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Linear formulation of nonlinear propagation of optical beams and pulses

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A new method of scaled complex rays is introduced to treat the nonlinear propagation in Kerr media. The analytical solution is made possible by assuming an appropriate parabolic variation of the refractive index and a Gaussian shape of a beam cross-section. Then, a proper scaling of the propagation distance, phase-front curvature and the beam amplitude reduces the soliton-like propagation to the well-known problem of Gaussian linear propagation. The Scaled Complex Ray Method is able to treat the propagation of higher order solitons in terms of higher order Hermite–Gaussian beams.

This communication presents basis of the Scaled Complex Ray Method (SCRM) invented [1] to treat the nonlinear propagation as the appropriately scaled linear process. It is shown that soliton [2] and Hermite–Gaussian (HG) [3] beams are interrelated through the complex ray equation [4], and that SCRM can be made [1] equivalent to the reduced variational calculus [5], [6].

In a nonlinear Kerr medium, a balance between the nonlinearity, diffraction and/or dispersion is described by Nonlinear Schrödinger Equation (NSE) [2]

$$\{i\partial_z+(1/2)\partial_x^2+|V|^2(x,z)\}\,V(x,z)=0. \tag{1}$$

The key point of SCRM is to properly scale the independent and dependent variables. A suitable preliminary scaling leading to Eq. (1) and adequate notation has been introduced elsewhere [3], [4], [7]. The transverse variable $x \equiv x/w_w$ is scaled by the beam (or pulse) radius w_w at the waist ($z=0$) and the propagation direction variable $z \equiv z/z_D$ is scaled by the (Rayleigh or Fresnel) diffraction length $z_D \simeq k_L w_w^2$, k_L being the wave number in the low power limit [3]. A Gaussian beam

$$V(x,z) = A(z)\exp[-(1/2)(x/v(z))^2], \tag{2}$$

$A = a\exp(i\varphi)$, provides a presumed form of a solution to NSE (1). The complex half-width v of the beam is defined by the real beam spot-size w normalized by w_w and real radius R of phase front curvature normalized by z_D [7]

$$v^{-2}(z) = w^{-2}(z)-i/R(z). \tag{3}$$

A nonlinear index distribution is modeled by a parabolic approximation

$$|V(x,z)|^2 \simeq (1/2)w^{-2}(z)\{[1-\gamma(z)]-[1-\eta^2(z)]w^{-2}(z)x^2\}, \tag{4}$$

in which the scale factors η and γ are introduced [1] to indicate the nonlinear ($\eta \neq 1 \neq \gamma$) and linear ($\eta = 1 = \gamma$) cases and will be specified later. A factor $(1/2)w^{-2}$ anticipates a field averaging effect in the transverse cross-section of the beam. Inserting Eqs. (2)–(4) into NSE (1) gives the nonlinear complex ray and complex amplitude equations ($,_z$ means $\partial/\partial z$):

$$-iv^{-4}(z)[(v^2(z)),_z - i] + [1-\eta^2(z)]w^{-4}(z) = 0, \tag{5}$$

$$2i(\ln A(z)),_z - iv^{-2}(z) + [1-\gamma(z)]w^{-2}(z) = 0. \tag{6}$$

An essence of SCRM is to recast the nonlinear problem (5), (6) into the adequate linear problem:

$$(\bar{v}(\bar{z})^2),_{\bar{z}} - i = 0, \tag{7}$$

$$2i(\ln \bar{A}(\bar{z})),_{\bar{z}} - \bar{v}^{-2}(\bar{z}) = 0, \tag{8}$$

by scaling x and z to $\bar{x}$ and $\bar{z}$, v and A to $\bar{v}$ and $\bar{A}$, and w_0 and z_F to $\bar{w}_0$ and $\bar{z}_F$, respectively [1]. Solutions to Eqs. (7), (8) are well known from the linear theory and have the form of HG beams [3], [8]. Therefore, knowing a proper scale transformation, one is able to use these functions as the scaled solutions to the nonlinear problem at hand.

In Equations (5), (6), the scale transformation has to incorporate η and γ into expressions of the new beam parameters $\bar{v}$ and $\bar{A}$. To this end, let us scale z_D by η what implies also the scaling of z:

$$\bar{z}_D = (\eta(z))^{-1} z_D, \quad \bar{z} = \eta(z) z, \tag{9}$$

and scale the phase front curvature and the beam phase

$$\bar{R}(\bar{z}) = \eta(z) R(z), \tag{10}$$

$$\bar{\varphi}(\bar{z}) = \varphi(z)\eta(z)/\gamma(z), \tag{11}$$

with x, w_w and a remaining unchanged. The scaling (9)–(11) transforms the nonlinear problem (5), (6) into the linear problem (7), (8) in a new, scaled space $(\bar{x},\bar{z})$. The result [1] of the scaling is the solution to the linear problem

$$V(x,z) = \bar{a}(\bar{z})\exp[(1(\gamma/\eta)\bar{\varphi}(\bar{z})]\exp[(1/2)\bar{x}^2(i\eta\bar{R}^{-1}(\bar{z}) - \bar{w}^{-2}(\bar{z})], \tag{12}$$

with the imposed scaled effects on z, R and φ, recognized as the self-shortening (9), self-focusing (10) and the phase self-modification (11), respectively.

In general, the factors η and γ slowly vary with z. It can be proved [9] that SCRM is valid when $\eta \simeq 1 \simeq \gamma$ or $\bar{z} \simeq 0$. The first case takes place in the linear limit when $|V|^2 \simeq 0$, while the second condition means $\eta \simeq 0$, *i.e.*, the self-trapped soliton case when the equation of the straight complex ray (7) is *exactly* fulfilled [9]. For the case of nonzero radiation contribution to the total field, η only approximates zero, ray equation is satisfied approximately and the SCRM works only in a sufficiently thin, transverse to the propagation direction section of the medium [9]. In each section the complex ray is straigth but, as scaling factors η and γ change with z, the ray in the whole medium is curved. In both cases the numerical implementation of SCRM is straightforward [9].

All that is left to find is a proper specification of η and γ. One such a specification can be found [1] on the grounds of the variational analysis of the problem [5], [6]. From the Euler–Lagrange equations for the reduced problem [5] and for w, R, a, φ as dependent variational variables, Eqs. (5), (6) appear with:

$$\eta^2(z) = 1 - w/w_s, \tag{13}$$
$$\gamma(z) = 1-(5/2)w/w_s, \tag{14}$$
$$w_s = 2^{1/2}a_0^{-2}w_0^{-1}, \tag{15}$$

a_0, w_0 being initial values of a and w at $z = z_0$, respectively. The field distribution (12) is now completely determined. Equations (13)–(15) are the compatibility conditions between the SCRM and the variational approach. The parameter w_s has a meaning of the beam or soliton width. Indeed, if w is constant, then $w = w_0 = w_w = w_s = 1$, $a = a_0 = a_s = 2^{1/4}$ and Eq. (12) exhibits the self-trapped soliton beam in variational approximation [5]

$$V(x,z) = 2^{1/4}\exp[(3/4)iz]\exp[-(1/2)x^2]. \tag{16}$$

The soliton field, as given by Eq. (12) fits well to the exact solution, as it has been shown elsewhere [6].

The variational specification (13)–(15) of the factors η and γ is by no means unique and other procedures could, in principle, be applied here as well (*e.g.*, see estimations in [10]–[13]). In spite of that, the variational approach is clearly compatible with the complex ray description. Both methods can be applied in analysis of other propagation phenomena like soliton propagation in multi-dimensions [2], [6], [9], beam interactions with nonlinear interfaces [3], [4], [7], [14] or propagation of higher order ($N > 1$) optical solutions.

In general, the scaled Equations (7) and (8) have solutions in the form of HG beams of an arbitrary order, built from the fundamental Gaussian (2) times the Hermite polynomial of n-th order

$$H_n(y) = (-1)^n\exp(y^2)(d^n/dy^n)\exp(-y^2),$$

with the argument and beam amplitude properly chosen [3]. For even HG beams the parabolic approximation (4) is still valid and the complex ray equation (7) remains unchanged. Therefore, SCRM can treat the propagation of the higher order HG beams as well. The question immediately arises as to whether the higher order HG beams model the higher order ($N > 1$) solitons. The first, preliminary attempt to verify this thesis has been made [15].

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