. $i\le 10‰ \Rightarrow \cos\Theta\ge 0.99995$) and that the limit transition $\Delta l\to 0$ we get (similarly as in equation (1)):

$$\frac{dQ}{dl} = q_b = \frac{2}{3}\mu\sqrt{2g}\,h^{3/2} = \frac{2}{3}\mu\sqrt{2g}\,(H - p)^{3/2}. \tag{31}$$

From the volume flow balance it follows that

$$Q = Q_d - \int_0^l q_b\,dl, \tag{32}$$

where Q_d is the volume flow in the channel at the beginning of the overflow ($l = 0$). Thus $dQ/dl = -q_b$ and after inserting it into equation (29) and ordering we get:

$$\frac{dH}{dl} = \frac{i - J - \dfrac{(2\beta\upsilon - \beta_b U)}{g\,A}\dfrac{dQ}{dl} - \dfrac{Q^2}{g\,A^2}\dfrac{d\beta}{dl}}{1 - \beta\dfrac{Q^2 b}{g\,A^3}}. \tag{33}$$

Equation (33) differs significantly from the equations of motion known so far, e.g. (9), in the momentum coefficient β_b next to U and in term

$$\frac{Q^2}{gA^2}\frac{d\beta}{dl}, \tag{34}$$

which expresses the influence of the variation in the momentum coefficient β along the overflow chamber on the value of the kinetic energy of the main liquid stream in the channel.

3. DIMENSIONLESS FORM OF THE EQUATION OF MOTION

The equation of motion in dimensionless form is convenient because its dimensionless coefficients constitute similarity numbers of the modelled and actual phenomena. If dimensionless variables in the form:

$$\zeta = \frac{H}{H_p}, \qquad \xi = \frac{l}{l_p}, \qquad q = \frac{Q(l)}{Q_d}, \tag{35}$$

are introduced, the appropriate derivatives, from formula (33), expressed as the new variables can be written as follows:

$$\frac{dH}{dl} = \frac{H_p}{l_p}\frac{d\zeta}{d\xi}, \qquad \frac{dQ}{dl} = \frac{Q_d}{l_p}\frac{dq}{d\xi}. \tag{36}$$

The area A of the overflow chamber's flow section can be transformed into a diagonal form in order to generalize the description, thus it is applicable to any shape of a cross-section. Therefore, it is assumed that the overflow chamber above the weir edge (crest) has a constant width equal to the width b of the inlet channel (or to its diameter $D = b$ if the channel is circular in shape), which agrees with the practice. Thus

$$A = A_0 + b(H - H_p) = A_0\left[1 + \frac{bH_p}{A_0}(\zeta - 1)\right] = A_0\left[K_0\zeta - (K_0 - 1)\right], \tag{37}$$

where:

A_0 – the surface area of the chamber's flow section at the beginning of the weir ($l = 0$);

K_0 – a coefficient which can be defined as a shape number ($K_0 = 1$ for a rectangular channel, whereas for other typical shapes of the channel $K_0 > 1$):

$$K_0 = \frac{bH_p}{A_0}. \tag{38}$$

After inserting (35)–(38) into (33) and ordering them we ultimately obtain the following dimensionless form of the modified equation of motion of a liquid in a channel with a side weir:

$$\frac{d\zeta}{d\xi} = \frac{L_0(i - J) - \left[\eta q \dfrac{dq}{d\xi} + q^2 \dfrac{d\beta}{d\xi}\right] \dfrac{Fr_0^2}{[K_0\zeta - (K_0 - 1)]^2}}{1 - \dfrac{\beta Fr_0^2 K_0 q^2}{[K_0\zeta - (K_0 - 1)]^3}}, \tag{39}$$

where:

L_0 – a relative length of the overflow edge: $L_0 = l_p/H_p$,

η – a substitute coefficient whose value should be determined experimentally (for a given weir and shape of the channel and set of motion parameters, $\eta = \eta(\xi)$):

$$\eta = 2\beta - k\beta_b, \tag{40}$$

q – a dimensionless volume flow of a liquid in the overflow chamber:

$$q = 1 - \frac{2}{3}\mu \frac{l_p H_p \sqrt{2gH_p}}{Q_d} \int_0^{\xi} (\zeta - P_0)^{3/2} d\xi, \tag{41}$$

μ – a weir discharge coefficient:

$$\mu = \frac{q_r}{\dfrac{2}{3}\dfrac{H_p^{\,5/2}}{Q_d}\sqrt{2g}\displaystyle\int_0^{L_o} (\zeta - P_0)^{3/2} d\xi}, \tag{42}$$

for $\xi = l/H_p$ and $\xi \in \langle 0, L_0 \rangle$; $q_r = Q/Q_d$ [20],

Fr_0 – the Froude number in the initial section of the overflow chamber ($l = 0$):

$$Fr_0^2 = \frac{Q_d^2}{A_0^{\ 2} g H_p}. \tag{43}$$

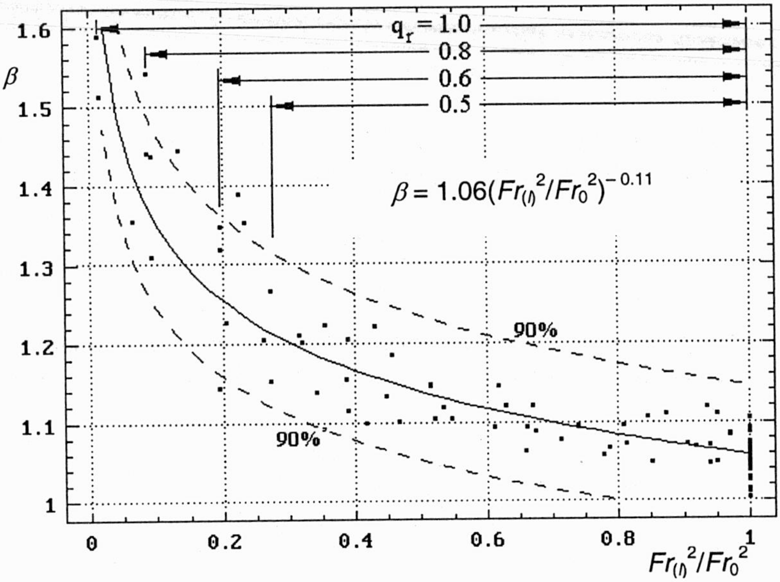

Fig. 2. Regression of coefficient β versus $Fr_{(l)}^{\ 2}/Fr_0^{\ 2}$ along overflow chamber [2]

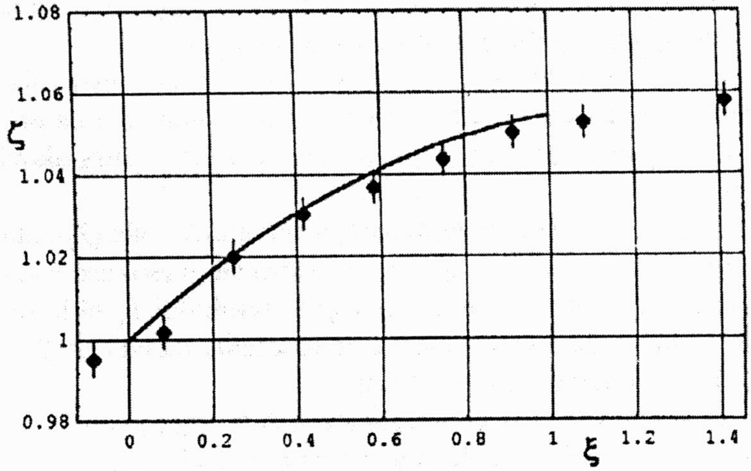

Fig. 3. Dimensionless elevation of free surface (ζ) along weir overflow chamber ($\xi \in$ <0; 1>), calculated (—) using equation of motion (39) and measured in the model (⧫) with marked measuring error [19]

In the case of subcritical flow over the side weir (the swelling curve), the dimensionless variables and coefficients (similarity numbers) in equation (39) satisfy the following relations:

$$\xi \geq 1 \quad \text{for} \quad 1 \geq (\xi = l/l_p) \geq 0 \quad \text{at} \quad 1 \geq q \geq 1 - q_r \quad \text{and} \quad Fr_0 < 1.$$

The suitability of equation (39) for the description of liquid flows in channels having side storm overflows with throttling pipe from the overflow chamber has been verified experimentally. Studies of local velocity distributions in overflow channels and chambers of different cross-sectional profiles have shown [2], [19] that the momentum coefficient β varies markedly, which applies both to β (figure 2) and the value of its derivative $d\beta/dl$, along the weir. This means that the new form of the equation of motion is not merely justified but necessary. Model studies on a scale of 1:5 have proved that equation (39) describes $d\zeta/d\xi$ accurately, i.e. dH/dl is within the measuring error of the elevation H in physical models (figure 3). A mathematical model describing the behaviour of weirs of this type and a numerical procedure for the hydraulic dimensioning of them have been developed [19], [21].

4. CONCLUSIONS

Hydraulic calculations of side weirs, i.e. the volumetric flow rate and the free-surface profile along the weir, still encounter difficulties since theoretical considerations lead to differential equations which cannot be solved analytically. One-dimensional equations of non-uniform flow with mass variation along the weir are an implicit function of the depth $H(l)$ and the volume $Q(l)$ in the channel and thus the direct integration of them for any shape of the overflow chamber's cross-section is practically impossible. The numerical solution of such problems has become attainable with the advent of high-speed computers.

Starting from the principle of conservation of momentum, a new form of the equation of motion (33) describing the free-surface profiles on the side weir has been derived. It differs significantly from the equations of motion known so far in the corrected mass decrement term and in the added momentum variation term $(Q^2/gA^2)(d\beta/dl)$. The equation has originated from experimental proof that the value of the momentum coefficient β of the main stream varies markedly along the overflow chamber ($\beta \in$ <1.0; 1.6> [2], [19]).

The dimensionless form of the modified differential equation of motion (39) constitutes a generalization of theoretical considerations concerning one-dimensional description of liquid flows with variation in mass and momentum and it is applicable to the hydraulic dimensioning of side weirs, including weirs with throttling pipe [21].

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WYPROWADZENIE RÓWNANIA RUCHU W KANAŁACH Z BOCZNYMI PRZELEWAMI

Przeprowadzono analizę znanych postaci równań ruchu cieczy niutonowskich w kanałach pryzmatycznych z bocznymi przelewami, wyprowadzonych z różniczkowej postaci równania Bernoulliego oraz z zasady zachowania pędu. Wychodząc z zasady ilości ruchu, wyprowadzono równanie ruchu nowej postaci z korektą w członie ubytku masy oraz dodatkowym członem zmiany pędu. Zmodyfikowane, bezwymiarowe równanie ruchu znajduje zastosowanie do opisu przepływów na przelewach bocznych, tj. do określania kształtu swobodnego zwierciadła cieczy oraz objętości przepływu.

równanie ruchu
kanały pryzmatyczne
przelewy boczne
równanie bernoulliego