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Effect of convergence acceleration of a series describing the wavefront phase on the accuracy of the hologram aberration estimation*

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Taking as an example an axial hologram recorded and reconstructed under the condition of limit of resolution the influence of the convergence acceleration of a series describing the wavefront phase on the convergence rate and thus on the accuracy of the aberration determination has been considered. Spherical aberration, coma and field curvature have been chosen for considerations, since they illustrate some extreme situations. It has been stated that the acceleration of series convergence is associated with a change in error distribution among the constant term of the expansion, the Gaussian sphere component and the rest responsible for the aberrations.

1. Introduction

The definition of the aberration is based on the expansion into binomial series of phases φ_q of wavefronts q taking part in imaging

$$\varphi_q = \frac{2\pi}{\lambda_q} z_q \left(\sqrt{1 + \left(\frac{\rho - \rho_q}{z_q}\right)^2} - \sqrt{1 + \left(\frac{\rho_q}{z_q}\right)^2} \right), \tag{1}$$

where q – index denoting the object wave o, reference wave r, reconstructing wave c or Gaussian wave G, respectively, ρ – coordinate in the hologram plane, ρ_q – coordinate of the source of the wave q in a plane parallel to the hologram plane.

The transformation of the square root suggested in [1]

$$\sqrt{1+\xi} = \sqrt{1+k}\,\sqrt{1 + \frac{\xi - k}{1+k}} \tag{2}$$

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plays a double role: by choosing k such that $|(\xi - k)/(1+k)| \leqslant 1$ it assures the summability of the series outside the classical binomial expansion and, simultaneously, it allows to accelerate the numerical calculations of aberrations.

By taking advantage of (2) the phase at a hologram point ρ of a wavefront emerging from the point source located at the position determined by the coordinate ρ_q may be expressed as follows [2]:

$$\varphi_q \approx \frac{2\pi}{\lambda_q} z_q \sqrt{1+k}\left\{\left[\binom{1/2}{1}\frac{1}{1+k} - \binom{1/2}{2}\frac{2k}{(1+k)^2} + \binom{1/2}{3}\frac{3k^2}{(1+k)^3}\right.\right.$$

$$\left. - \binom{1/2}{4}\frac{4k^3}{(1+k)^4}\cdots\right]\left[\left(\frac{\rho-\rho_q}{z_q}\right)^2 - \left(\frac{\rho_q}{z_q}\right)^2\right] + \left[\binom{1/2}{2}\binom{2}{0}\frac{1}{(1+k)^2}\right.$$

$$\left. - \binom{1/2}{3}\binom{3}{1}\frac{k}{(1+k)^3} + \binom{1/2}{4}\binom{4}{2}\frac{k^2}{(1+k)^4}\cdots\right]\left[\left(\frac{\rho-\rho_q}{z_q}\right)^4 - \left(\frac{\rho_q}{z_q}\right)^4\right]$$

$$+\left[\binom{1/2}{3}\binom{3}{0}\frac{1}{(1+k)^3} - \binom{1/2}{4}\binom{4}{1}\frac{k}{(1+k)^4}\cdots\right]\left[\left(\frac{\rho-\rho_q}{z_q}\right)^6 - \left(\frac{\rho_q}{z_q}\right)^6\right]$$

$$+\left[\binom{1/2}{4}\binom{4}{0}\frac{1}{(1+k)^4}\cdots\right]\left[\left(\frac{\rho-\rho_q}{z_q}\right)^8 - \left(\frac{\rho_q}{z_q}\right)^8\right]$$

$$\left.\cdots\cdots\cdots\cdots\right\}. \tag{3}$$

By expanding the binomials $(\rho - \rho_q)^{n+1}$, and next summing up the phases $\varphi_c \mp \varphi_r \pm \varphi_e - \varphi_G$, each of which being of the form (3), we obtain the aberration expressions.

2. The convergence of both the aberration sums and aberrations of various kinds

In the paper [2] the aberration structure has been shown in the form of a Pascal triangle. The sums of the terms in horizontal lines are sums of all the aberrations of given order, while the infinite sums taken in definite (diagonal) directions define the complete aberrations of given kind (spherical aberration, coma, etc.). Hence, the convergence rate may be understood in two ways: as a convergence rate of aberration sum (horizontal convergence), as a convergence rate of single aberrations (diagonal convergence). These mutual relations in the case of hologram aberrations are the subject of study in this work. The separation of the particular kinds of aberrations is reduced to respective transposition of the series terms. As it follows from the great transposition theorem the quick convergence of a row does not necessarily mean an equally quick convergence of the column of the series being summed [3].

The complete aberrations are determined by the infinite series. In practice, the calculations of the aberrations are resticted to several initial terms of the respective expansions, the rest being, at most, estimated. The transformation (2) and the expression (3) based on the latter, offer some convenience in calculations of complete aberrations allowing to calculate them relatively quickly and with high accuracy on the base of several initial terms of the series.

As it may be easily noticed, for any arbitrary k the infinite sums of polynomials occurring in (3) tend respectively to 1/2, -1/8, 1/16, etc., i.e., to the coefficients of the series expansion of the square root $\sqrt{1+\xi}$. For $k \to 0$ they are convergent to the same values. It seems, that for numerical purposes some compromise may be established between the choice of k and the greatest order n of aberration which would assure the required accuracy of the calculations.

The polynomials occurring in the expression (3) perform the part of the numerical weights ascribed to the aberrations of particular orders which appear in the sum defining the complete aberration of the given kind. For n and k fixed on the base of relation (3) it may be said that the reduction of the aberration error is gained at the expense of the Gaussian sphere error introduced purposely (Fig. 1). The introduction of k results in a distribution of the error among the constant term of expansion (1), the Gaussian term $(1/2\xi^2)$ and the rest creating the aberrations. Obviously, in numerical calculations the coordinates of the reference sphere are not determined anew to load them with an error.

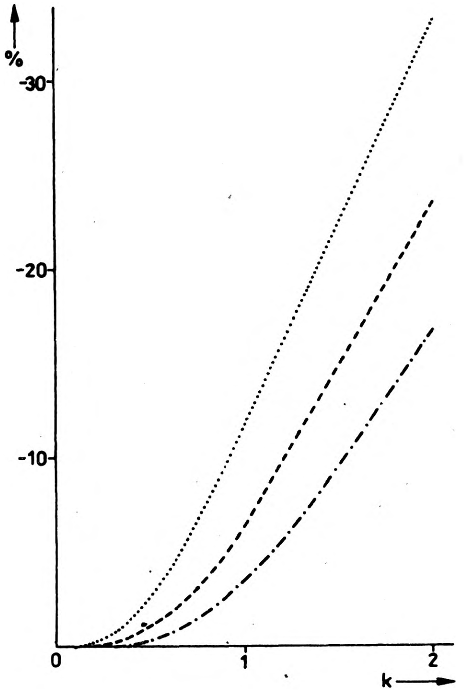


Fig. 1. The error of the part of the expansion of the wavefront phase Φ_q responsible for the Gaussian imaging, introduced as a result of the procedure accelerating the series convergence. The Roman numbers denote the highest order aberration taking part in the partial sum (.-.-.-.- XIII, - - - - XI, ... IX)

3. Example of an axial hologram

An acceleration of the convergence is advisable in the case of large apertures and thus in the case of great aberrations. In order to illustrate the problems discussed we have chosen a system of great aberrations operating at the resolution limit: $z_o = 1$, $z_r = 1.1\ z_o$, $z_c = 1.2\ z_o$, $\rho = z_o$, $\mu = 1$. The analysis performed concerned the aberration of extreme behaviour. The spherical aberration, coma and effective field curvature have been chosen to illustrate the extreme cases of dependences upon transversal coordinate of the wavefront sources $\rho_q(x_q, y_q)$. The Table 1 illustrates the convergence rate of aberrations and the advisability of its acceleration. All the aberrations are given in the $2\pi/\lambda_0$-units and concern the wavefront Φ_R.

Table 1. $x_o = 0.01\, z_e,\ y_e = 0$

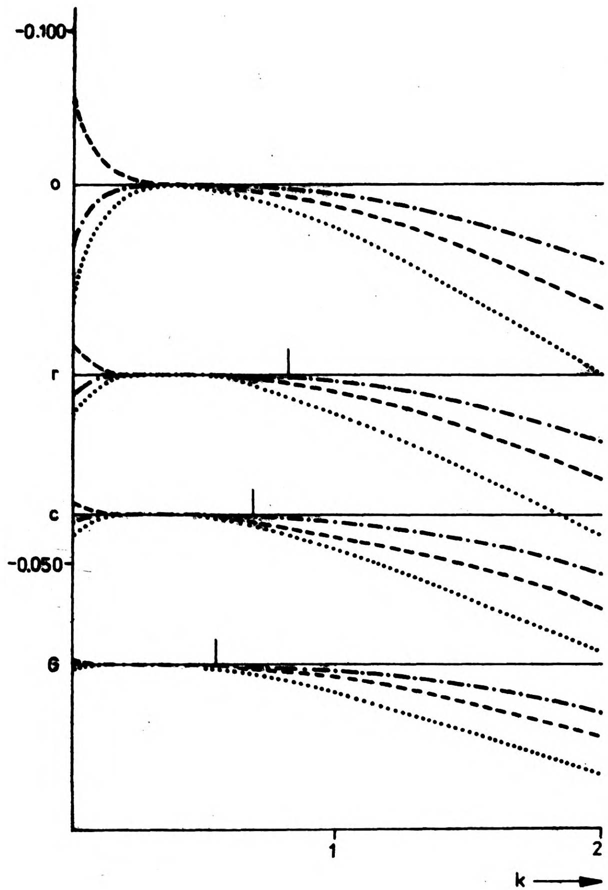
	Number of expansion terms (aberration order)	Value of aberration	Accurate value	Remarks
Spherical aberration	23 XLVII 24 XLIX	0.004 979 3 0.002 710 3	0.003 811 2	Accurate value has been estimated on the base of geometrical calculations
Coma	23 XLVII 24 XLIX	-0.100 081 6 -0.098 936 1	-0.099 450	Value estimated by extrapolation
	52 CV	-0.099 114 4 -0.099 870 0	-0.099 498 1	Value estimated according to the Euler scheme
Effective field curvature	11 XXIII	0.102 482 3	0.102 482 3	Rest less than 10^{-7} resulting from the truncated terms

3.1. Spherical aberration

From the fact that this aberration does not depend on the coordinates φ_q but only on ρ it follows that there is no need to distinguish the said two kinds of convergence, as they are identical. Figure 2 illustrates the k-dependent parts of the phase φ_q determining the aberrations, when aberrations from III to IX, and from III to XI, and from III to XIII, inclusively, are taken into account. The spherical aberration as a function of k is shown in Fig. 3, where some its fragments are enlarged. The accurate value of aberration given in the figure may be obtained by geometrical or analytical calculations, since it is the case of a series which may be summed up analytically [4].

As it may be seen from Fig. 3 this aberration is very slow converging. This is visible when three graphs for $k = 0$ are compared. With the increasing number of terms in partial sums describing the aberration the amplitude of oscillation around the accurate value lowers but is still great. However, this amplitude decreases quickly with growing k. If the error due to the truncated rest of the series is coarsly estimated on the base of the value and the sign of the first of the rejected terms, then in the case of calculations restricted to the IX, XI, and XIII orders inclusive the error is about $\pm 300\%$.

The application of the procedure accelerating the convergence makes the errors of scarcely few percents, starting from $k = 0.3$ to $k = 2$.



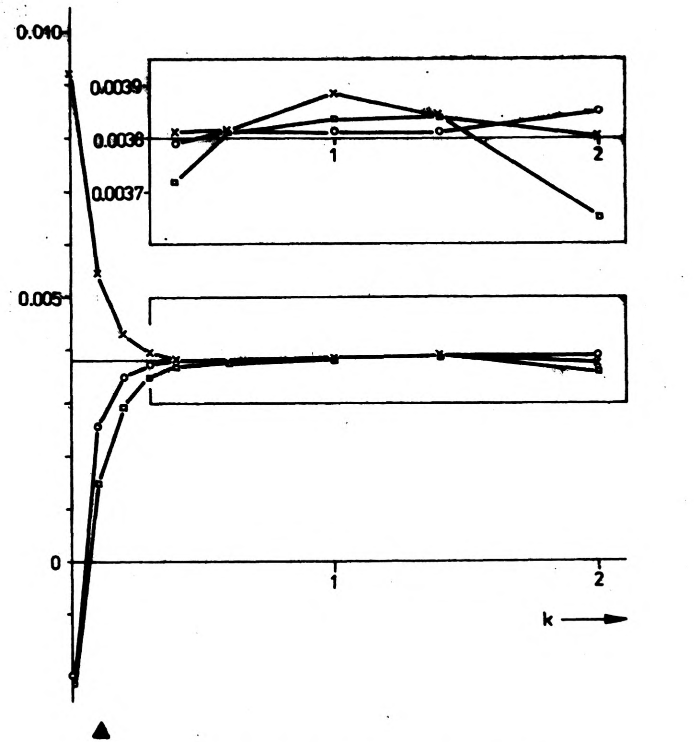


Fig. 3. Spherical aberration - influence of the acceleration of the series convergence on the aberration convergence. The horizontal continuous line denotes the accurate value (o——o XIII, x——x XI, ▫——▫ IX)

Fig. 2. The influence of the acceleration of the "horizontal" convergence of the series on the convergence of the part of the wavefront phases: object (o), reference (r), reconstructing (c), and Gaussian (G) ones, deciding about the spherical aberration. The continuous horizontal line denotes the accurate value obtained by geometrical calculation. The vertical lines denote the value of k, for which the horizontal convergence is theoretically the quickest (.-.-.- XIII, - - - - - XI, ... IX)

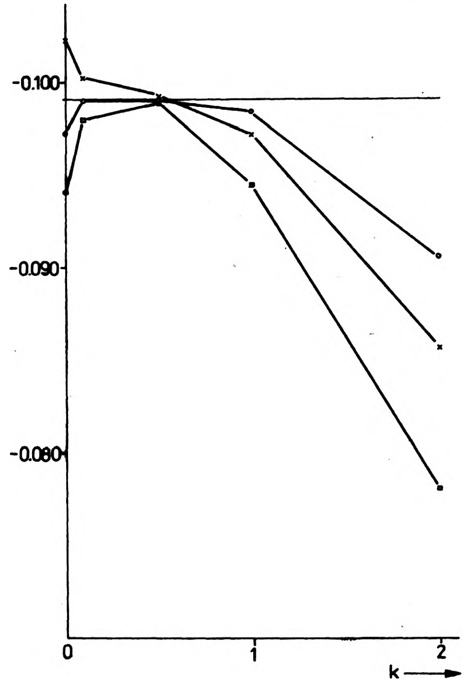


Fig. 4. Influence of the horizontal convergence acceleration on the convergence coma (o——o XIII, x——x XI, ▪——▪ IX)

3.2. Coma

In Table 1 the very slow convergence of this aberration is observed. The consistenee of two first meaning cyphres in the consecutive orders of aberration starts with n = 65 (partial sum composed of 32 terms). In comparison with the spherical aberration the amplitudes of the oscillations of the partial sum are small. If only the aberrations up to IX order are taken into account the accuracy of few percent may be achieved (comp. Fig. 4, k = 0), while a rough estimation of the error based on the first term of the rejected rest gives the value of 8.7%.

The effect of the horizontal convergence acceleration on the aberration convergence is illustrated in Fig. 4. The accurate value of aberration marked by horizontal continuous line and obtained by the way of extrapolation has been confirmed by numerical calculations according to Euler scheme [3] with the highest difference Δ^9 encountered.

3.3. Effective field curvature

This aberration is of rapid convergenee for a given position x_0 = 0.01 z_0 of the object (comp. Tab. 1). A quick increment of the constant numerical coefficients with the increase of aberration order (6, 15, 28 ..., $(n^2+n)/2$) is strongly damped by the factor x_q^{n-1}. Both the series, i.e., those describing the contribution to the aberration coming from either the object wave phase or the Gaussian sphere phase are highly convergent. Because of some inconveniences connected with the scale choice of the aberration errors (ranging from several percent to a fraction of promille), they are not presented graphically but listed in Table 2, giving their percentage values. However, double

Table 2

k	III-IX	III-XI	III-XIII
0.65	-9.2	-4.7	-2.4
0.55	-0.8	-0.25	-0.08
0.41	-0.25	-0.18	-0.17
0.29	-0.061	-0.19	-0.045
0.19	-0.012	-0.0032	-0.0006
0	-0.62	+0.16	-0.042

increase of the object point distance from axis $x_o = 0.02\ s_o$ results in a diametrally different behaviour of this aberration. Since, the lateral magnification of the system amounts to $M_{lat} = 66$, and the position of the point, being Gaussian conjugated with the object point amounts to $x_G = 1.32\ s_o$, a part of the Gaussian phase responsible for this aberration will be extremely slowly convergent ($s_G = 1.346939\ s_o$) and just this aberration will decide about the convergence rate for the total aberration. The situation may be easily estimated on the base of Fig. 5, for k = 0. In the figure the continuous horizontal line is used to mark the accurate value of aberration estimated according to the Euler scheme. The greatest difference taken for calculations is Δ^9.

The estimated value of aberration is equal to 0.236 271 5. In order to illustrate the slowness of the convergence the following data may be used: the partial sum composed of 76 terms (the highest order of the aberration taken into account n = 153) amounts to 0.154 461, but when the next term is in-

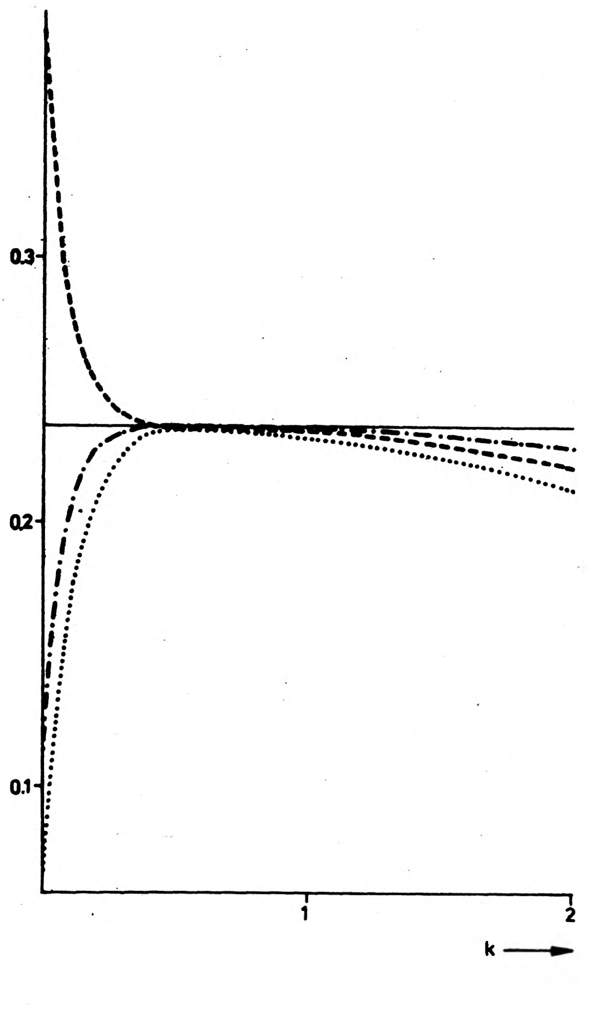


Fig. 5. Effective field curvature - influence of the horizontal convergence of aberration (.-.-.- XIII, - - - - XI, ... IX)

cluded it becomes 0.314 676. For a rough estimation of the series rest the rejection of further terms starting with the 77th one causes the error to become less than +68%.

The advantages offered by applying the accelerating procedure are visible in Fig. 5.

4. Modified coefficients of aberration

When restricting the calculations of aberrations to several orders, we face the problem of non-equal contribution of particular phases φ_q to a given aberration. The proportionality of the contribution of all the phases φ_q is assured only when the total aberration is calculated, i.e., when the infinite summation is performed.

It seems (as it was suggested in the work [2]) that it would be convenient to introduce the modified aberration coefficients which would allow an equal contribution of each of phases φ_q to the corresponding aberrational expressions. The method of convergence acceleration described in [1] may be employed to equalize the convergence rates of the series.

If we have two series

$$\sqrt{1+\xi} \approx \binom{1/2}{0} + \binom{1/2}{1}\xi + \binom{1/2}{2}\xi^2 + \ldots,$$

$$\sqrt{1+\xi'} \approx \binom{1/2}{0} + \binom{1/2}{1}\xi' + \binom{1/2}{2}\xi'^2 + \ldots,$$

the equal convergence rate is achieved when choosing k such that $(\xi'-k)/(1+k) = \xi$. The ratio of the rests truncated at the same place of the series to their sums will be the same in both the cases.

Generally, the aberration coefficients are of the form

$$W_n^{RV} = \sum_q \frac{\rho_q^p x_q^r y_q^s}{z_q^n}, \quad p, r, s = 0, 1, 2, \ldots \quad . \tag{4}$$

The coefficients modified in the above sense will have the form

$$W_{n_{mod}}^{RV} = \sum_q \alpha_q^{(n)} \frac{\rho_q^p x_q^r y_q^s}{s_q^n}, \qquad (5)$$

where $\alpha_q^{(n)} = \sqrt{1 + k_q}\, w_{n+1}$, and w_{n+1} are the polynomials occurring in the development (3).

The values k_q fitted individually to each of the phases φ_q will make their expansion into series satisfying the condition of equalizing both the convenience rates. Figures 6 and 7 illustrate the errors of spherical aberration and coma calculated according to the modified

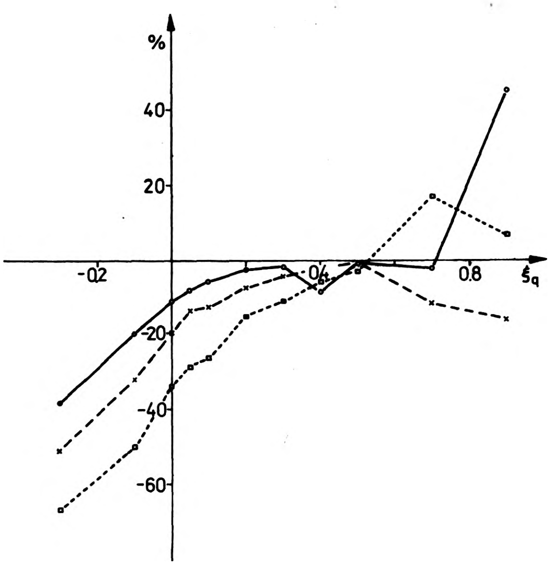


Fig. 6. The errors in calculation of the spherical aberration with the help of aberration coefficients modified according to (5) (o——• XIII, ▫——▫ XI x——x IX)

coefficients for the system like in Section 3. The results seem not to be so satisfactory, as it might be expected, though they are consistent with the intuition. The intervals of the values ξ_q, for which the errors of aberrations are of order of several percent are rather narrow, while the requirement that the errors for different aberrations do not exceed some single percent simultaneously, makes them even narrower.

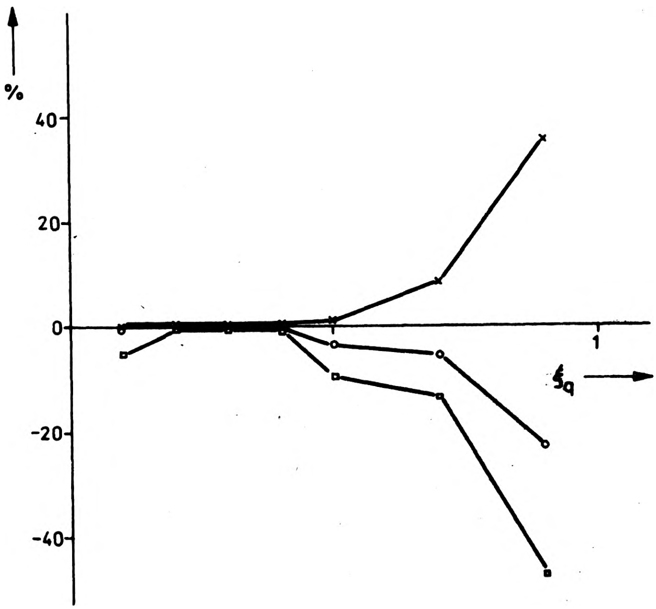


Fig. 7. Errors in calculation of coma by using the aberration coefficients modified according to (5) (o——o XIII, ×——× XI, ▫——▫ IX)

5. Conclusions

Theoretically, the quickest horizontal convergence occurs when $k = \xi_q$. In Fig. 2 the vertical lines denote these values for particular phases. In order to achieve the accurate values at these points it is necessary to take account of the infinite number of terms. In practice, when only the partial sums composed of several terms are calculated the accurate values are achieved for smaller values of k.

A comparison of Figs. 2 and 3 seems to be interesting (attention should be paid to different scales). An acceleration of horizontal convergence of the series describing all the phases o, r, c and G occurs within the whole range of values k. As it follows from Fig. 2 the absolute values of the errors of the components of the spherical aberration diminish at the beginning, and then grow up to exceed even the original values for k = 0. On the other hand, after performing the summation (Fig. 3) over the whole range of variation for k the aberration errors are much less. Thus, it may be spoken about as the local reduction of the error.

The usage of the modified coefficients of aberrations seems to be reasonable in the case when wavefront phases differ considerably. In the example discussed the values of the phases are relatively close to each other and the results may be considered to be satisfactory in the narrow range of values of ξ_q. Let us remark, however, that though the truncated rests of the series are proportional (with the same proportionality factor) to the corresponding phases, but this fact is associated additionally with the different error distribution between the constant term of the expansion and the Gaussian term.

It seems that at the resolution limit the interval of the values of k from 0.2 to 1 gives satisfactory results for all the aberrations considered, even when we confine ourselves to the partial sum containing IX order of aberration. A comparison with the aberration values for $k = 0$ illustrates the advantages which are offered when accelerating the convergence.

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ВЛИЯНИЕ УСКОРЕНИЯ СХОДИМОСТИ РЯДА, ОПИСЫВАЮЩЕГО ФАЗУ ВОЛНОВОГО ФРОНТА НА ТОЧНОСТЬ ОПРЕДЕЛЕНИЯ АБЕРРАЦИИ

На примере осевой голограммы в условиях записи и реконструкции на границе разделимости рассмотрено влияние ускорения сходимости ряда, описывающего фазу волнового фронта на скорость сходимости, а следовательно, и на точность определения аберрации. В качестве представляющего крайнее поведение была избрана сферическая аберрация, кома и эффективная кривизна поля. Было выявлено, что ускорение сходимости ряда сопровождается изменением распределения ошибок между постоянным членом разложения, компонентом гауссовой сферы и остальной частью, ответственной за аберрацию.