Optica Applicata, Vol. XXIV, No. 3, 1994

Analytical relations for optical constant determination knowing a special angle and a reflectance value at the surface of an absorbing medium

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Simple forms of equations for the pseudo-Brewster angle and the principal angle of reflection at the surface of an absorbing medium are obtained. Analytical relations are presented for the rectangular coordinates in the plane of the complex dielectric constant corresponding to a special angle and a reflectance value such as the value of the normal incidence reflectance, the *p*-polarization reflectance, or the ratio of *p*- and *s*-polarization reflectances at the respective special angle.

1. Introduction

The reflection of a collimated light beam at the planar interface between a transparent medium of incidence and an absorbing medium of refraction is determined by the complex Fresnel coefficients [1]

$$r_v = |r_v|\exp(j\delta_v), \quad v = p,\ s \tag{1}$$

where p and s identify the linear polarizations parallel and perpendicular to the plane of incidence, respectively. $|r_v|$ and δ_v ($v = p$, s) are the magnitude and absolute phase angle for the complex reflection coefficient r_v. The intensity (or power) reflectance is given by

$$R_v = r_v r_v^* = |r_v|^2, \quad v = p,\ s. \tag{2}$$

The term r_v^* is the complex conjugate of r_v, as usual. Two special angles associated with the external reflection of light at the surface of an absorbing medium are well known [2]. They are the pseudo-Brewster angle of minimum parallel reflectance R_p and the principal angle at which the differential phase shift $\Delta = \delta_p - \delta_s = 90°$. Relations between the special angles and the complex refractive index $\hat{n} = n - jk$ of the absorbing medium have been derived by different authors [3]–[7]. The algebra is very laborious and there may be routes to simplify the final results. This paper uses simplified relations for the Fresnel reflection coefficients at normal and oblique incidence [8]. Following this procedure simpler relations are obtained for the special angles. Then, relations are inferred for the contours of constant special angles in the complex plane of the dielectric constant ε ($\varepsilon = \hat{n}^2 = \varepsilon_r - j\varepsilon_i$) [9].

Nomograms of lines of constant special angles and lines of constant reflectances are frequently used for the determination of optical constants n and k of the absorbing medium of refraction [8], [10]. In this paper we present analytical relations for the rectangular coordinates (ε_r, ε_i) in the plane of the complex dielectric constant corresponding to a special angle and another reflection parameter such as the normal incidence reflectance R_0, the p-polarization reflectance R_p, and the ratio R_p/R_s of the p- and s-polarization reflectances at the respective special angle. By using these relations the optical constants n and k can be determined directly from two measured reflection parameters, with no need to use nomograms. Even if the nomograms are used, the errors inherent in the graphical methods can be avoided by checking the results analytically.

2. Relations for the Fresnel reflection coefficients at normal and oblique incidence

The complex amplitude Fresnel reflection coefficients r_p and r_s of the p- and s-polarized monochromatic plane waves of light at the planar interface between a transparent medium of incidence and an absorbing medium of refraction are given in [1] as:

$$r_p = [\varepsilon\cos\Phi - (\varepsilon - \sin^2\Phi)^{1/2}/[\varepsilon\cos\Phi + (\varepsilon - \sin^2\Phi)^{1/2}], \tag{3a}$$

$$r_s = [\cos\Phi - (\varepsilon - \sin^2\Phi)^{1/2}/[\cos\Phi + (\varepsilon - \sin^2\Phi)^{1/2}], \tag{3b}$$

Φ is the angle of incidence, and $\varepsilon = \varepsilon_1/\varepsilon_0$ is the complex relative dielectric constant, where ε_0 (real, usually $\varepsilon_0 = 1$) and ε_1 (complex) are the dielectric constants of the media of incidence and refraction, respectively. This paper uses the Nebraska–Muller convention, $r_p = -r_s$ at normal incidence.

The amplitude Fresnel reflection coefficient r_0 at normal incidence is given in [1] as

$$r_0 = (\varepsilon^{1/2} - 1)/(\varepsilon^{1/2} + 1) \tag{4}$$

and it can be written in the form

$$r_0 = (1 - P_0)/(1 + P_0) \tag{5}$$

where $P_0^2 = 1/\varepsilon$. Then, the Fresnel reflection coefficients r_v, $v = p$, s, which are given by Eqs. (3), can be expressed in the simple form

$$r_v = (1 - P_v)/(1 + P_v), \quad v = p,\ s \tag{6}$$

where

$$P_p = P_0[1 + (1 - P_0^2)u]^{1/2}, \quad (v = p), \tag{7a}$$

$$P_s = [1 + (1 - P_0^2)u]^{1/2}/P_0, \quad (v = s), \tag{7b}$$

and $u = \tan^2\Phi$. One can see that

$$P_p = P_s P_\varepsilon^2. \tag{8}$$

From Equations (5) and (6) the following inverse relations are obtained:

$$P_0 = (1 - r_0)/(1 + r_0), \tag{9a}$$

$$P_v = (1 - r_v)/(1 + r_v), \quad v = p,\ s. \tag{9b}$$

With r_v, $v = p$, s, expressed in the form of Eq. (6), we find for the ratio $\rho = r_p/r_s$ of the complex Fresnel reflection coefficient the following simple expression:

$$\rho = (u - P_s)/(u + P_s). \tag{10}$$

In some cases it is easier to work with

$$Q = (1-\rho)/(1+\rho). \tag{10}$$

Then, one obtains

$$Q = P_s/u. \tag{12}$$

3. Equations for the special angles

Taking $u = \tan^2\Phi$ as independent variable, the condition for the minimum of R_p is

$$\partial R_p/\partial u = 0. \tag{13}$$

Evaluating this derivative gives the following simple cubic equation:

$$\alpha\beta u^3 + 3\alpha u^2 - u - 1 = 0 \tag{14}$$

for the pseudo-Brewster angle Φ_{pB} ($u = \tan^2\Phi_{pB}$) at given complex relative dielectric constant ε of the absorbing medium of refraction, $\varepsilon = \varepsilon_1/\varepsilon_0 = (n-jk)^2/\varepsilon_0 = \varepsilon_r - j\varepsilon_i$, and $|\varepsilon|^2 = \varepsilon_r^2 + \varepsilon_i^2$, where:

$$\alpha = (P_0 P_0^*)^2 = 1/|\varepsilon|^2, \tag{15a}$$

$$\beta = 1 - (P_0^2 + P_0^{*2}) = 1 - 2\varepsilon_r/|\varepsilon|^2. \tag{15b}$$

At the principal angle Φ_p the differential phase shift $\Delta = \delta_p - \delta_s = 90°$. This means that $\mathrm{Re}(\rho) = 0$, and then $QQ^* = 1$, where Q is defined by Eq. (11). Using Equation (12) gives at the principal angle

$$P_s P_s^* = u^2 \tag{16}$$

where P_s is given by Eq. (7b), and the following cubic equation is obtained:

$$u^3 - u^2 - (\beta/\alpha)u - 1/\alpha = 0 \tag{17}$$

where $u = \tan^2\Phi_p$, and α and β are given by Eqs. (15). Equations for the special angles written in more complex forms can be found in [2]–[8]. Algebraic methods for the solving of cubic and fourth order equations can be found, for example, in [9].

4. Constant special angle contours in the complex ε plane

If α and β (Equations (15)) are substituted into Equations (14) and (17), one obtains the following equations for the contours of constant special angles in rectangular coordinates $(\varepsilon_r, \varepsilon_i)$ of the complex ε plane:

– for the pseudo-Brewster angle Φ_{pB}, $u = \tan^2\Phi_{pB}$

$$|\varepsilon|^4(1+u) - |\varepsilon|^2 u^2(3+u) + 2\varepsilon_r u^3 = 0, \tag{18}$$

– for the principal angle Φ_p, $u = \tan^2\Phi_p$

$$[\varepsilon_r - u/(1+u)]^2 + \varepsilon_i^2 = [u^2/(1+u)]^2. \tag{19}$$

The contour of constant principal angle is a circle of radius $u^2/(1+u)$ with the centre at $\varepsilon_r = u/(1+u)$ and $\varepsilon_i = 0$. The contour of constant pseudo-Brewster angle resembles a cardioid [6].

5. Relations for the coordinates $(\varepsilon_r, \varepsilon_i)$ in the complex ε plane knowing a special angle and the normal incidence reflectance

The values of optical constants n and k corresponding to a point $(\varepsilon_r, \varepsilon_i)$ in the complex ε plane are given by relations:

$$n^2 = (|\varepsilon| + \varepsilon_r)/2, \tag{20}$$
$$k^2 = \varepsilon_i^2/[2(|\varepsilon| + \varepsilon_r)]. \tag{21}$$

One method for the determination of n and k is based on the measurement of the normal incidence intensity (power) reflectance R_0 and the pseudo-Brewster angle Φ_{pB}. Nomograms of lines of constant Φ_{pB} and lines of constant R_0 are then used [6]. In this section we present analytical relations for the rectangular coordinates $(\varepsilon_r, \varepsilon_i)$ in the complex ε plane corresponding to a known special angle and a specified value of the normal incidence reflectance R_0. We use the following notation:

$$f_0 = (1+R_0)/(1-R_0). \tag{22}$$

At known pseudo-Brewster angle Φ_{pB} and reflectance R_0, the coordinate ε_r in the complex ε plane is determined from the following quadratic equation:

$$a\varepsilon_r^2 + b\varepsilon_r + c = 0 \tag{23}$$

with coefficients given by:

$$a = 4(1+u)f_0^2, \tag{24a}$$
$$b = 2[u^3 - f_0(2+2u+3u^2+u^3)], \tag{24b}$$
$$c = 1+u+3u^2+u^3, \tag{24c}$$

and $u = \tan^2\Phi_{pB}$.

At known principal angle Φ_p and reflectance R_0, ε_r is determined from the relation

$$2\varepsilon_r = [1+u-u^2(1-u)]/[(1+u)f_0 - u] \tag{25}$$

where $u = \tan^2\Phi_p$.

The coordinate ε_i is determined for every known special angle from the relation

$$\varepsilon_i = (2\varepsilon_r f_0 - \varepsilon_r^2 - 1)^{1/2}. \tag{26}$$

6. Relations for the coordinates $(\varepsilon_r, \varepsilon_i)$ in the complex ε plane knowing a special angle and the p-polarization reflectance at this special angle

Another method for the determination of n and k of an absorbing medium is based on the measurement of the p-polarization reflectance R_p at the pseudo-Brewster angle Φ_{pB} or the principal angle Φ_p [2]. We use the notation

$$g_p = (1-R_p)/(1+R_p). \tag{27}$$

At known pseudo-Brewster angle Φ_{pB} and R_p at Φ_{pB} one obtains the following quadratic equation for $x = |\varepsilon|^2 \geqslant u^2/(1+u)$:

$$a_4x^4+a_3x^3+a_2x^2+a_1x+a_0 = 0 \tag{28}$$

with coefficients given by:

$$a_0 = u^7g_p^2, \tag{29a}$$
$$a_1 = -2u^5g_p^2(1+2u), \tag{29b}$$
$$a_2 = u^3[4u+g_p^2(1+2u)^2], \tag{29c}$$
$$a_3 = -u^2(1+u)(5+u), \tag{29d}$$
$$a_4 = (1+u)^2. \tag{29e}$$

Inserting the positive roots of Equation (28) into Equation (18) gives:

$$\varepsilon_r = x[3+u-x(1+u)/u^2]/(2u), \tag{30a}$$
$$\varepsilon_i = (x-\varepsilon_r^2)^{1/2}. \tag{30b}$$

At known principal angle Φ_P and R_p at Φ_P, one obtains the following quadratic equation for the coordinate ε_r:

$$\varepsilon_r^2+c_1\varepsilon_r+c_0 = 0 \tag{31}$$

with coefficients given by:

$$c_0 = u[u^3+(1-u)(1+u)^2/(2g_p^2)], \tag{32a}$$
$$c_1 = 2u^2-(1+u)^3/(2g_p^2), \tag{32b}$$

and $u = \tan^2\Phi_P$. Using Equation (19) gives

$$\varepsilon_i = \{u[2\varepsilon_r+u(u-1)]/(1+u)-\varepsilon_r^2\}^{1/2}. \tag{33}$$

7. Relations for the coordinates $(\varepsilon_r, \varepsilon_i)$ in the complex ε plane knowing a special angle and the ratio R_p/R_s at this special angle

Optical constants n and k of an absorbing medium of refraction can also be determined knowing the ratio R_p/R_s at Φ_{pB} or Φ_P [2]. We use the notation

$$g_{ps} = (1-R_p/R_s)/(1+R_p/R_s). \tag{34}$$

At known pseudo-Brewster angle Φ_{pB} and R_p/R_s at Φ_{pB}, one obtains the following quadratic equation for $x = |\varepsilon|^2 \geqslant u^2/(1+u)$:

$$x^2+a_1x+a_0 = 0 \tag{35}$$

with coefficients given by

$$a_0 = u^4[4u+g_{ps}^2(1-u)^2]/[(1+u)^2(u+g_{ps}^2)], \tag{36a}$$
$$a_1 = -u^2[1+g_{ps}^2/(u+g_{ps}^2)] \tag{36b}$$

and $u = \tan^2\Phi_{pB}$. Then, the coordinates ε_r and ε_i are obtained by substituting the positive roots of Eq. (35) into Eqs. (30).

At known principal angle Φ_p and R_p/R_s at Φ_p one obtains the following relation for the coordinate ε_r:

$$\varepsilon_r = u(1-u+2ug_{ps}^2)/(1+u) \tag{37}$$

and then ε_i is determined by substituting Eq. (37) into Eq. (33). One can see that, at a given special angle, the relations corresponding to a specific value of R_p/R_s are simpler than those for R_p.

8. Conclusions

Simple forms of equations for the special angles are presented in this paper. Analytical relations are presented for the rectangular coordinates $(\varepsilon_r, \varepsilon_i)$ in the complex ε plane corresponding to a special angle and another reflection parameter such as the normal incidence reflectance R_0, the p-polarization reflectance R_p, and the ratio R_p/R_s of the p- and s-polarization reflectances at the respective special angle. These relations are useful for optical constant determination. A detailed analysis of the sensitivity of these methods has been performed numerically in [10].

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Received June 27, 1994