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Quasi-plane electromagnetic wave of optical frequency*

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In this paper an initial value problem is formulated for electromagnetic field in a homogeneous, isotropic, lossy medium for the case of one space variable. It is possible to look for the solution of the mentioned problem in the form of an asymptotic series with respect to reciprocal large wave number. It turns out that the first asymptotic expression constitutes a quasi-plane electromagnetic wave. The introduced concept finds its applications in the laser theory.

1. Introduction

An initial value problem is formulated for the electromagnetic field in a homogeneous, isotropic, lossy medium for the case of one space variable. The external current and initial data are assumed in the following forms:

$$J(x, t) = J^{+}(x, t)e^{ik(x-vt)} + J^{-}(x, t)e^{ik(x+vt)}, \tag{1a}$$

$$\nabla J(x, t) = 0, \tag{1b}$$

$$E(x, 0) = E_0(x)e^{ikx}, H(x, 0) = H_0(x)e^{ikx},$$

$$E_0(x) = 1_x = 0, \; H_0(x)1_x = 0, \tag{2}$$

(1_x – denotes unit vector along x-axis), where the amplitudes are independent of the wave number k, $k = 2\pi/\lambda$.

The sourceless (1b) external current J includes that

$$J^{+}\cdot 1_x = 0, J^{-}\cdot 1_x = 0. \tag{3}$$

In the following E, H, J, A denote perpendicular vectors to x-axis.

By assuming (1b) Maxwell's equations are equivalent to the wave equation for vector potential function $A(x, t)$. The electric and magnetic fields are expressed by

$$E = -\frac{\partial A}{\partial t}, H = \frac{1}{\mu}1_x \times \frac{\partial A}{\partial x}. \tag{4}$$

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The electromagnetic initial value problem can be formulated for wave equation in the following form:

$$L[A] = -\mu J, \tag{5a}$$

$$\left.\frac{\partial A}{\partial x}\right|_{t=0} = -\mu 1_x \times H_0(x) e^{ikr}, \quad \left.\frac{\partial A}{\partial t}\right|_{t=0} = -E_0(x) e^{ikr} \tag{5b}$$

where L is the differential operation

$$L = \frac{\partial^2}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \frac{2a}{v}\frac{\partial}{\partial t}, \tag{6}$$

and v denotes wave propagation velocity, $v = (\mu\varepsilon)^{-1/2}$, α — permittivity, μ — permeability. The parameter a is defined by

$$a = \frac{\sigma}{2}\eta, \ \eta = \sqrt{\mu/\varepsilon}, \tag{7}$$

σ — conductivity, η — intrinsic impedance of free space.

This initial value problem may be solved by Riemann-Green's function for operation (6) (see [1]). In the paper [1] the following theorem was proved:

Theorem: Let $J^+(x, t)$, $J^-(x, t)$, $E_0(x)$, $H_0(x)$ be an analytic function in the domain $t \geqslant 0$, $-\infty < x < \infty$. Then the solution $A(x, t)$ of initial value problem (5) may be expanded into the asymptotic series of wave number

$$A(x, t; k) \sim e^{ik(x-vt)} \sum_{n=1}^{\infty} \frac{A_n^+(x, t)}{(ik)^n} + e^{ik(x-vt)} \sum_{n=1}^{\infty} \frac{A_n^-(x, t)}{ik^n}. \tag{8}$$

This asymptotic convergence is uniform in any bound domain of variables x, t. The series may be differentiated term by term with respect to variables (x, t).

2. Expansion in asymptotic series

Although there exists an exact solution of initial value problem (5) in the form of an integral [1], it is too complex to be used. Assuming a large wave number, i.e., a very short wave (optical frequencies) we may consider the solution in the form of asymptotic series. According to *Theorem* we substitute the series (8) into the wave equation and into the initial conditions (5). By comparing the terms $(ik)^{-n}$ we obtain the following partial differential equations of first order and initial values for coefficients of asymptotic series:

$$M^{\pm}[A_1^{\pm}] = -\frac{1}{2}\mu J^{\pm},$$

$$A_1^{\pm}|_{t=0} = -\frac{1}{2}\left(\varphi(x) \mp \Psi(x)\right) \tag{9}$$

where

$$\varphi(x) = \mu 1_x \times H_0(x),\ \Psi(x) = \frac{1}{v} E_0(x), \tag{10}$$

and differential operation

$$M^{\pm} = \frac{\partial}{\partial x} \pm \frac{1}{v}\frac{\partial}{\partial t} \pm a. \tag{11}$$

For higher coefficients ($n = 2, 3, \dots$) we obtain the following recursive system of equation with the same differential operations:

$$M^{\pm}[A^{\pm}_{n+1}] = -\frac{1}{2} L[A_n],$$

$$A^{\pm}_{n+1}|_{t=0} = -\frac{1}{2}\left(\frac{\partial A^+_n}{\partial x} \mp \frac{1}{v}\frac{\partial A^+_n}{\partial t} + \frac{\partial A^-_n}{\partial x} \mp \frac{1}{v}\frac{\partial A^-_n}{\partial t}\right)_{t=0}, \tag{12}$$

$n = 1, 2, \dots$.

The Cauchy's problems (9), (12) are of a common form

$$\frac{\partial u}{\partial x} \pm \frac{1}{v}\frac{\partial u}{\partial t} \pm au = -f(x, t), \tag{13}$$

$$u|_{t=0} = u_0(x).$$

This initial value problem is very simple to solve. The solution obtained by using the characteristics [2] is of the following form:

$$u(x, t) = -\int_0^{vt} f\left(x \mp \xi, t - \frac{\xi}{v}\right) e^{-a\xi} d\xi + u_0(x \mp vt) e^{-avt}. \tag{14}$$

Obviously, the procedure of recursive equations (9), (12) requires the assumption used in the formulation of *Theorem.*

3. Definition of quasi-plane wave

According to *Theorem* and to formulae (4) we obtain asymptotic series for electric and magnetic fields

$$E \sim e^{ik(x-vt)} \sum_{n=0}^{\infty} \frac{E^+_n(x, t)}{(ik)^n} + e^{ik(x+vt)} \sum_{n=0}^{\infty} \frac{E^-_n(x, t)}{(ik)^n},$$

$$H \sim e^{ik(x-vt)} \sum_{n=0}^{\infty} \frac{H_n(x, t)}{(ik)^n} + e^{ik(x+vt)} \sum_{n=0}^{\infty} \frac{H^-_n(x, t)}{(ik)^n} \tag{15}$$

where the coefficients are of the forms

$$E_n^{\pm} = -\left(\frac{\partial A_n^{\pm}}{\partial t} \pm v A_{n+1}^{\pm}\right),\ H_n^{\pm} = \frac{1}{:\mu} 1_x \left(\frac{\partial A_n^{\pm}}{\partial x} + A_{n+1}^{\pm}\right), n = 0, 1, \ldots, \tag{16a}$$

and in particular

$$A_0^{\pm} \equiv 0. \tag{16b}$$

The first terms of asymptotic expansions (15) yield [3]:

$$E(x, t; k) = e^{ik(x-vt)} E_0^{\pm}(x, t) + e^{ik(x+vt)} E_0^{-}(x, t) + 0(k^{-1}), \tag{17}$$
$$H(x, t; k) = e^{ik(x-vt)} H_0^{+}(x, t) + e^{ik(x+vt)} H_0^{-}(x, t) + 0(k^{-1}).$$

We recall that the amplitudes $E_0(x, t)$, $H_0(x, t)$ are independent of the wave number k.

These first terms (17) of asymptotic expansion (15) constitute a quasi-plane electromagnetic wave.

According to (16) and to equation (9) we obtain Cauchy's problems for electric field of quasi-plane wave

$$\frac{\partial E_0^{+}}{\partial x} + \frac{1}{v}\frac{\partial E_0^{+}}{\partial t} + \frac{\sigma}{2}\eta = -\frac{1}{2}\eta J^{+}(x, t), \tag{18a}$$

$$E_0^{+}(x, 0) = \frac{1}{2}[E_0(x) - \eta 1_x \times H_0(x)] - \frac{\partial E_0^{-}}{\partial x} + \frac{1}{v}\frac{\partial E_0^{-}}{\partial t} + \frac{\sigma}{2}\eta E^{-} =$$
$$= -\frac{1}{2}\eta J^{-}(x, t), \tag{18b}$$
$$E_0^{-}(x, 0) = \frac{1}{2}[E_0(x) + \eta 1_x \times H_0(x)].$$

Amplitudes of magnetic field of quasi-plane wave are:

$$H_0^{+} = \frac{1}{\eta} 1_x \times E_0^{+}(x, t),\ H_0^{-} = -\frac{1}{\eta} 1_x \times E_0^{-}(x, t). \tag{19}$$

They express the dependence of $H_0^{\pm}(x, t)$ on $E_0^{\pm}(x, t)$ just the same as for the plane wave.

Quasi-plane wave is a good approximation of exact solution of initial value problem (5), (4) (see formula (17)) in the region of optical frequencies.

If, however, the initial values (2) or the amplitudes of external current suffer a considerable change along one wavelength and within one wave period, i.e., the variability of the above functions is compared with the phase variability $k(x \mp vt)$, the quasi-plane wave is not regular enough to be considered as an electromagnetic wave. Then it is necessary to take account of higher terms of asymptotic expansion (15).

The introduced concept of a quasi-plane electromagnetic wave finds its application in the laser theory [4].

References

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