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Semiapertures quadrature *

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This paper contains the results obtained from the application of the Hilbert transforms to present the diffraction patterns of real semiapertures in analytical form with a quadrature component of the aperture function. The mentioned results have been compared with those obtained for the amplitude-apodized optical systems.

The Fourier transform of a full aperture described by a real transmittance function $t(x, y)$ is a real distribution of aperture in the Fourier plane $F(\omega, v)$ [1]

$$\mathscr{F}_F\{t(x, y)\} = F(\omega, v). \tag{1}$$

The Fourier transform of a semiaperture $t_s(x, y)$ (i.e., the full aperture function $t(x, y)$ multiplied by the Heaviside function $H(x)$, i.e., $t_s(x, y) = t(x, y) H(x)$) is an analytical function in the form

$$\mathscr{F}_F\{t(x, y)\ H(x)\} = \frac{1}{2}[F(\omega, v) + i\hat{F}(\omega, v)]. \tag{2}$$

Here, the real and imaginary parts (the latter being the quadrature component) constitute a pair of Hilbert transforms. Thus, the Fraunhofer diffraction pattern generated by a real semiaperture $t_s(x, y)$, distributed along the line parallel to the normal to the edge of the aperture semiplane, has been represented in an analytic form in the image plane (comp. [2]). If the aperture transmittance is described by a complex function, such that the parts of aperture are symmetric, the Fourier transform is given by doubled real part of the analytic function

$$t(x, y) = t^*(-x, -y),\ F_F \to 2\mathrm{Re}\{\mathscr{F}_F\}. \tag{3}$$

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If the aperture is antisymmetric, the Fourier transform is equal to doubled imaginary parts of the function

$$t(x, y) = -t^*(-x, -y), \; F_F \to 2\mathrm{Im}\{\mathscr{F}_F\}. \tag{4}$$

A modified amplitude distribution in the image plane may be obtained also by convolving the spatial frequency distribution of the object function $F(\omega, v)$ with the reciprocal function $(-i/\pi\omega)$

$$F_m(\omega, v) = F(\omega, v) \otimes \big(-i/(\pi\omega)\big). \tag{5}$$

By definition the Hilbert transform of the function $f(x)$ is a convolution of this function $f(x)$ with the reciprocal function $\big(-1/(\omega x)\big)$

$$\mathscr{F}_{\mathrm{Hi}}(x) = \frac{1}{\pi} \int\limits_{-\infty}^{\infty} \frac{f(x')dx'}{x'-x} = f(x) \otimes \left(-\frac{1}{\pi x}\right). \tag{6}$$

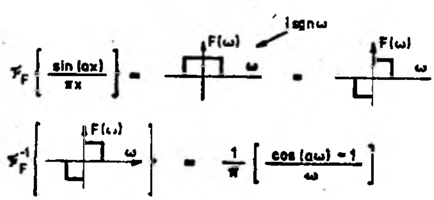
The Fourier transform $(-\pi x)^{-1}$ is equal to $i\,\mathrm{sgn}\,\omega$, i.e., it is equal to $+i$ for positive ω and to $-i$ for negative ω. The Hilbert transform is thus equivalent to a special filtering in which the amplitudes of spectral components remain unchanged, while their phases are shifted by $\pi/2$ in the positive or negative direction, in accordance with the sign of ω.

If, for instance, the aperture function is $f(x) = \sin(ax)/(\pi x)$ its Hilbert transform may be produced by:

i) calculating the Fourier transform of the function $f(x)$,

ii) multiplying this transform by $i\,\mathrm{sgn}\,\omega$,

iii) performing an inverse Fourier transform



(7)

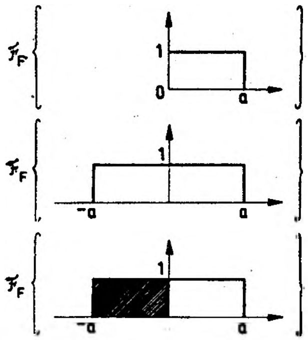
The result obtained is also a Fourier transform of a rectangle aperture function (see Table) with halves of the aperture areas being in antiphase with respect to each other. It is a quadrature component of the analytic spatial frequency distribution of the rectangle aperture function multiplied by the Heaviside function (see Table, example 1).

The amplitude distribution corresponding to the Hilbert transform of a rectangle function is presented in Fig. 1a, the corresponding intensity distribution being shown in Fig. 1b.

The result obtained is a quadrature component of the spatial frequency distribution of the aperture function, sinc (ax), multiplied by the Heaviside function (Table, example 3).

Analytic functions describing the semiapertures. Examples

1



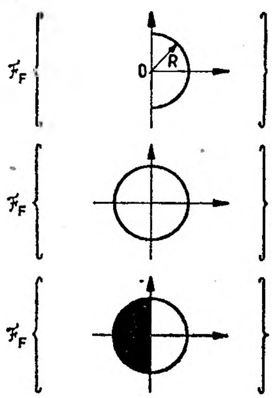
$$\operatorname{sinc}(a\omega)+\frac{i}{\omega}\left[\cos(\omega a)-1\right],$$

$$2\operatorname{sinc}(a\omega)$$

$$\downarrow\uparrow\ {}^{-}\mathscr{F}_{\mathrm{Hi}}$$

$$\frac{2i}{\omega}\left[\cos(\omega a)-1\right]$$

2



$$\frac{J_1(kR)}{kR}+i\,\frac{H_1(kR)}{kR},$$

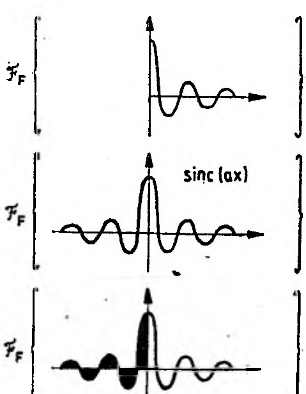
(J_1—first order Bessel function, H_1—first order Struve function)

$$\frac{2J_1(kR)}{kR}$$

$$\downarrow\uparrow\ {}^{-}\mathscr{F}_{\mathrm{Hi}}$$

$$\frac{2iH_1(kR)}{kR}$$

3



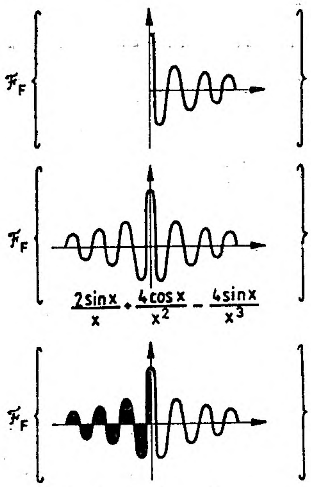
$$\frac{\pi}{a}\prod\left(\frac{\pi\omega}{a}\right)+i\,\ln\left|\frac{\omega-a}{\omega+a}\right|$$

$$2\,\frac{\pi}{a}\prod\left(\frac{\pi\omega}{a}\right)$$

$$\downarrow\downarrow\ {}^{-}\mathscr{F}_{\mathrm{Hi}}$$

$$2i\,\ln\left|\frac{\omega-a}{\omega+a}\right|$$

4



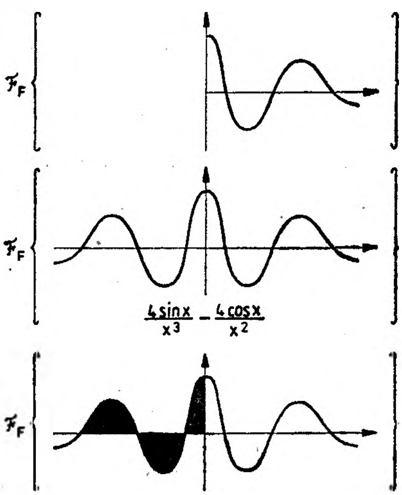
$$\omega^2 + i\left[\frac{\omega^2}{\pi}\ln\left|\frac{\omega-a}{\omega+a}\right| + \frac{2a\omega}{\pi}\right]$$

$$2\omega^2$$

$$\downarrow\ \uparrow\ -\mathcal{F}_{\mathrm{Hi}}$$

$$2i\left[\frac{\omega^2}{\pi}\ln\left|\frac{\omega-a}{\omega+a}\right| + \frac{2a\omega}{\pi}\right]$$

5



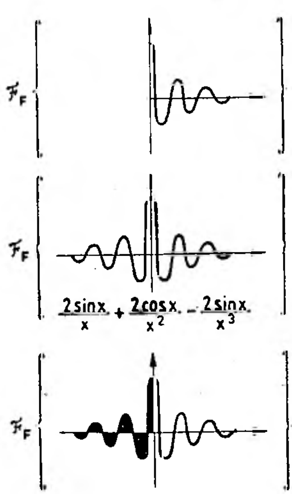
$$(1-\omega^2) + i\left[(1-\omega^2)\frac{1}{\pi}\ln\left|\frac{\omega-a}{\omega+a}\right| - \left|\frac{2\omega a}{\pi}\right|\right]$$

$$2(1-\omega^2)$$

$$\downarrow\ \uparrow\ -\mathcal{F}_{\mathrm{Hi}}$$

$$2i\left[(1-\omega^2)\frac{1}{\pi}\ln\left|\frac{\omega-a}{\omega+a}\right| - \frac{2\omega a}{\pi}\right]$$

6



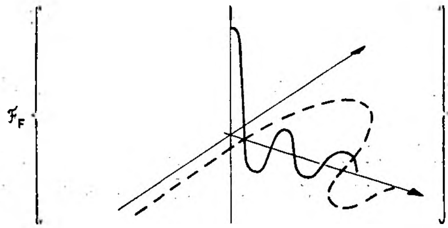
$$\frac{1}{2}(1+\omega^2) + i\left[\frac{1}{2}\frac{(1+\omega^2)}{\pi}\ln\left|\frac{\omega-a}{\omega+a}\right| + \frac{a\omega}{\pi}\right]$$

$$2\left[\frac{1}{2}(1-\omega^2)\right]$$

$$\downarrow\ \uparrow\ -\mathcal{F}_{\mathrm{Hi}}$$

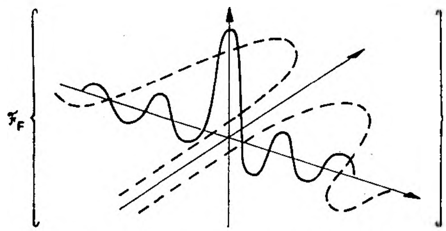
$$2i\left[\frac{1}{2}\frac{(1+\omega^2)}{\pi}\ln\left|\frac{\omega-a}{\omega+a}\right| + \frac{a\omega}{\pi}\right]$$

7



$$\left\{\cos(a\omega) - \frac{a}{\pi}\left[\omega^2 \ln\left|\frac{\omega-a}{\omega+a}\right| + 2a\omega\right]\right\}$$

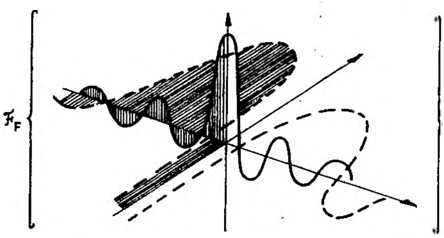
$$+ i\left[\frac{1}{\pi}\ln\left|\frac{\omega-a}{\omega+a}\right| + \sin(a\omega)\right.$$



$$2\left\{\cos(a\omega) - \frac{a}{\pi}\left[\omega^2 \ln\left|\frac{\omega-a}{\omega+a}\right| + 2a\omega\right]\right\}$$

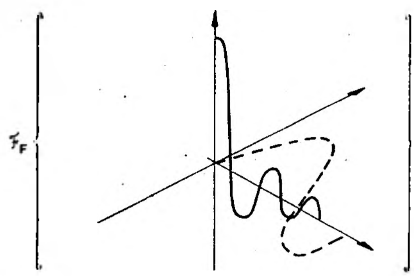
$\downarrow \uparrow \; {}^{-}\mathscr{F}_{\mathrm{Hi}}$

$\frac{2\sin x}{x} - \frac{\alpha}{x^3}\left[e^{ix}(2-x^2-2ix) - e^{-ix}(2-x^2+2ix)\right]$



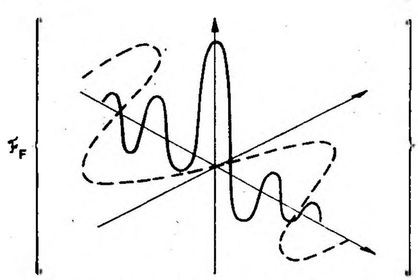
$$2i\left[\frac{1}{\pi}\ln\left|\frac{\omega-a}{\omega+a}\right| + \sin(a\omega)\right]$$

8



$$\left[\cos(\sin(\omega_0\omega)) - \cos(\omega_0\omega)\right|$$

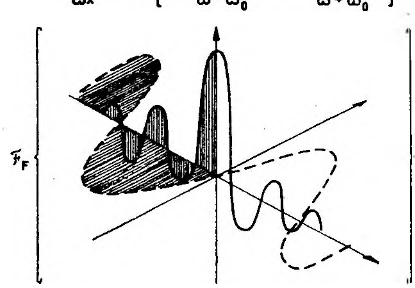
$$+ i\left[\frac{1}{\pi}\ln\left|\frac{\omega-a}{\omega+a}\right| + \sin(\sin(\omega_0\omega))\right]$$



$$2\left[\cos(\sin(\omega_0\omega)) - \mathrm{soc}(\omega_0\omega)\right]$$

$\downarrow \uparrow \; {}^{-}\mathscr{F}_{\mathrm{Hi}}$

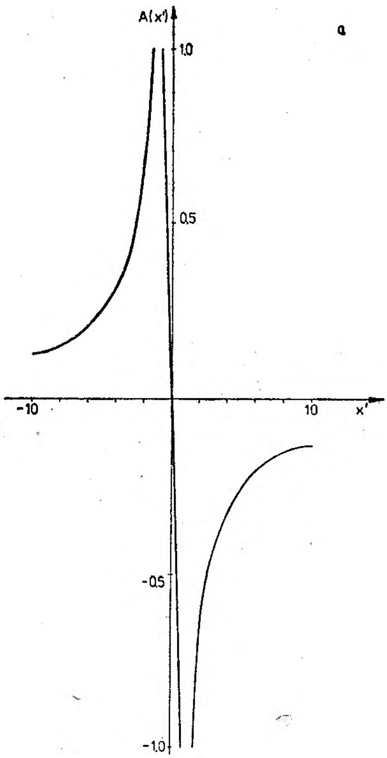
$\frac{2\sin(\omega x)}{\omega x} + i\left[\frac{\sin(\omega_0-\omega)x}{\omega-\omega_0} - \frac{\sin(\omega_0+\omega)x}{\omega+\omega_0}\right]$



$$2i\left[\frac{1}{\pi}\ln\left|\frac{\omega-a}{\omega+a}\right| + \sin(\sin(\omega_0\omega))\right]$$

The results obtained for other aperture functions are given also in the Table.

In Figures 2a, b the amplitude and intensity distributions generated by a Hilbert transform of the apodized rectangle aperture are shown and com-



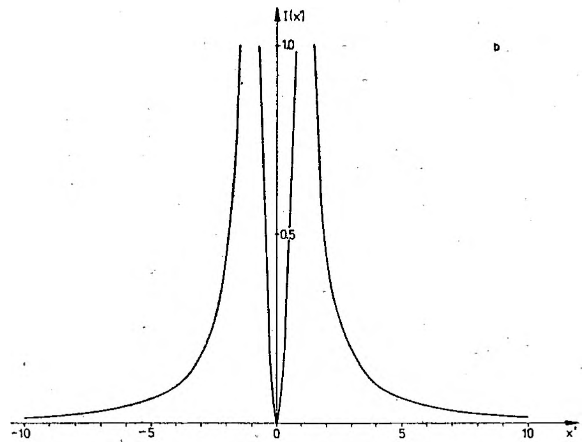
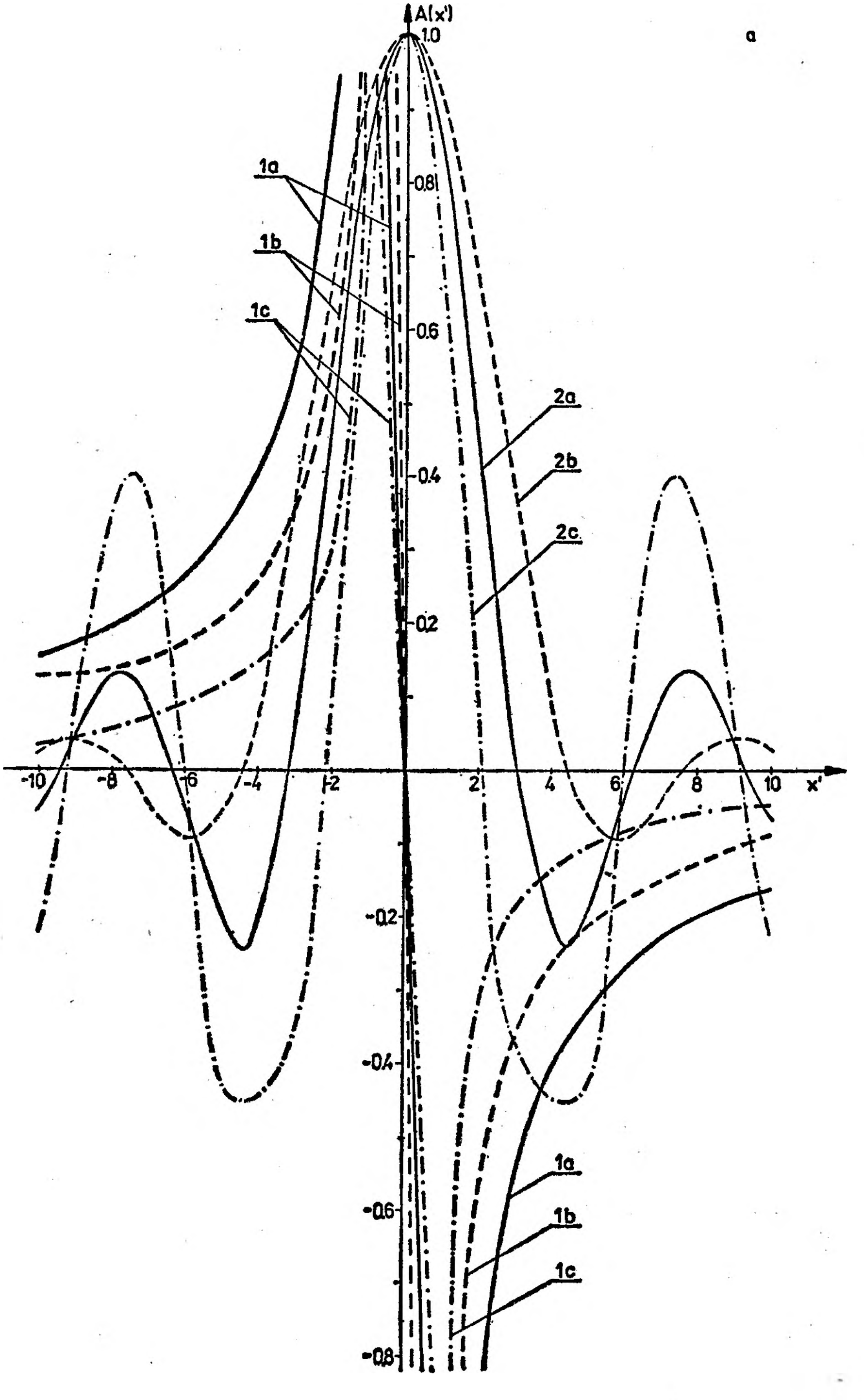


Fig. 1. Hilbert transform of a rectangle aperture function: a – amplitude distribution, b – intensity distribution

pared with the amplitude and intensity distributions generated by a Fourier transform of the same aperture [3]. The results obtained indicate that by employing the Hilbert transform the positive and negative parts are separated. This is especially useful for two-channel incoherent optical processors, in the case of Hilbert transform, as it allows to define the spread function in the form of two positive filters in accordance with the relation $h(x) = h_+(x) - h_-(x)$, and moreover, there is no energy loss in the secondary maximum. It is also well known that the function and its Hilbert transform have the same autocorrelation function [4]

$$\int_{-\infty}^{\infty} f^*(x) f(x-x')\,dx \rightleftarrows \int_{-\infty}^{\infty} F^*_{\mathrm{Hi}}(x) F_{\mathrm{Hi}}(x-x')\,dx. \tag{8}$$

This means that the total energies in the object and the image are identical. An application of the Hilbert transform is particularly important for visualization of the phase objects. Two examples ($f(x) = e^{ix^2}$, $f(x) = e^{ia\sin(\omega_0 x')}$ shown in


A(x')
1.0
a
0.8
0.6
0.4
0.2
-0.2
-0.4
-0.6
-0.8
-10
-8
-6
-4
-2
2
4
6
8
10
x'
1a
1b
1c
2a
2b
2c
1a
1b
1c

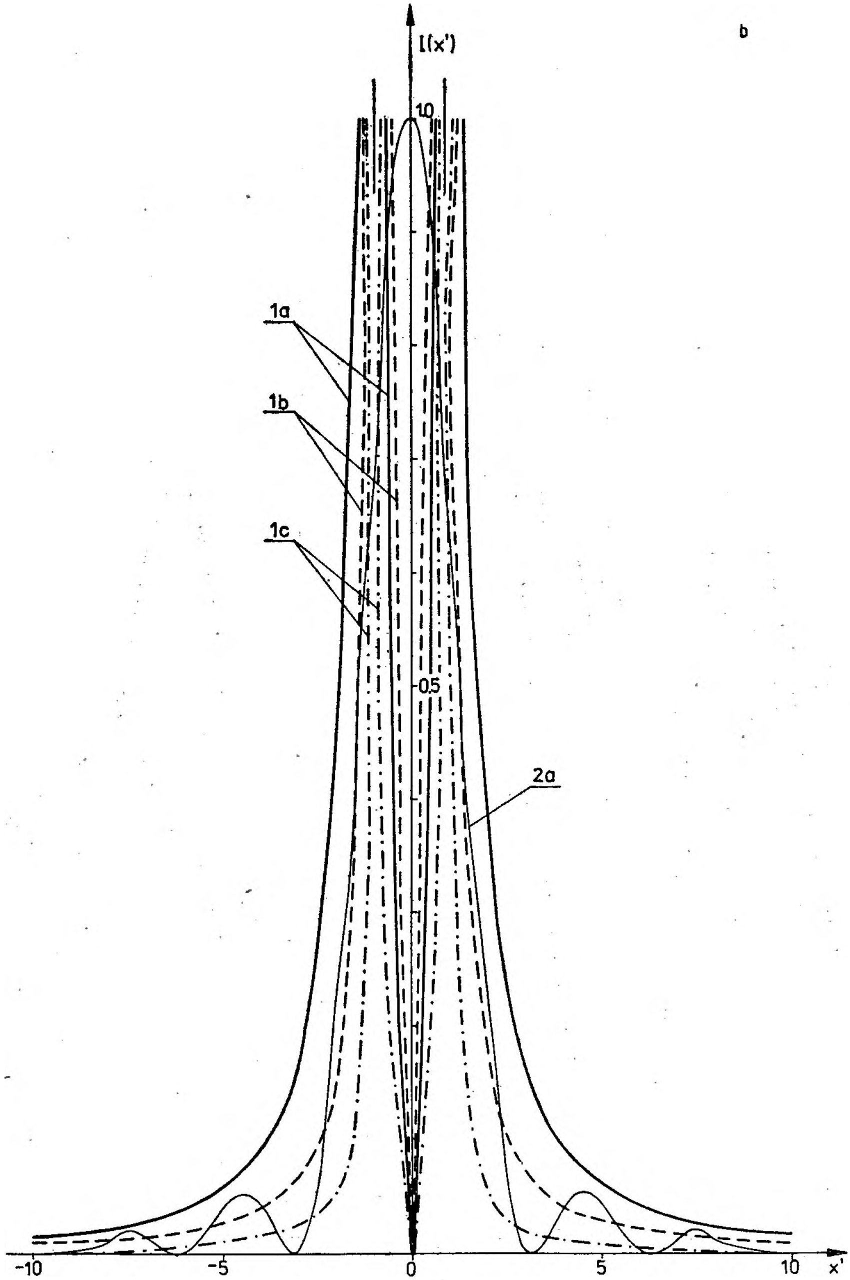
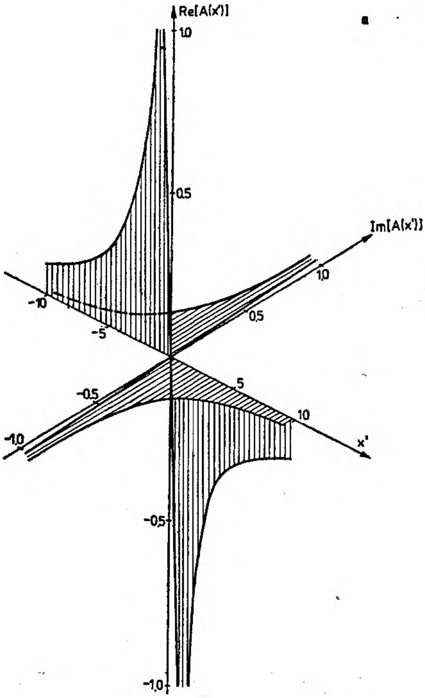


Fig. 2. Hilbert transform (curves 1 *abc*) and Fourier transform (curves 2 *abc*) of an apodized rectangular aperture: a – amplitude distribution, b – intensity distribution: (————) $t(x) = 1$, (– – – –) $t(x) = 1-x^2$, (– · – · – · – · –) $t(x) = 0,5\,(1+x^2)$



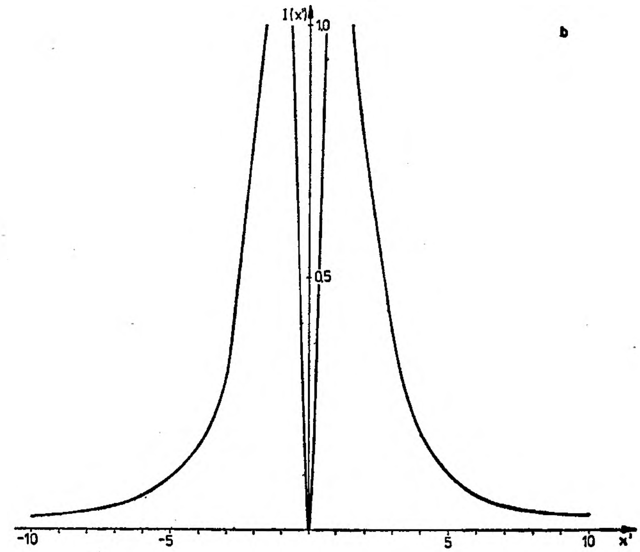
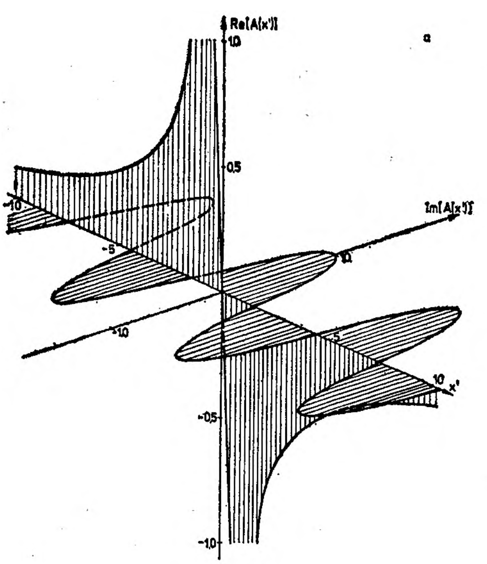


Fig. 3. Hilbert transform of the e^{ix^2} function: a — amplitude distribution, b — intensity distribution



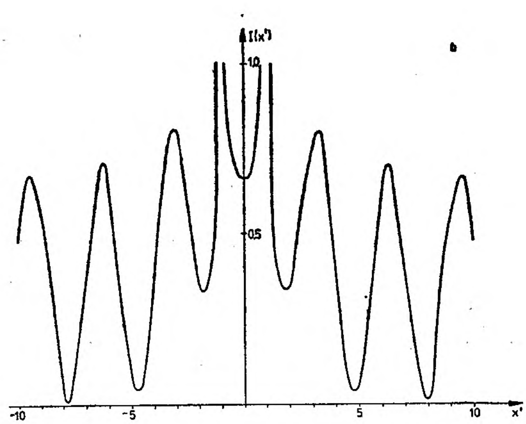


Fig. 4. Hilbert transform of the $f(x) = e^{ia\sin(\omega_0 x)}$ function: a — amplitude distribution, b — intensity distribution

Figs. 3a, b and 4a, b illustrate the situation. Actually, we work on a construction of a filter representing the three-dimensional kernel of Hilbert transform and on its application to produce the quadratures of aperture functions in an incoherent processor.

Translated by Ireneusz Wilk

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Single-wavelength coding of colour in one-step rainbow holography

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1. Introduction

The basic tricks used in the rainbow holography consist in restricting the object beam with a narrow slit and next recording this beam after its passage through an imaging system. Thus, the wavefronts generated by the examined object are recorded together with the light beam diffracted by the slit on a rainbow hologram. Consequently, in the reconstruction step the image of the object is reconstructed together with the image of the slit. In monochromatic light the reconstructed image is visible from the position where the slit image is reconstructed. When white light is used to reconstruction, the image of the slit becomes spectrally diffused, but from the given point of this image a monochromatic image of the object may be seen. By locating the eye in another point of the slit image the image of the object will appear in another colour.