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# Apodization by a set of slits in one- and two-dimensional optical systems *

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In the paper [1] the influence of selected amplitude apodizers on the resolving power of a two-point image was analysed. The critical values of two-point resolution distance were determined when using the Rayleigh criterion. In this paper the internity distributions of slits image obtained by using apodizers in an incoherent optical system are analysed as a supplement to the previous considerations.

A stationary and linear optical systems with incoherent illuminator may be described as a convolution of intensity distribution $I(x)$ in the object with an intensity point-spread function $S(x)$ of the optical system

$$I(x') = \int_{-\infty}^{\infty} I(x)\ S(x'-x)\,dx. \tag{1}$$

The relation between the point spread function $S(x, y)$ and the pupil function $T(\xi, \eta)$ is given by the squared modulus of the two-dimensional Fourier transform. This relation may be simplified to the squared of the one dimensional Fourier transform in the case of a one-dimensional system

$$S(x') = \left| \int_{-\infty}^{\infty} T(\xi)\exp\left(\frac{ikx'\xi}{f}\right) d\xi \right|^2 \tag{2}$$

* This work was carried on under the Research Project M.R. I.5.

and to the squared modulus of the Hankel transform in the case of rotational symmetry of the system

$$S(r') = \left| \int_{-\infty}^{\infty} T(\varrho) J_0\left(\frac{k\varrho r'}{f}\right) \varrho d\varrho \right|^2. \tag{3}$$

For the amplitude apodizers of the type

$$\begin{aligned}
&\text{I} \quad T(\xi) = \begin{cases} 1 & |\xi| \leqslant a \\ 0 & |\xi| > a \end{cases}, \\
&\text{II} \quad T(\xi) = \begin{cases} \frac{1}{2}(1+\xi^2) & |\xi| \leqslant a \\ 0 & |\xi| > a \end{cases}, \\
&\text{III} \quad T(\xi) = \begin{cases} 1-\xi^2 & |\xi| \leqslant a \\ 0 & |\xi| > a \end{cases}.
\end{aligned} \tag{4}$$

the intensity point-spread functions take the forms [2]:

— *for the slit pupil*

$$\begin{aligned}
S_{\text{I}}(x') &= \left[\frac{2\sin x'}{x'}\right]^2, \\
S_{\text{II}}(x') &= \left[\frac{2\sin x'}{x'} + \frac{2\cos x'}{x'^2} - \frac{2\sin x'}{x'^3}\right]^2, \\
S_{\text{III}}(x') &= \left[\frac{4\sin x'}{x'^3} - \frac{4\cos x'}{x'^2}\right]^2,
\end{aligned} \tag{5}$$

— *for the circular pupil*

$$\begin{aligned}
S_{\text{I}}(r') &= \left[\frac{2J_1(r')}{r'}\right]^2, \\
S_{\text{II}}(r') &= \left\{J_1(r')\left[\frac{1}{r'} - \frac{4}{r'^3}\right] + \frac{2}{r'^2} J_0(r')\right\}^2, \\
S_{\text{III}}(r') &= \left[\frac{8J_1(r')}{r'^3} - \frac{4J_0(r')}{r'^2}\right]^2.
\end{aligned} \tag{5a}$$

In view of Eq. (1) the image intensity distribution for an incoherently illuminated slit-object of width $2b$ takes the forms:

— *in the one-dimensional system*

$$I(x') = I_0 \int_{x'-b}^{x'+b} S^2(x') dx' \tag{6}$$

where: $x' = (kX'a)/f$ – normed image coordinate (the aperture a was normed to 1 at the edge), $S$ – the point spread function of an apodizer

— *in the two-dimensional system*

$$I(r') = I_0 \int\limits_{r'-b}^{r'+b} S^2(r')dr'. \tag{7}$$

The numerical results concerning the intensity distribution in the slit image depending on the type of amplitude apodization are shown in Figs. 1a–d and 4a–d for one-dimensional and two-dimensional systems, respectively, the respective slit width $2b$ being equal to 1, 2, 3 and 6. In Figs. 1e–h the intensity distributions in the image of a slit of the widths: $2b = 1, 2, 3, 6$ are shown for apodizers I, II, III in the one-dimensional system (the intensities being normed to the value 1 at the slit centre), while in Figs. 4e–h – those for the two-dimensional system.

For the object consisting of two slits, equal in width their spacing being equal to the width, described by the function

$$I(x) = \begin{cases} 0 & |x| < b \\ I_0 \ \ 0 \leqslant |x| \leqslant 3b \\ 0 & |x| > 3b \end{cases} \tag{8}$$

the intensity distribution in the image is equal to:

— *for one-dimensional case*

$$I(x') = I_0 \int\limits_{x'-3b}^{x'+3b} S^2(x')dx' - I_0 \int\limits_{x'-b}^{x'+b} S^2(x')dx', \tag{9}$$

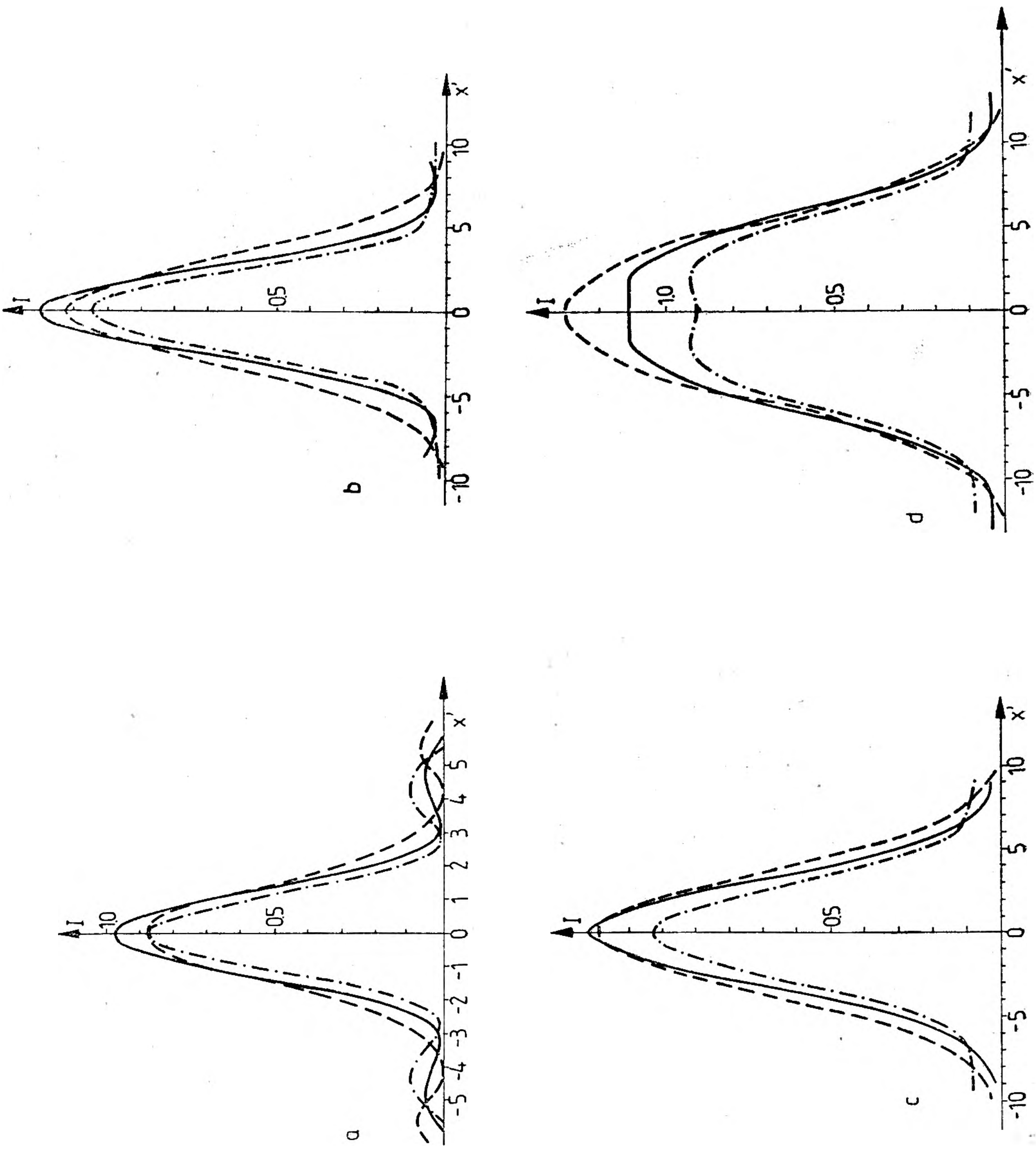
— *for two-dimensional case*

$$I(r') = I_0 \int\limits_{r'-3b}^{r'+3b} S^2(r')dr' - I_0 \int\limits_{r'-b}^{r'+b} S^2(r')dr'. \tag{10}$$

In the Figs. 2a–d the intensity distributions in the image of two slits for $2b = 1, 2, 3$ and 6 are shown for one-dimensional system and apodization I, II, III, for the two-dimensional system being given in Figs. 5a–d. The correspondingly normalized intensity distributions for the one-dimensional system are shown in Figs. 2e–h for one-dimensional system and in Figs. 5e–h – for two-dimensional system.

The light intensity distribution in the image of three-slit object described by the function

$$I(x) = \begin{cases} I_0 & |x| \leqslant b \\ 0 \ \ 0 < & |x| \leqslant 3b \\ I_0 \ \ 3b \leqslant & |x| \leqslant 5b \\ 0 & |x| > 5b \end{cases} \tag{11}$$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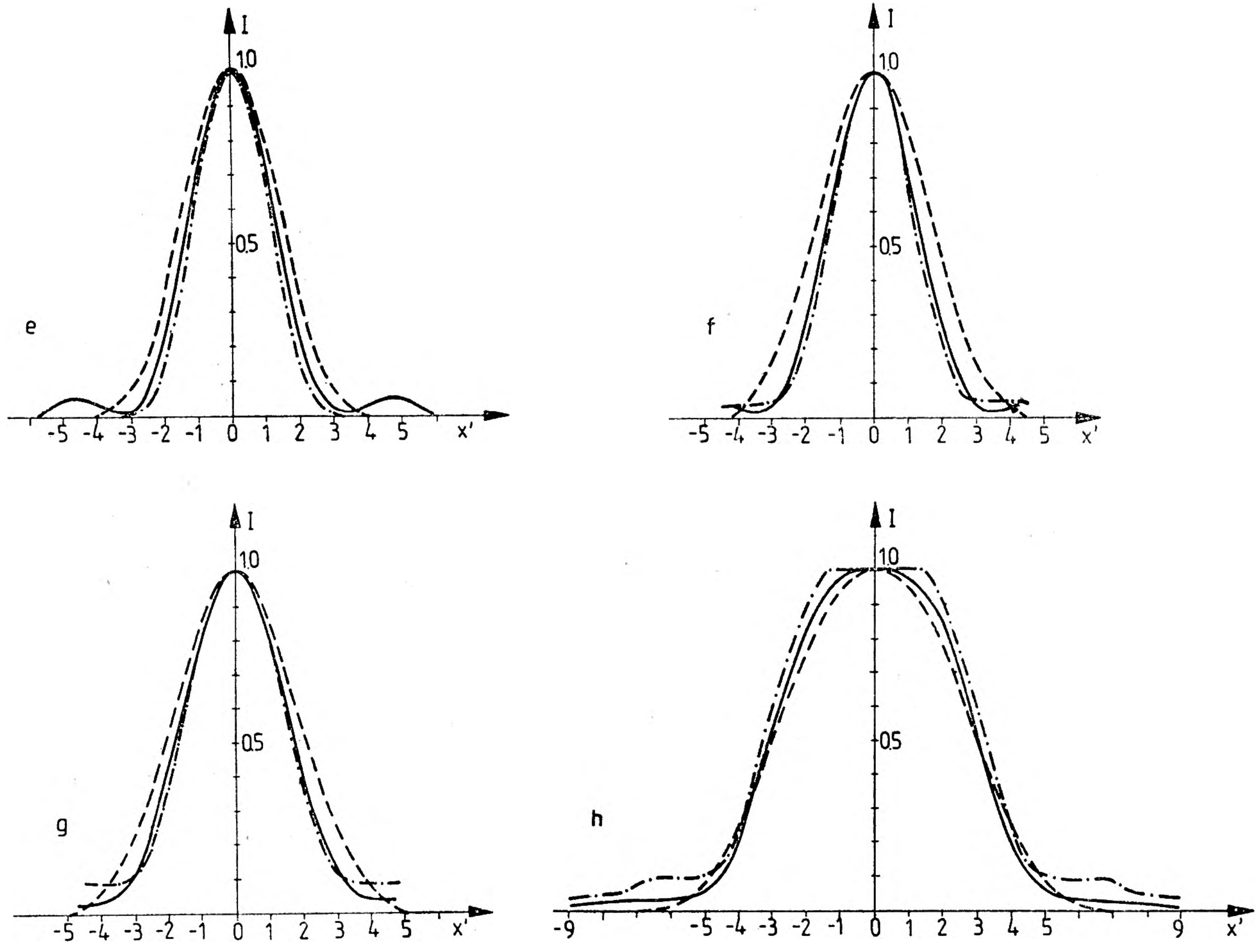
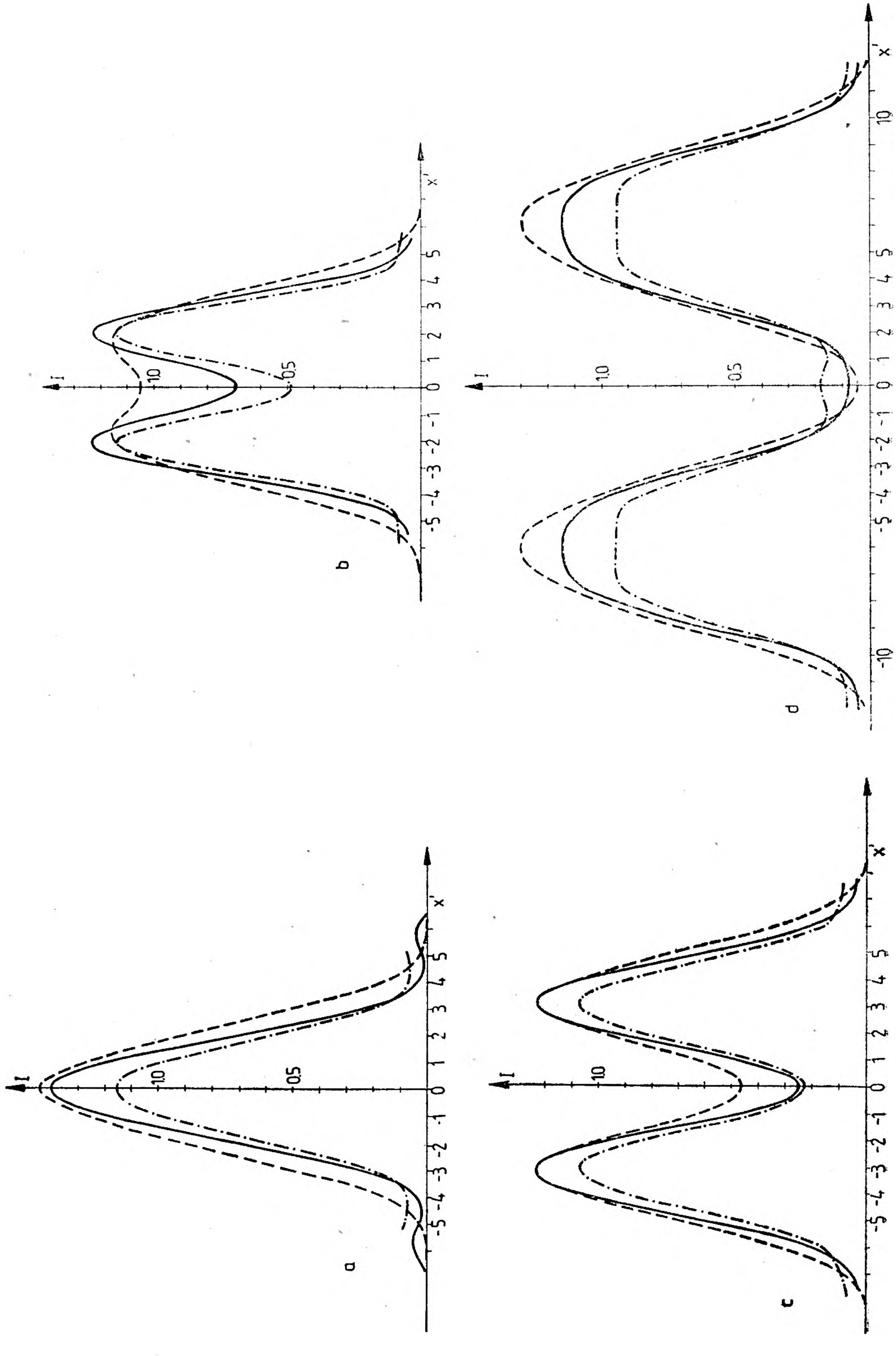



Fig. 1. The light intensity distribution in the image of a slit of the width: a. $2b = 1$, b. $2b = 2$, c. $2b = 3$, d. $2b = 6$ in one-dimensional system. Normed intensity distribution (light intensity in the slit middle being equal to 1) in the slit image of the width: e. $2b = 1$, f. $2b = 2$, g. $2b = 3$, h. $2b = 6$. Apodization: I(————), II(— . — . — .), III(— — — —)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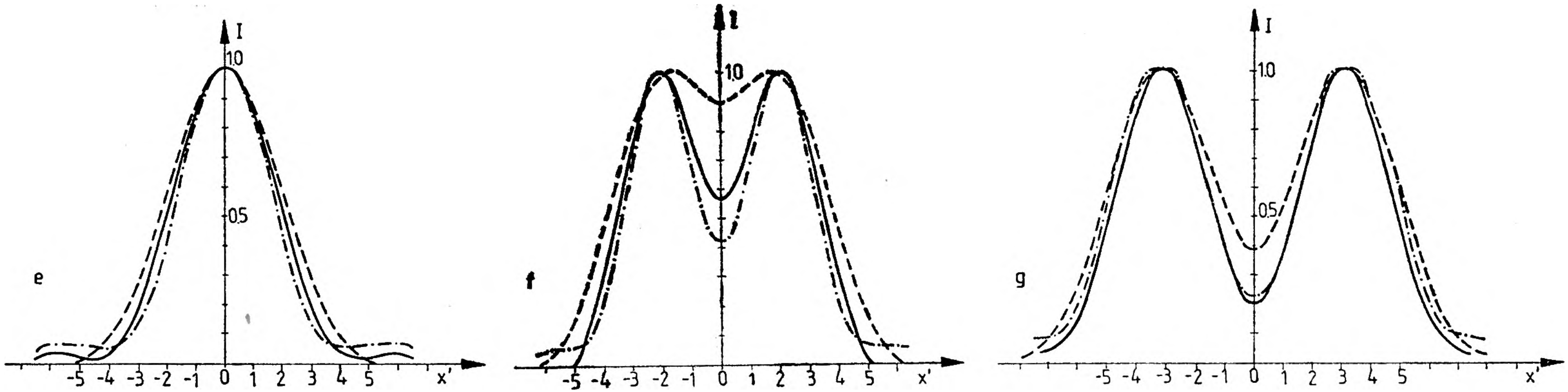


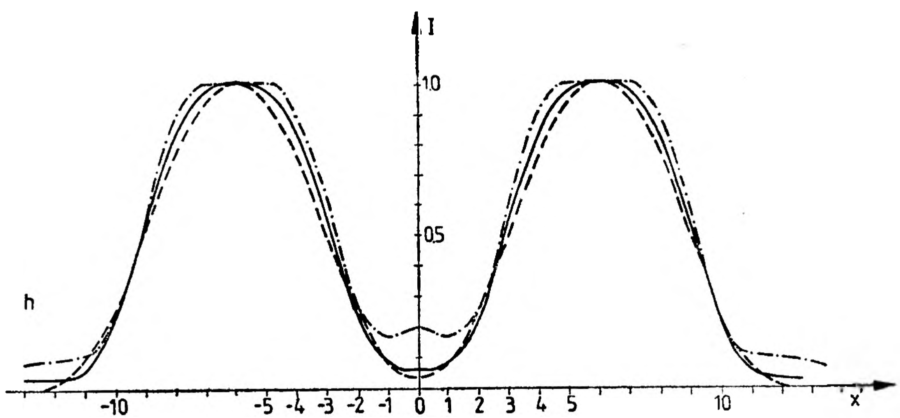
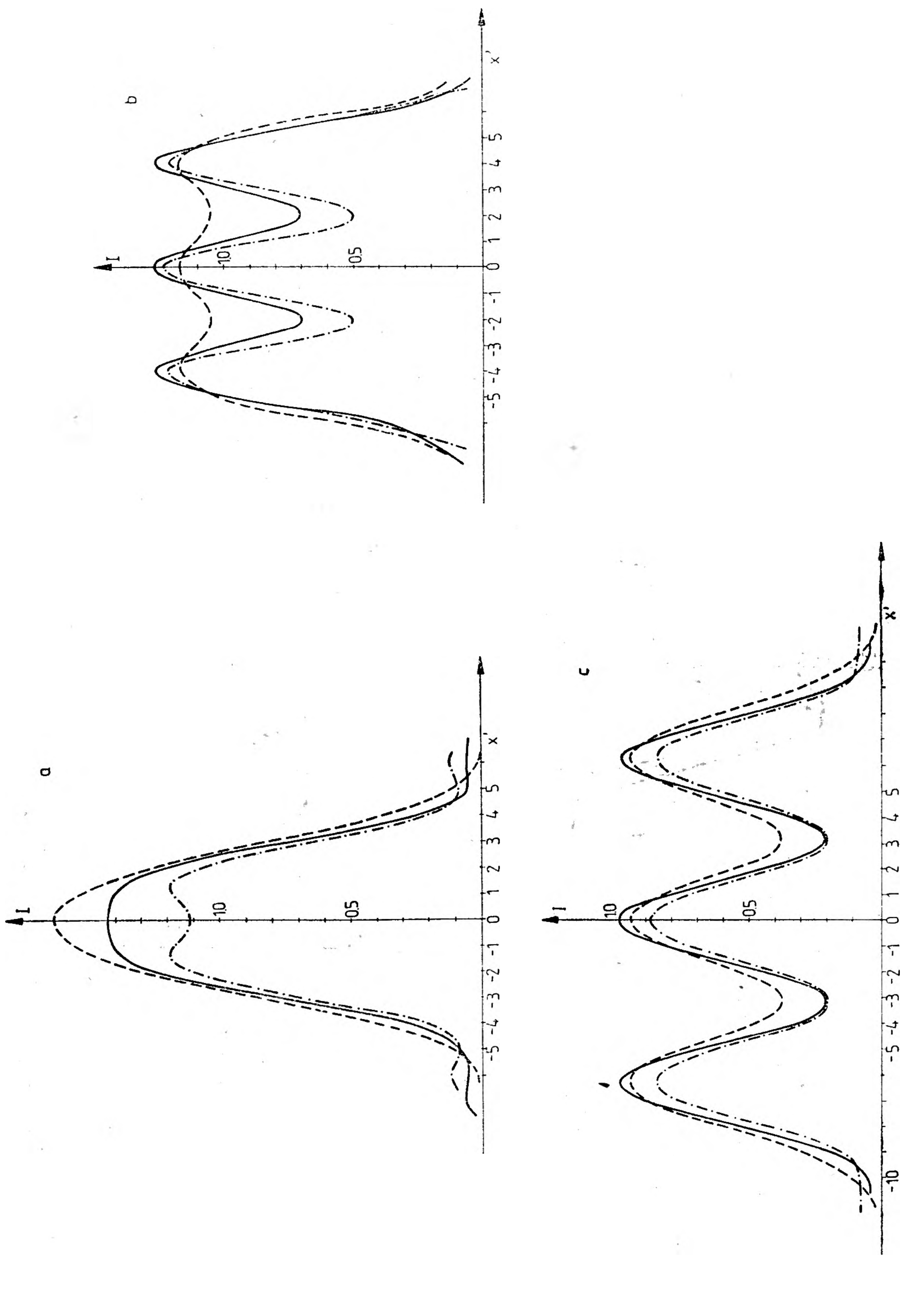




Fig. 2. The light intensity distribution in the image of two slits of widths and spacings equal, respectively, to: a. $2b = 1$, b. $2b = 2$, c. $2b = 3$, d. $2b = 6$ in one-dimensional system. Normed intensity distribution in the image of two slits of widths and spacings equal, respectively, to: e. $2b = 1$, f. $2b = 2$, g. $2b = 3$, h. $2b = 6$, in one-dimensional cases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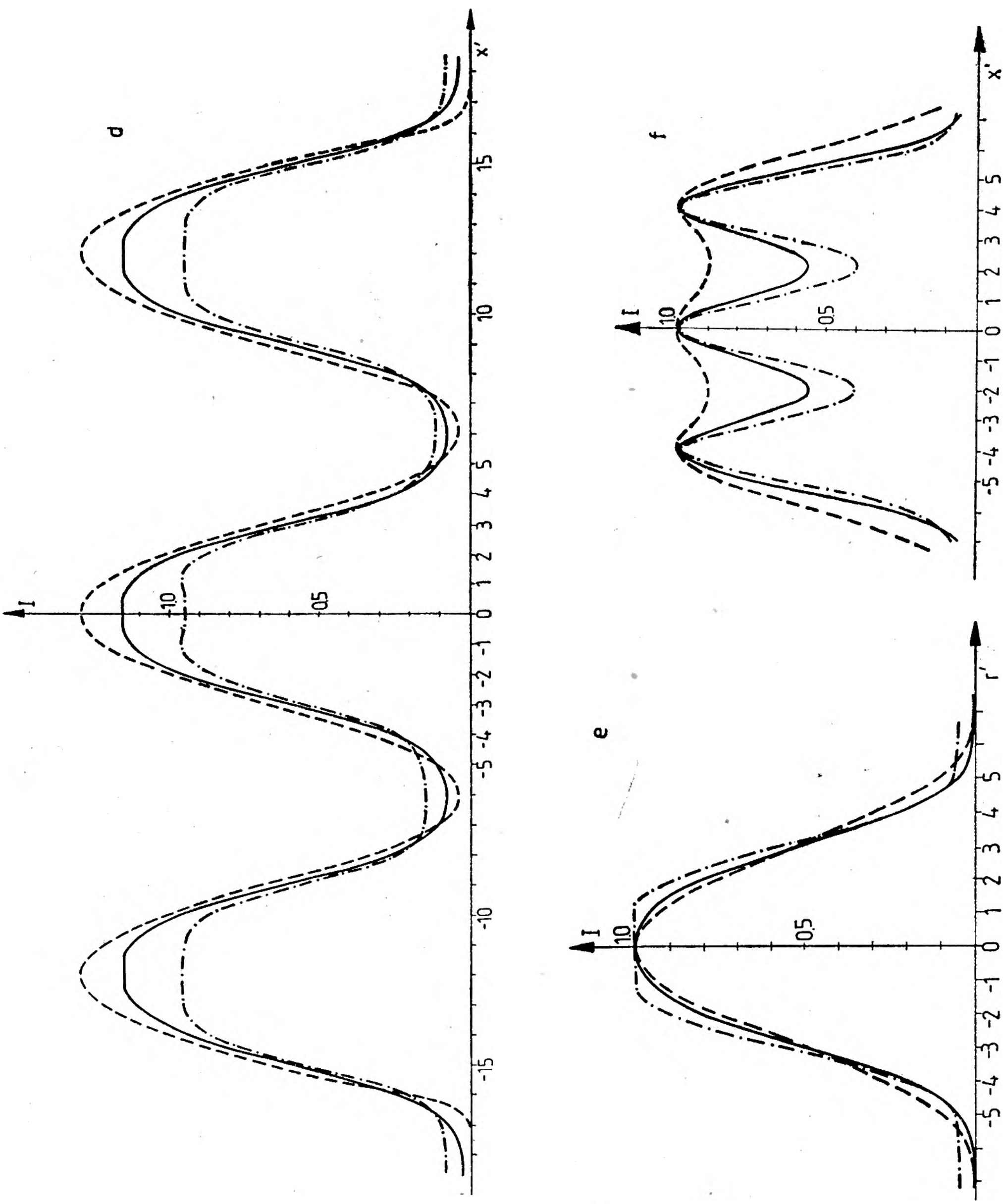
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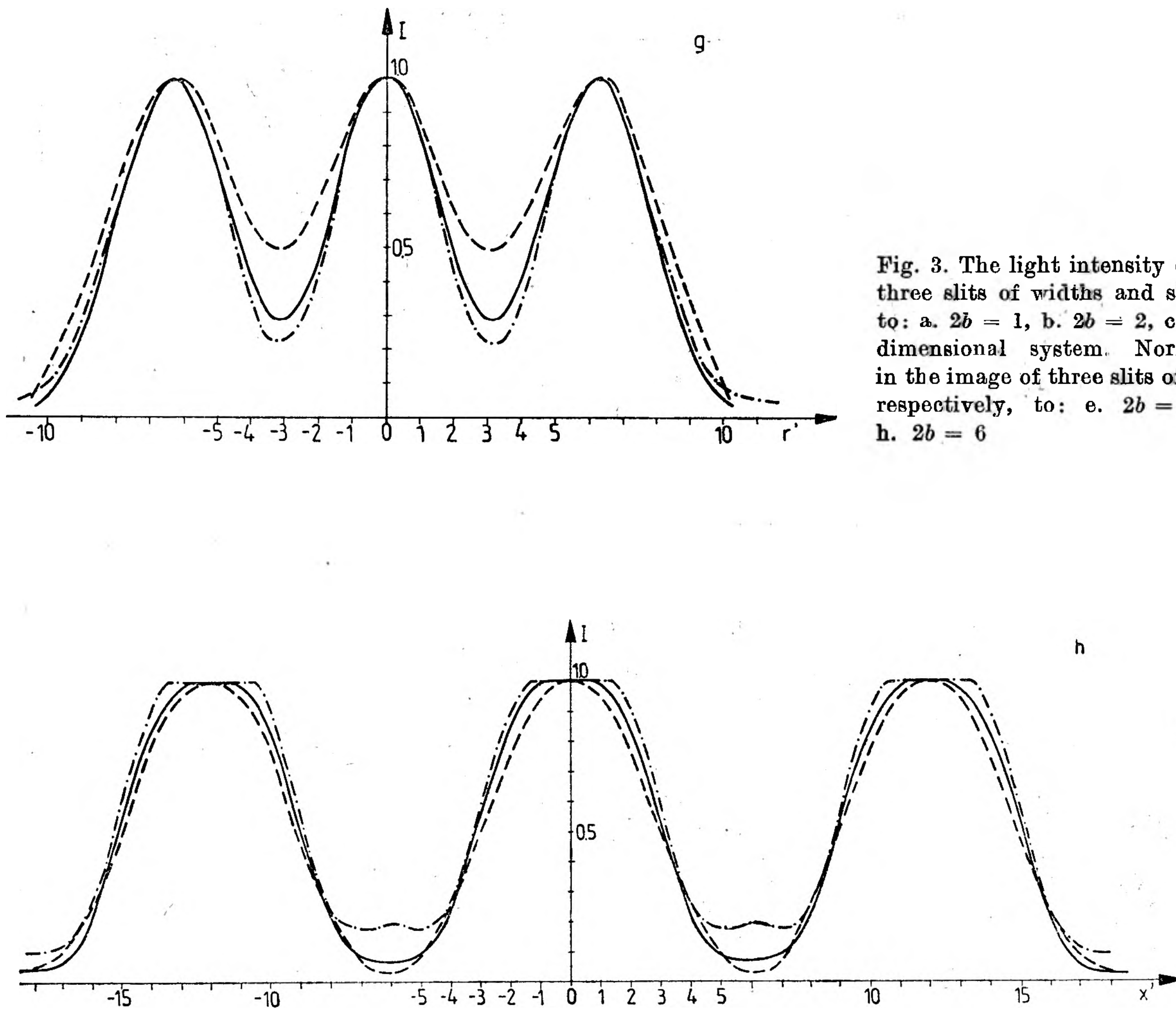
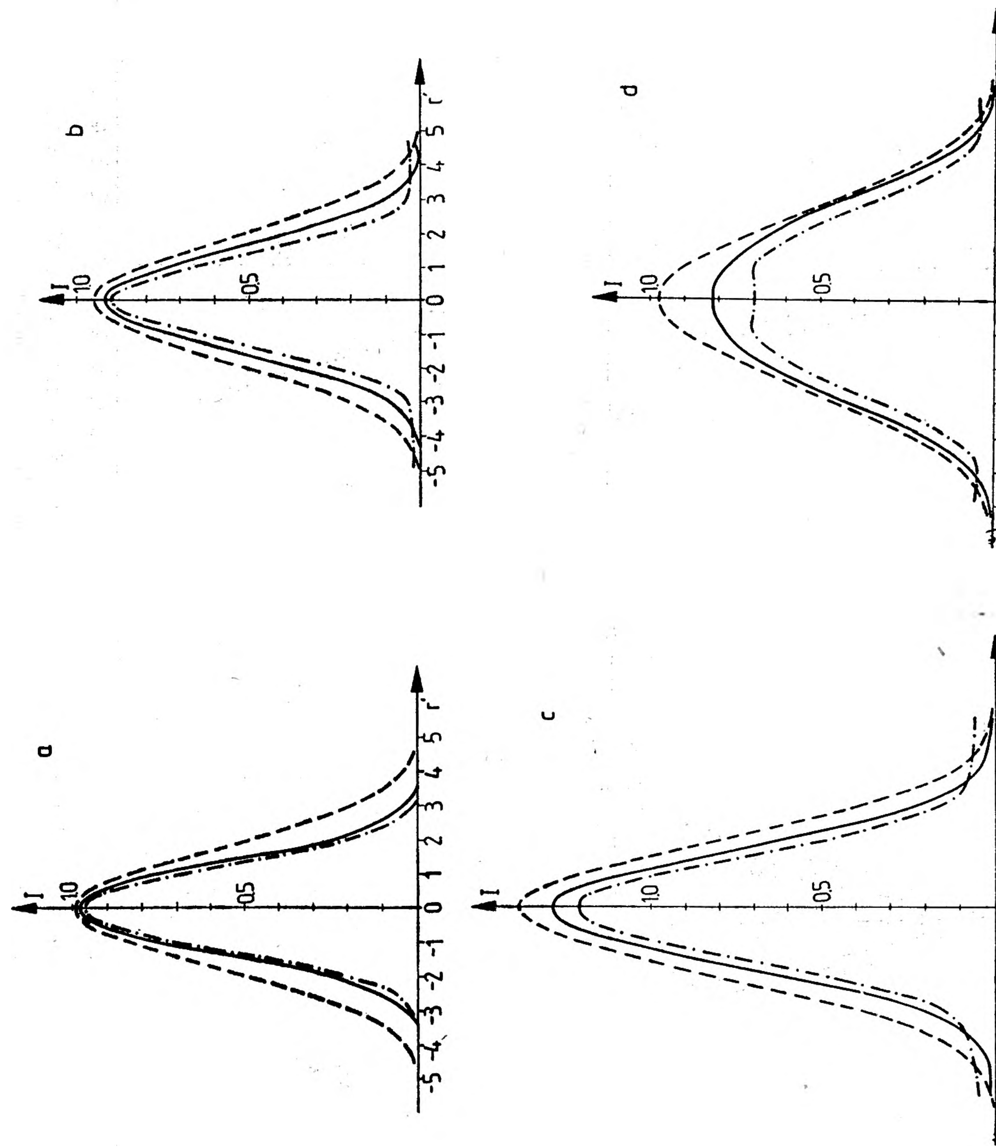



Fig. 3. The light intensity distribution in the image of three slits of widths and spacings equal, respectively, to: a. $2b = 1$, b. $2b = 2$, c. $2b = 3$, d. $2b = 6$ in one-dimensional system. Normed intensity distribution in the image of three slits of widths and spacings equal, respectively, to: e. $2b = 1$, f. $2b = 2$, g. $2b = 3$, h. $2b = 6$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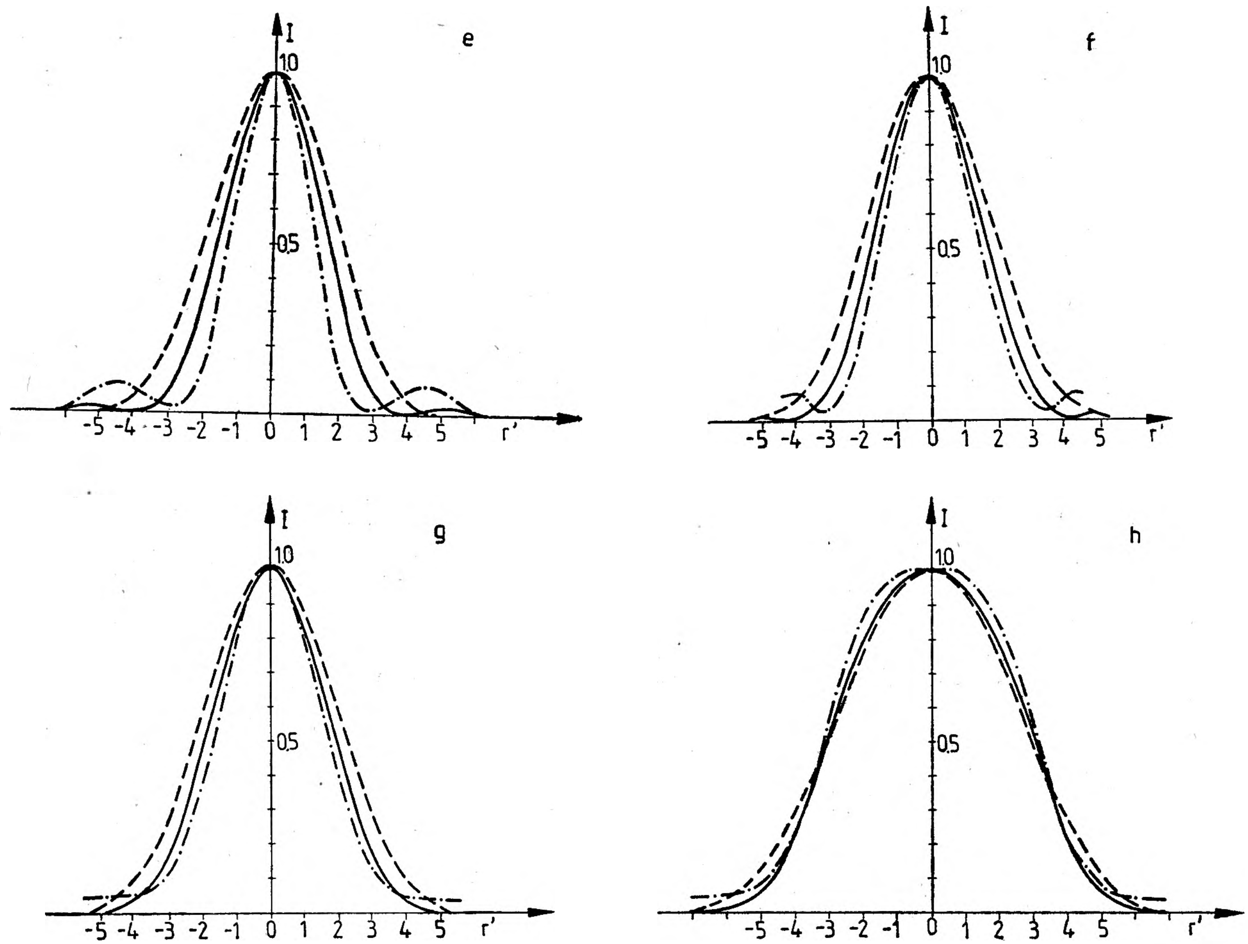
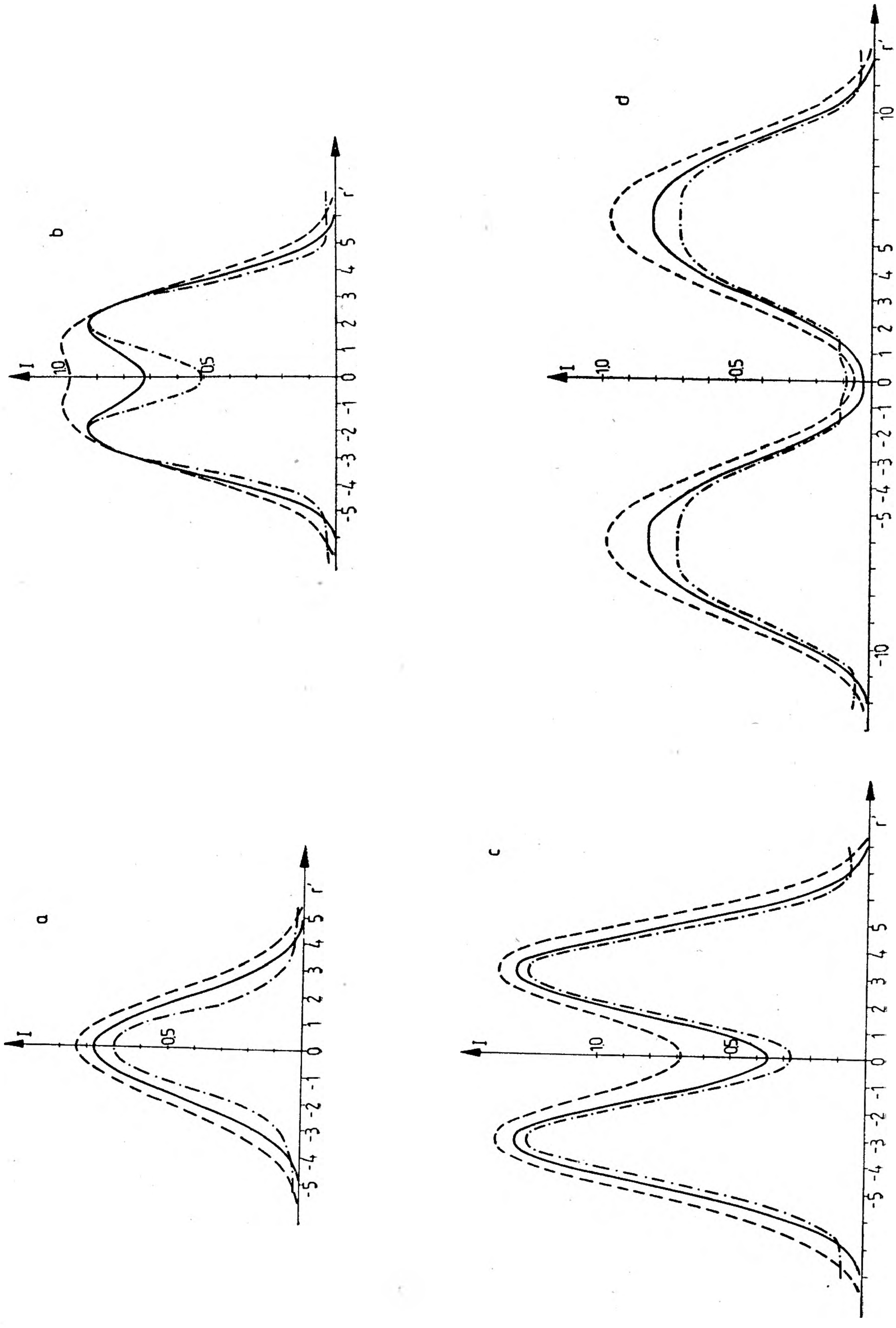



Fig. 4. The light intensity distribution in the image of a slit of widths: a. $2b = 1$, b. $2b = 2$, c. $2b = 3$, d. $2b = 6$ in two-dimensional systems. Normed intensity distribution in the image of a slit of the widths e. $2b = 1$, f. $2b = 2$, g. $2b = 3$, h. $2b = 6$ in two-dimensional systems



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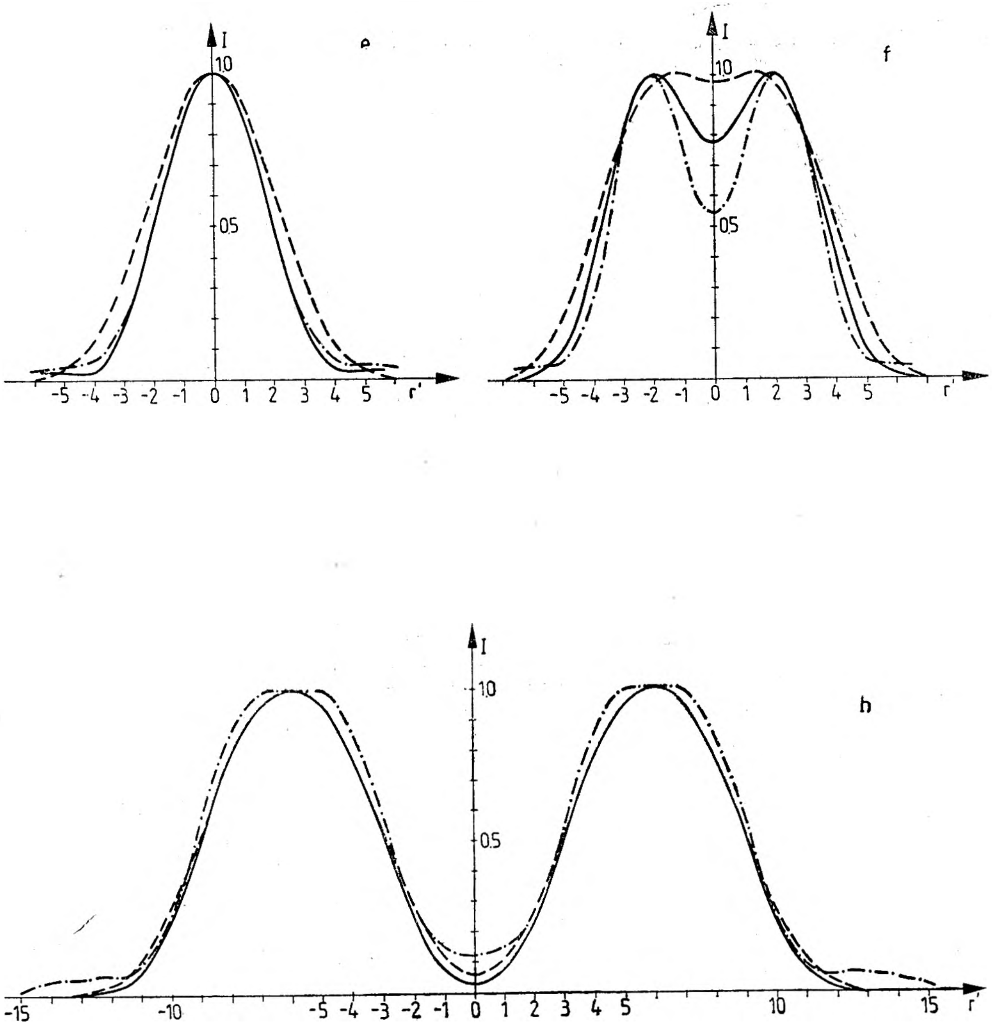
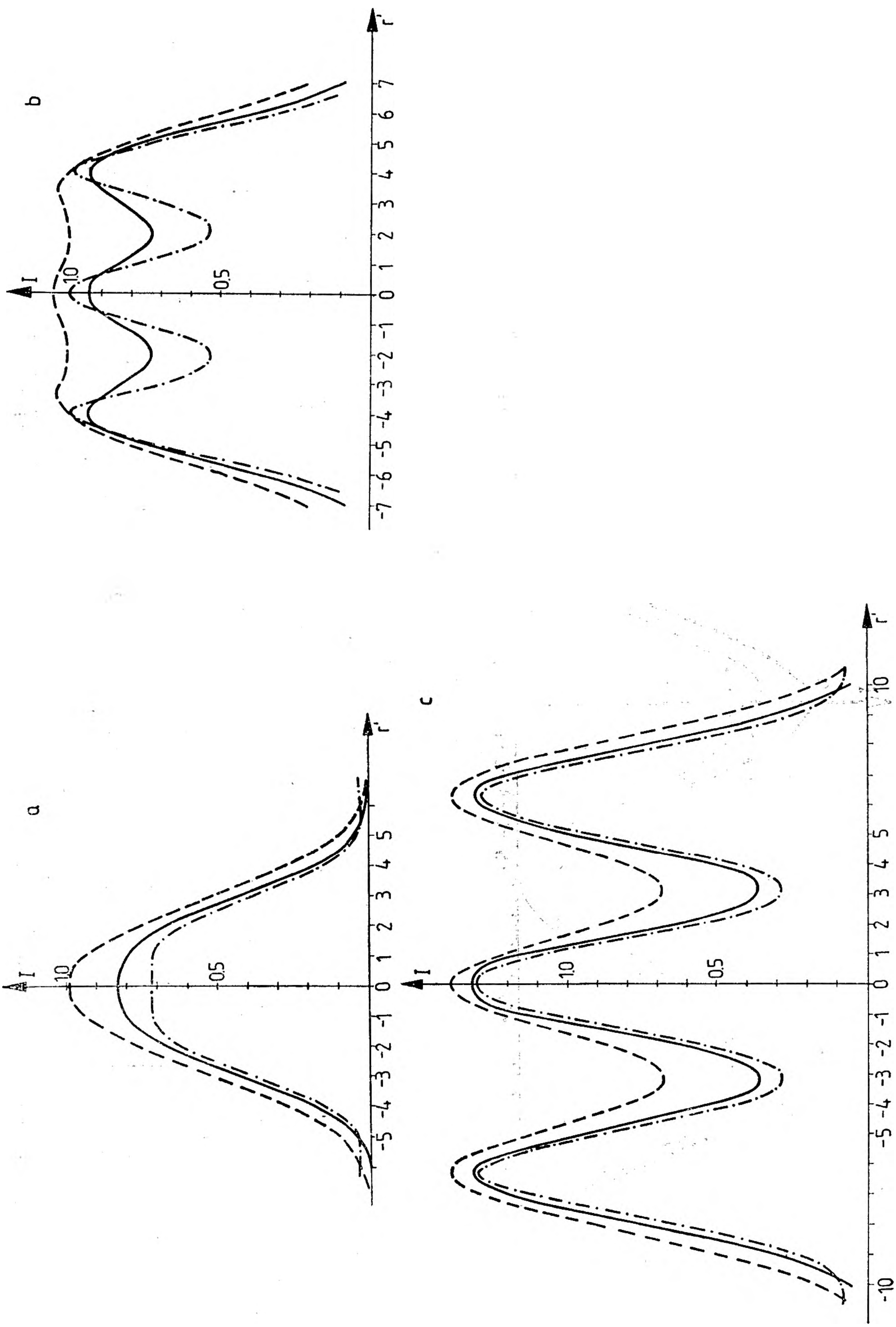




Fig. 5. The light intensity distribution in the image of two slits of widths and spacings equal, respectively, to: a. $2b = 1$, b. $2b = 2$, c. $2b = 3$, d. $2b = 6$ in two-dimensional system. Normed intensity distribution in the image of two slits of width and spacings equal, respectively, to: e. $2b = 1$, f. $2b = 2$, g. $2b = 3$, h. $2b = 6$ in two-dimensional systems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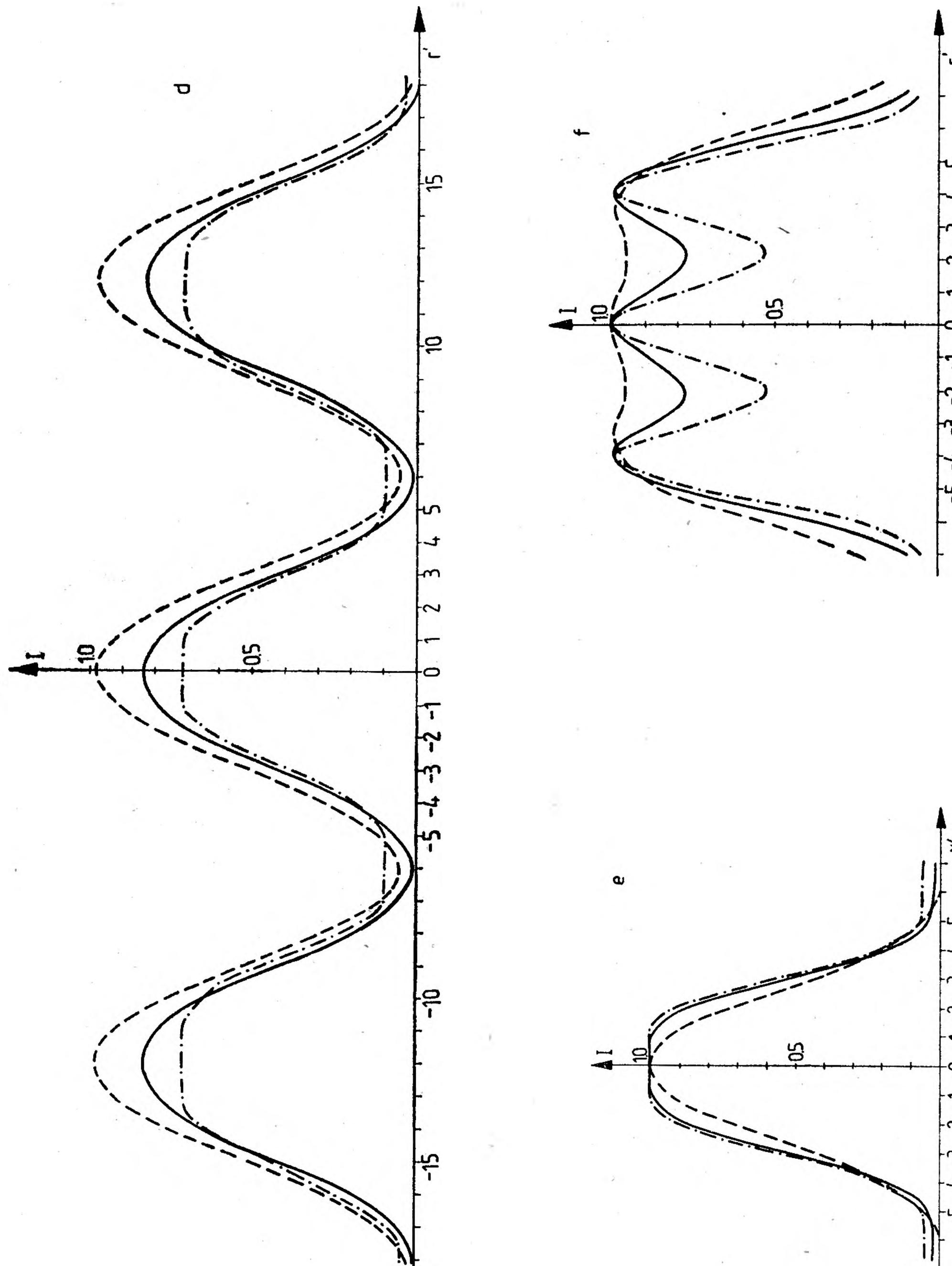
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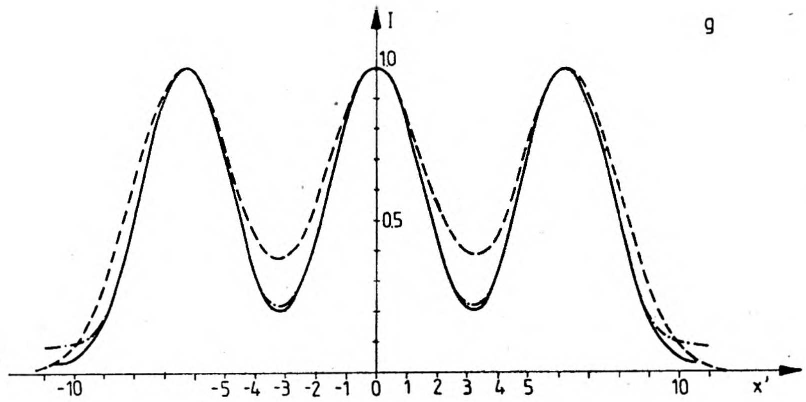


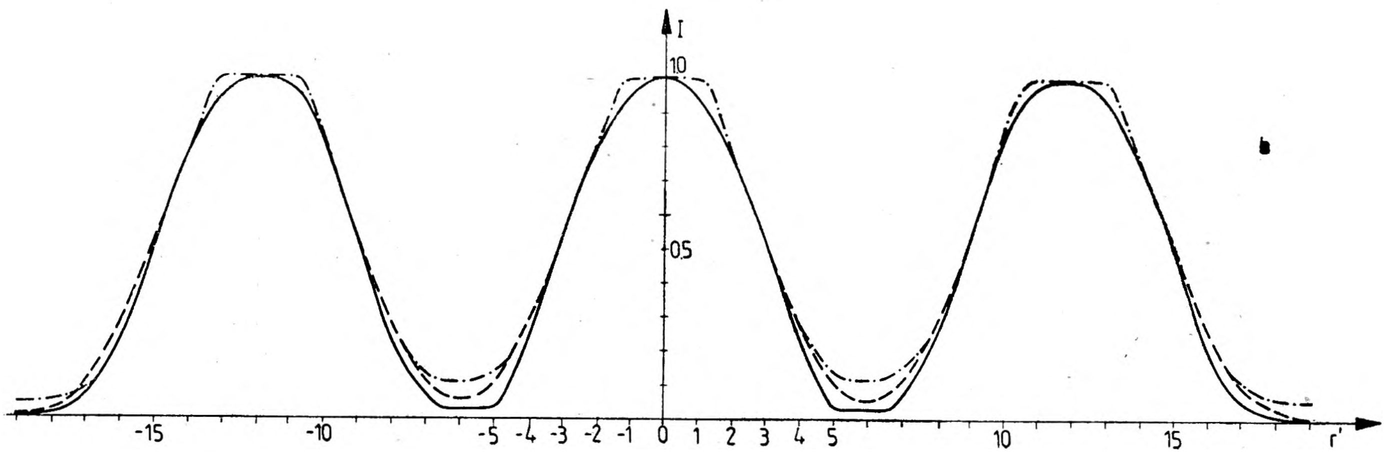




Fig. 6. The light intensity distribution in the image of three slits of widths and spacings equal, respectively, to: a. $2b = 1$, b. $2b = 2$, c. $2b = 3$, d. $2b = 6$ in two-dimensional system. Normed intensity distribution in the image of three slits of width and spacings equal, respectively, to: e. $2b = 1$, f. $2b = 2$, g. $2b = 3$, h. $2b = 6$ in two-dimensional system

is equal:

— *for one-dimensional system*

$$I(x') = I_0 \int\limits_{x'-5b}^{x'+5b} S^2(x')dx' - I_0 \int\limits_{x'-3b}^{x'+3b} S^2(x')dx' + I_0 \int\limits_{x'-b}^{x'+b} S^2(x')dx', \tag{12}$$

— *for two-dimensional system*

$$I(r') = I_0 \int\limits_{r'-5b}^{r'+5b} S^2(r')dr' - I_0 \int\limits_{r'-3b}^{r'+3b} S^2(r')dr' - I_0 \int\limits_{r'-b}^{r'+b} S^2(r')dr'. \tag{13}$$

The light intensity distributions in the image of three slits of $2b = 1, 2, 3, 6$ for the one-dimensional system with apodization I, II, III are shown in Figs. 3a–d, while those for the two-dimensional case are presented in Figs. 6a–d. Correspondingly normalized intensity distributions for the one-dimensional system are given in Figs. 3e–h and for the two-dimensional system – in Figs. 6e–h.

From the results obtained it follows that for $2b = 1$ the image of a single slit (Figs. 1a, 4a) is identical with the spread function (the object is undistinguishable). For $2b = 2$ the image of a single slit (Figs. 1b, 4b) and the images of two slits (Figs. 2b, 5b) and three slits (Figs. 3b, 6b) are comparable with the results obtained for the point, two point, and three-point objects [1]. For $2b = 3$ the images are already distinguishable but the intensity between them is not equal to zero. For an apodizing filter II and $2b = 3$, in the one-dimensional system the slit is recognizable (the two-point Rayleigh distance for this apodizer amounts to 2.8). For the two-dimensional system the limiting Rayleigh distance is 3.5. For $2b = 6$ the shapes of images are recognizable, while from Figs. 1d, h, 2d, h, 3d, h, 4d, h, 5d, h it follows that the image is best recognizable for an apodizer of type II. From the earlier analysis [1] it follows that the apodization of this type improves the resolution. The apodizer of type III behaves oppositely. When analysing the graphs in Figs. 2b–d, 3b–d, 5b–d, 6b–d the contrast $C = (I_{\max} - I_{\min})/(I_{\max} + I_{\min})$ may be determined, depending on spatial frequencies ($\omega$) in the object. In particular, for a one-dimensional system we obtain:

| $\omega = 1.57$ | $\omega = 1.005$ | $\omega = 0.52$ |
|---|---|---|
| $C_{(1-x)} \ll C_{(1)}$<br>$C_{(0.5(1+x^2))} \gg C_{(1)}$ | $C_{(1-x)} \ll C_{(1)}$<br>$C_{(0.5(1+x^2))} > C_1$ | $C_{(1-x)} \approx C_{(1)}$<br>$C_{(0.5(1+x^2))} \ll C_{(1)}$ |

while for two-dimensional system:

| $\omega = 1.57$ | $\omega = 1.005$ | $\omega = 0.52$ |
|---|---|---|
| $C_{(1-x^2)} \ll C_{(1)}$<br>$C_{(0.5(1+x^2))} \gg C_{(1)}$ | $C_{(1-x^2)} < C_{(1)}$<br>$C_{(0.5(1+x^2))} \approx C_{(1)}$ | $C_{(1-x^2)} > C_{(1)}$<br>$C_{(0.5(1+x^2))} < C_{(1)}$ |

*Translated by Ireneusz Wilk*

If the aperture is antisymmetric, the Fourier transform is equal to doubled imaginary parts of the function

$$t(x, y) = -t^*(-x, -y), \quad F_F \rightarrow 2\mathrm{Im}\{\mathscr{F}_F\}. \tag{4}$$

A modified amplitude distribution in the image plane may be obtained also by convolving the spatial frequency distribution of the object function $F(\omega, v)$ with the reciprocal function $(-i/\pi\omega)$

$$F_m(\omega, v) = F(\omega, v) \otimes \big(-i/(\pi\omega)\big). \tag{5}$$

By definition the Hilbert transform of the function $f(x)$ is a convolution of this function $f(x)$ with the reciprocal function $\big(-1/(\omega x)\big)$

$$\mathscr{F}_{\mathrm{Hi}}(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(x')dx'}{x'-x} = f(x) \otimes \left(-\frac{1}{\pi x}\right). \tag{6}$$

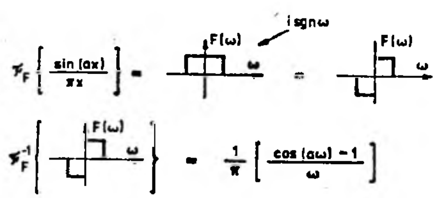
The Fourier transform $(-\pi x)^{-1}$ is equal to $i\,\mathrm{sgn}\,\omega$, i.e., it is equal to $+i$ for positive $\omega$ and to $-i$ for negative $\omega$. The Hilbert transform is thus equivalent to a special filtering in which the amplitudes of spectral components remain unchanged, while their phases are shifted by $\pi/2$ in the positive or negative direction, in accordance with the sign of $\omega$.

If, for instance, the aperture function is $f(x) = \sin(ax)/(\pi x)$ its Hilbert transform may be produced by:

i) calculating the Fourier transform of the function $f(x)$,

ii) multiplying this transform by $i\,\mathrm{sgn}\,\omega$,

iii) performing an inverse Fourier transform




(7)

The result obtained is also a Fourier transform of a rectangle aperture function (see Table) with halves of the aperture areas being in antiphase with respect to each other. It is a quadrature component of the analytic spatial frequency distribution of the rectangle aperture function multiplied by the Heaviside function (see Table, example 1).

The amplitude distribution corresponding to the Hilbert transform of a rectangle function is presented in Fig. 1a, the corresponding intensity distribution being shown in Fig. 1b.

The result obtained is a quadrature component of the spatial frequency distribution of the aperture function, sinc $(ax)$, multiplied by the Heaviside function (Table, example 3).

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## Semiapertures quadrature *

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This paper contains the results obtained from the application of the Hilbert transforms to present the diffraction patterns of real semiapertures in analytical form with a quadrature component of the aperture function. The mentioned results have been compared with those obtained for the amplitude-apodized optical systems.

The Fourier transform of a full aperture described by a real transmittance function $t(x, y)$ is a real distribution of aperture in the Fourier plane $F(\omega, v)$ [1]

$$\mathscr{F}_F\{t(x,y)\} = F(\omega, v). \tag{1}$$

The Fourier transform of a semiaperture $t_s(x, y)$ (i.e., the full aperture function $t(x, y)$ multiplied by the Heaviside function $H(x)$, i.e., $t_s(x, y) = t(x, y)$ $H(x)$) is an analytical function in the form

$$\mathscr{F}_F\{t(x, y)\ H(x)\} = \frac{1}{2}[F(\omega, v)+i\hat{F}(\omega, v)]. \tag{2}$$

Here, the real and imaginary parts (the latter being the quadrature component) constitute a pair of Hilbert transforms. Thus, the Fraunhofer diffraction pattern generated by a real semiaperture $t_s(x, y)$, distributed along the line parallel to the normal to the edge of the aperture semiplane, has been represented in an analytic form in the image plane (comp. [2]). If the aperture transmittance is described by a complex function, such that the parts of aperture are symmetric, the Fourier transform is given by doubled real part of the analytic function

$$t(x,y) = t^*(-x, \ -y), \ F_F \to 2\mathrm{Re}\{\mathscr{F}_F\}. \tag{3}$$

* This work was carried on under the Research Project M.R. I.5.