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Minimization of the second moment of the image intensity distribution*

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1. Introduction

Minimization of the second moment of the image intensity distribution produced by a point object leads to an improvement of the imaging quality (telescopes, microscope objectives, mirrors). In order to minimize the second moment of the intensity distribution in the image it is necessary to define an apodizing pupil function which may be done basing on the variational method [1-3]. In this paper an algorithm is given to determine a generalized pupil function minimizing the second moment of the intensity distribution in the image for a fixed value of energy transferred through the optical system with spherical aberration.

2. Theory

The complex amplitude $U(x_0, y_0)$ in the image plane is a Fourier transform $\mathscr{F}$ of the generalized pupil function $T(x_1, y_1)$ [4]:

$$U_0(x_0, y_0) = C\mathscr{F}\left\{T(x_1, y_1)\right\}. \tag{1}$$

Here, the pupil function is a product of the pupil function $U(x_1, y_1)$ and the phase factor

$$T(x_1, y_1) = U(x_1, y_1)\exp\left[\frac{ik}{2z}(x_1^2 + y_1^2) + i\,\Phi(x_1, y_1)\right], \tag{1a}$$

where x_0, y_0 and x_1, y_1 - the coordinates in the image and pupil planes, respectively, z - the distance of the exit pupil from the image plane, k - the wave number, Φ - the wave aberration, C - the coefficient of the form

$$C = \frac{1}{i\lambda z}\exp(ikz)\exp\left[\frac{ik}{2z}(x_0^2 + y_0^2)\right]. \tag{2}$$

The pupil function is assumed in the form

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$$U(x_1, y_1) = \begin{cases} U_0(x_1, y_1) & \text{in the pupil,} \\ 0 & \text{beyond the pupil,} \end{cases} \tag{3}$$

which may be also written as follows

$$U(x_1, y_1) = U_0(x_1, y_1)P_0(x_1, y_1), \tag{4}$$

where $P_0(x_1, y_1)$ is an aperture function.

After simple rearrangements we obtain from (1)

$$\int\limits_{-\infty}^{+\infty}\int\left\{\left|\frac{\partial T(x_1, y_1)}{\partial x_1}\right|^2 + \left|\frac{\partial T(x_1, y_1)}{\partial y_1}\right|^2\right\}dx_1\, dy_1 = \int\limits_{-\infty}^{+\infty}\int(x_0^2 + y_0^2)\left|U(x_0, y_0)\right|^2 dx_0\, dy_0 = \sigma \tag{5}$$

The right hand side of the equation (5) is an expression determining the second moment of intensity in the image.

If we restrict our attention to a point object positioned on the axis and to the system with circular aperture of the radius a, then it is convenient to write the aperture function (5) in the polar coordinates

$$\int\limits_0^\infty \left|\frac{\partial T(r)}{\partial r}\right|^2 r dr = \int\limits_0^\infty |U(s)|^2 s^3\, ds. \tag{6}$$

After substituting (1) and (4) into (6) we obtain

$$\sigma = \int\limits_0^\infty \left|\frac{\partial T(r)}{\partial r}\right|^2 rdr = \left\{\int\limits_0^a\left|\frac{\partial U_0(r)}{\partial r}\right|^2 + U_0^2(r)\left[\frac{\partial}{\partial r}\left(\exp\left(\frac{ikr^2}{2z} + i\Phi(r)\right)\right)\right]^2\right\}$$

$$+ aU_0^2(r)_{r=a}\,\delta(0) + \left(U_0(r)\,\frac{\partial U_0(r)}{\partial r}\right)^2_{r=a} a\,\delta(0) + \frac{a}{2}\,\frac{\partial}{\partial r}\left(U_0^2(r)\right)_{r=a} \tag{7}$$

$$+ aU_0^2(r)\,\frac{\partial}{\partial r}\left[\exp\frac{ikr^2}{2z} + i\Phi(r)\right]_{r=a}$$

Since σ must take a finite value, condition necessary for the pupil function resulting from (7) is

$$U_0(a) = 0. \tag{8}$$

When assuming the passivity of the system we get

$$\left|U_0(r)\right| \leqslant 1. \tag{9}$$

After applying the Parseval theorem to equation (1) the expression for energy passing through the aperture has the form

$$E = \int\limits_0^1 \left|U_0(r)\right|^2 rdr \tag{10}$$

The spherical wave aberration is well approximated by the polynomial

$$\Phi = \sum_{i=1}^{4} a'_{2i}\, r^{2i} \, . \tag{11}$$

Finally, from (7) and (11) we obtain

$$\sigma = \int_0^1 \left\{ \left| \frac{\partial U_0(r)}{\partial r} \right|^2 + U_0^2(r)\left[2a_2 r + 4a_4 r^3 + 6a_6 r^5 + 8a_8 r^7\right]^2 \right\} r\,dr, \tag{12}$$

where

$$a_2 = a'_2 + \frac{\pi}{\lambda z}\,, \quad a_4 = a'_4, \quad a_6 = a'_6, \quad a_8 = a'_8. \tag{12a}$$

Thus, the problem is reduced to finding a function $U_0(r)$ satisfying the conditions (8) and (9) for the specified value E minimizing the moment σ.

From the variational calculus [5] it follows that the function

$$F = \frac{1}{2}\left\{\left(\frac{\partial U_0(r)}{\partial r}\right)^2 + \frac{1}{2} U_0^2(r)[(2a_2 r + 4a_4 r^3 + 6a_6 r^5 + 8a_8 r^7)^2 - \lambda] + \mu_1(r)[1 - U_0(r)] + \mu_2(r)(1 + U_0(r))\right\} r, \tag{13}$$

(where λ, $\mu_1(r)$, $\mu_2(r)$ are Lagrange factors), satisfies the Euler-Lagrange equation

$$\frac{\partial F(r, U_0(r), U'_0(r))}{\partial U_0(r)} - \frac{d}{dr}\frac{\partial F(r, U_0(r), U'_0(r))}{\partial U'_0(r)} = 0, \quad r \neq r_0, \tag{14}$$

where $U'_0(r) = \partial U_0(r)/\partial r$, except for the boundary point r_0.

At the point r_0 the boundary condition

$$\lim_{r \to r_0 + 0} \left| \frac{\partial F}{\partial U'_0} \right| = \lim_{\to r_0 - 0} \left| \frac{\partial F}{\partial U'_0} \right| \tag{15}$$

must be fulfilled. The Lagrange factors must satisfy the following relations:

$$\begin{aligned} &\lambda \geqslant 0, \\ &\mu_1(r) \leqslant 0, \quad \mu_1(r)[1 - U_0(r)] \equiv 0, \\ &\mu_2(r) \leqslant 0, \quad \mu_2(r)[1 + U_0(r)] \equiv 0, \end{aligned} \tag{16}$$

Let us consider the following cases:

High energy losses in the system

When the total energy passing through the system is small, then $\mu_1(r) = 0$, $\mu_2(r) = 0$, as it follows from the condition (16), and the disturbance $U_0(r)$ fulfils the following differential equation

$$r^2 U''_0(r) + r U'_0(r) - [(2a_2 r + 4a_4 r^3 + 6a_6 r^5 + 8a_8 r^7)^2 + \lambda]\, U_0(r) = 0\,. \tag{17}$$

The solution of this equation is given by the following pupil function

$$U_0(r) = AU_1(r) + BU_1(r)\left[\ln r + \sum_{\nu=1}^{\infty}\left(\left(-\frac{1}{2\nu}\right)\frac{1}{c_{2\nu}}\, r^{-2\nu}\right)\right], \tag{18}$$

where

$$U_1(r) = \sum_{\nu=0}^{\infty}\left(k_{2\nu}\, r^{2\nu}\right), \quad k_0 = 1, \tag{18a}$$

and the expansion coefficients are of the form

$$k_2 = \begin{cases} \dfrac{1}{(2\nu)^2} \displaystyle\sum_{i=1}^{\nu}\left(b_{2i}\, k_{2(\nu-i)}\right), & 2\nu \leqslant 16, \\[2ex] \dfrac{1}{(2\nu)^2} \displaystyle\sum_{i=1}^{8}\left(b_{2i}\, k_{2(\nu-i)}\right), & 2\nu > 16, \end{cases} \tag{18b}$$

$b_2 = -\lambda$, $b_4 = 4a_2^2$, $b_6 = 16a_2a_4$, $b_8 = 24a_2a_6 + 16a_4^2$, $b_{10} = 32a_2a_8 + 48a_4a_6$, $b_{12} = 64a_4a_8 + 36a_6^2$, $b_{14} = 96a_6a_8$, $b_{16} = 64a_8^2$,

$$c_{2\nu} = \sum_{i=1}^{\nu}\left(k_{2i}k_{2(\nu-i)}\right),$$

The second solution of the equation (18) tends to ∞, when $r \to 0+$, and thus the solution of the equation (17) within the interval $0 \leqslant r \leqslant 1$ is the pupil function

$$U_0(r) = A\sum_{\nu=0}^{\infty}\left(k_{2\nu}\, r^{2\nu}\right). \tag{19}$$

The constant A is calculated from the energy conditions, and is equal to

$$A = \sqrt{\frac{E}{\pi \displaystyle\sum_{\nu=0}^{\infty}\left(c_{2\nu}\frac{1}{(\nu+1)}\right)}}\,. \tag{19a}$$

From the formula (19) it follows that $U_0(r)$ changes within the limits

$$0 = U_0(1) \leqslant U_0(r) < U_0(0) = \sqrt{\frac{E}{\pi \displaystyle\sum_{\nu=0}^{\infty}\left(\frac{c_{2\nu}}{\nu+1}\right)}}\ \sum_{\nu=0}^{\infty}(k_{2\nu}) < 1; \tag{20}$$

thus the energy will be changed within this interval as follows

$$0 \leqslant E < \frac{2\pi \sum_{\nu=0}^{\infty} \frac{c_{2\nu}}{2\nu+2}}{\left(\sum_{\nu=0}^{\infty} (k_{2\nu})\right)^2} . \tag{21}$$

It remains to find the solution for the energy interval

$$\frac{\pi \sum_{\nu=0}^{\infty} \frac{c_{2\nu}}{\nu+1}}{\left(\sum_{\nu=0}^{\infty} (k_{2\nu})\right)^2} \leqslant E < 1/2. \tag{22}$$

Low energy losses in the system

Let us choose the following representation of the pupil function

a. $0 \leqslant r \leqslant r_0$, where $U_0(r) = 1$,

b. $r_0 < r < 1$, where $|U_0(r)| < 1$.

In the interval $0 \leqslant r \leqslant r_0$, $U_0'(r) = 0$ and for the equation (16) we have

$$\mu_2(r) = 0,$$

$$\frac{d}{dr}(rU'(r)) = \left\{-\lambda U(r) - \mu_1(r) + \mu_2(r)\right\} r,$$

$$\mu_1(r) = -\lambda .$$

At the point $r = r_0$ the right hand side of the equation (15) is equal to zero, i.e.

$$\lim_{r \to r_0 - 0} \left(\frac{\partial F}{\partial U_0'}\right) = 0.$$

From the equation $U_0(r) = 1$ it follows

$$\lim_{r \to r_0 - 0} U_0(r) = 1. \tag{23}$$

In the interval $r_0 \leqslant r < 1$ the pupil function fulfils the passivity condition $|U_0(r)| < 1$, and the Lagrange variables are equal to zero. The solution of the Euler-Lagrange equation gives then the following expression for the pupil function

$$U_0(r) = AU_1(r) + BU_1(r)\left[\ln r + \sum_{\nu=1}^{\infty} \left(\frac{1}{c_{2\nu}} \frac{r^{-2\nu}}{(-2\nu)}\right)\right], \tag{24}$$

where

$$U_1(r) = \sum_{\nu=0}^{\infty} \left(k_{2\nu} r^{2\nu}\right).$$

Now, the constants A, B, and λ must be determined.

From the condition $U(1) = 0$ we obtain

$$A \sum_{\nu=0}^{\infty} (k_{2\nu}) - B \sum_{\nu=0}^{\infty} (k_{2\nu}) \sum_{\nu=1}^{\infty} \frac{1}{2\nu\, c_{2\nu}} = 0. \qquad (25)$$

From the condition

$$\lim_{r \to r_0 + 0} \frac{\partial F}{\partial U_0'} = \lim_{r \to r_0 + 0} \left\{ r\, U_0'(r) \right\} = 0$$

it follows

$$A \sum_{\nu=0}^{\infty} \left(2\nu\, k_{2\nu}\, r_0^{2\nu-1}\right) - B \left\{ \sum_{\nu=0}^{\infty} \left(2\nu\, k_{2\nu}\, r_0^{2(\nu-1)}\right) \left[\ln r_0 + \sum_{\nu=1}^{\infty} \left(\frac{r_0^{-2\nu}}{2\nu\, c_{2\nu}} \right) \right] \right.$$

$$\left. + \sum_{\nu=0}^{\infty} \left(k_{2\nu}\, r_0^{2\nu}\right) \left[\frac{1}{r_0} + \sum_{\nu=1}^{\infty} \left(\frac{r^{-2\nu-1}}{2\nu\, c_{2\nu}} \right) \right] \right\} = 0. \qquad (26)$$

By letting the determinant of the expressions (25) and (26) be equal to zero the value of λ may be determined

$$\det \left| \quad \right| = 0 \longrightarrow \lambda.$$

The condition

$$\lim_{r \to r_0 + 0} U_0(r) = 1$$

gives

$$A \sum_{\nu=0}^{\infty} \left(k_{2\nu}\, r_0^{2\nu}\right) - B \sum_{\nu=0}^{\infty} \left(k_{2\nu}\, r_0^{2\nu}\right) \left[-\ln r_0 + \sum_{\nu=1}^{\infty} \left(\frac{1}{2\nu\, c_{2\nu}}\, r_0^{-2\nu} \right) \right] = 1. \qquad (27)$$

From the system of equations (25), (26) and (27) the coefficients A and B in the equation (24) may be determined. They are equal respectively to:

$$A = \frac{\tilde{N}_1}{\tilde{J}_0 \tilde{N}_1 - \tilde{J}_1 \tilde{N}_0}, \quad B = \frac{\tilde{J}_1}{\tilde{J}_0 \tilde{N}_1 - \tilde{J}_1 \tilde{N}_0} \qquad (28)$$

where $\tilde{N}_1$ is the aberrational Neuman function of first order equal to (28a)

$$\tilde{N}_1 = \sum_{\nu=0}^{\infty} \left(2\nu\, k_{2\nu}\, r_0^{2\nu-1}\right) \left[\ln r_0 + \sum_{\nu=1}^{\infty} \left(\frac{r_0^{-2\nu}}{2\nu\, c_{2\nu}} \right) \right] + \sum_{\nu=0}^{\infty} \left(k_{2\nu}\, r^{2\nu}\right) \left[\frac{1}{r_0} + \sum_{\nu=1}^{\infty} \left(\frac{r_0^{-2\nu-1}}{2\nu\, c_{2\nu}} \right) \right],$$

$\tilde{J}_1$ denotes the aberrational Bessel function of first order

$$\tilde{J}_1 = \sum_{\nu=1}^{\infty} \left(2\nu\, k_{2\nu} r^{2\nu-1}\right). \tag{28b}$$

$\tilde{N}_0$ and $\tilde{J}_0$ denote, respectively, the aberrational Neuman function and aberrational Bessel function both of zero order, i.e.

$$\tilde{N}_0 = \sum_{\nu=0}^{\infty}{}' \left(k_{2\nu}\, r_0^{2\nu}\right) \left[-\ln r_0 + \sum_{\nu=1}^{\infty} \left(\frac{r_0^{-2\nu}}{2\nu\, c_{2\nu}}\right)\right], \tag{28c}$$

$$\tilde{J}_0 = \sum_{\nu=0}^{\infty} \left(k_{2\nu}\, r_0^{2\nu}\right), \tag{28d}$$

r_0 is determined by the energy expression

$$E = \int_0^{r_0} r\,dr + \int_{r_0}^{1} U_0^2(r)\,r\,dr, \tag{29}$$

for $U_0(r)$ given by the equation (24) with the coefficients determined by the equation (28). Thus we obtain

$$E = \frac{1}{2}\, \frac{\tilde{\tilde{J}}_1 \tilde{N}_1 - \tilde{\tilde{N}}_1 \tilde{J}_1}{\tilde{J}_0 \tilde{N}_1 - \tilde{J}_1 \tilde{N}_0}, \tag{30}$$

where the constants $\tilde{\tilde{J}}_1$, $\tilde{\tilde{N}}_1$ are given by

$$\tilde{\tilde{J}}_1 = \sum_{\nu=0}^{\infty} \left(2\nu\, k_{2\nu}\right). \tag{30a}$$

and respectively

$$\tilde{\tilde{N}}_1 = \sum_{\nu=0}^{\infty} \left(2\nu\, k_{2\nu}\right) \sum_{\nu=1}^{\infty} \left(\frac{1}{-2\nu\, c_{2\nu}}\right) \left[1 + \sum_{\nu=1}^{\infty} \left(\frac{1}{c_{2\nu}}\right)\right] \sum_{\nu=0}^{\infty} \left(k_{2\nu}\right). \tag{30b}$$

A generalized analytical form of the pupil function has been obtained for the optical system with spherical aberration. In the case of low energy transmitted by the system this function is represented by the equation (19), while for the case of high energy passing through the system it is represented by a constant pupil function $U(r) = 1$ within the interval $0 \leqslant r \leqslant r_0$, and by a function determined by the equation (24) with the coefficients given by the equation (28) within the interval $r_0 \leqslant r \leqslant 1$.

In the far field case of an aberration-free system with $\sum_{\nu=1}^{8} b_{2\nu} r^{2\nu}$ there remains only $b_2 = \lambda$ and the equation (17) takes the form

$$r^2 U_0''(r) + r U_0'(r) - U_0(r)\lambda = 0. \tag{31}$$

Thus the following has been obtained:

i) For small values of energy the expression (reported earlier by Asakura) for an optimal pupil function minimizing the second moment in the interval $0 \leqslant r \leqslant 1$ has the form

$$U_0(r) = A \sum_{\nu=0}^{\infty} a_k r^k = A \sum_{\nu=0}^{\infty} \frac{(-1)^k}{k!}\left(\frac{\sqrt{-\lambda}\, r}{2}\right)^{2k} = AJ_0(\sqrt{-\lambda}\, r). \qquad (32)$$

From the condition $U_0(1) = 0$ the Legendre constant is evaluated, while from the condition for energy conservation the constant A is estimated to be

$$A = \sqrt{\frac{2E}{J_1(p_1)}}\,. \qquad (32a)$$

Thus, within the interval $0 \leqslant r \leqslant 1$ the energy changes within the limits

$$0 \leqslant E < \frac{1}{2} J_1^2(p_1). \qquad (33)$$

ii) For high energies, i.e., for

$$\frac{1}{2} J_1^2(p_1) \leqslant E < \frac{1}{2} \qquad (34)$$

the pupil function is constant $U_0(r) = 1$ within the interval $0 \leqslant r \leqslant r_0$.

For $r_0 \leqslant r \leqslant 1$ the pupil function is defined by the Bessel and Neuman functions of zero order

$$U_0(r) = AJ_0(\sqrt{\lambda}\, r) - BN_0(\sqrt{\lambda}\, r), \qquad (35)$$

while the coefficients A and B are expressed by the Bessel and Neuman functions of zero and first orders:

$$A = \frac{N_1(\sqrt{\lambda}\, r_0)}{J_0(\sqrt{\lambda}\, r_0)N_1(\sqrt{\lambda}\, r_0) - J_1(\sqrt{\lambda}\, r_0)N_0(\sqrt{\lambda}\, r_0)}, \qquad (35a)$$

$$B = \frac{J_1(\sqrt{\lambda}\, r_0)}{J_0(\sqrt{\lambda}\, r_0)N_1(\sqrt{\lambda}\, r_0) - J_1(\sqrt{\lambda}\, r_0)N_0(\sqrt{\lambda}\, r_0)}, \qquad (35b)$$

is estimated from the equations (25) and (26), which for this case give the condition

$$J_0(\sqrt{\lambda})N_1(\sqrt{\lambda}\, r_0) - J_1(\sqrt{\lambda}\, r_0)N_0(\sqrt{\lambda}) = 0. \qquad (36)$$

r_0 should be evaluated from the expression for energy

$$E = \frac{1}{2}\,\frac{J_1(\sqrt{\lambda})N_1(\sqrt{\lambda}\,r_0) - J_1(\sqrt{\lambda}\,r_0)N_1(\sqrt{\lambda})}{J_0(\sqrt{\lambda}\,r_0)N_1(\sqrt{\lambda}\,r_0) - J_1(\sqrt{\lambda}\,r_0)N_0(\sqrt{\lambda}\,r_0)}. \tag{37}$$

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