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The usable form of the left semiprojection of the displacement-gradient tensor in the fringe visibility method and its applications to evaluation of holographic interferograms*

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The usable form of the left semiprojection of the displacement-gradient tensor occurring in the fringe visibility method of holographic interferometry is given for small deformations of solids. Application of this form to computing the strain and rotation tensors is presented.

1. Introduction

The fringe-visibility method of holographic interferometry [1], [2] gives the possibility to determine the quantities relating to the displacement and strain fields without using differential methods.

As it results from the analysis of the fundamental equations of holographic interferometry [2], [3], the direct information concerning the strain and rotation of the object surface is contained only in the argument of the aperture function of the imaging system used during the reconstruction of a double-exposed hologram.

The light-intensity distribution I in the image plane of the imaging system is described by the following expression [4]:

$$I = I_0\left\{1+V\cos\left[D+\frac{\pi}{2}(1-\operatorname{sgn}P)\right]\right\}, \tag{1}$$

where I_0 is the light-intensity distribution in the object image, $V=|P|$ is the fringe-visibility function, D is the phase-difference function, and P — the aperture function, being the Fourier transform of the pupil function which describes the shape of the diaphragm in the imaging system

$$P(\vec{x}) = \iint\limits_{-\infty}^{+\infty} p(\vec{r})\exp(-ik\vec{x}\cdot\vec{r})d\vec{r} \tag{2}$$

where $\vec{r}$ is the radius-vector in the diaphragm plane, k is the wave number, and $\vec{x}$ —

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the vector being the aperture-function argument. If the object surface is illuminated by a plane wave, this argument takes the following form [4]:

$$\vec{x} = \frac{1}{L}N_a[N_v\vec{u}+(L_v-L)S_vN_s(\vec{\nabla}\otimes\vec{u})\vec{g}] \tag{3}$$

where L and L_v are the focusing distance of the imaging system, and the distance of the object surface from the diaphragm centre on the viewing direction, respectively, $\vec{u}$ is the vector of the displacement of the observed point of the object surface between two holographic exposures, $\vec{g}$ is the sensitivity vector of the holographic arrangement, i.e., the sum of the unit vectors related to the illumination and viewing directions, respectively, N_a, N_s and N_v are the operators of the normal projection onto the diaphragm plane of the imaging system, the plane tangent to the object surface and the plane perpendicular to the viewing direction, respectively, S_v denotes the oblique projection, along the vector normal to the object surface in the investigated object point, onto the plane perpendicular to the viewing directions, $\vec{\nabla}\otimes\vec{u}$ is the tensor product of the nabla and displacement vectors, called the displacement-gradient tensor, and $N_s(\vec{\nabla}\otimes\vec{u})$ is its left semiprojection.

The solution of Eq. (3) with respect to the tensor $N_s(\vec{\nabla}\otimes\vec{u})$ requires that the following system of three equations be solved

$$N_s(\vec{\nabla}\otimes\vec{u})\vec{g}^{(k)} = \vec{w}^{(k)}, \quad k = 1, 2, 3 \tag{4}$$

where

$$\vec{w}^{(k)} = \frac{1}{L_v-L}N_sA_v\left\{L\vec{x}^{(k)}-\frac{1}{L-L'}[(L_v-L')L\vec{x}^{(1)}-(L_v-L)L'\vec{x}'^{(1)}]\right\}, \tag{5}$$

under condition that three different holographic interferograms with different sensitivity vector $\vec{g}^{(k)}$ have an identical viewing direction. The vectors $\vec{x}^{(k)}$, $(k = 1, 2, 3)$, should be experimentally determined from the measurements of fringe visibility (see Eqs. (1) and (2)) for different interferograms relating to different illumination vectors and, consequently, for different sensitivity vectors. The vector $\vec{x}'$ is related to the focusing distance L', and the other $\vec{x}$-vectors — to L. The operator A_v denotes the oblique projection, along the normal to the diaphragm plane, onto the plane perpendicular to the viewing direction.

The purpose of this paper is to find the usable form of the left semiprojection of the displacement-gradient tensor to enable the preparation of appropriate numerical algorithms for computer calculations of the strain and rotation tensors.

2. Solution of the problem

Let $\vec{f}_1, \vec{f}_2, \vec{f}_3 = \vec{n}_s$ and $\vec{f}^1, \vec{f}^2, \vec{f}^3 = \vec{n}_s$ be covariant and contravariant bases, respectively, originating in the investigated object point, where $\vec{n}_s$ is the normal of the object surface. Then the semi-interior tensor $N_s(\vec{\nabla}\otimes\vec{u})$ can be defined by six components $U_{\alpha i}$ ($\alpha = 1, 2$; $i = 1, 2, 3$) in the contravariant base $\{\vec{f}^i\}_{i=1,2,3}$

$$N_s(\vec{\nabla}\otimes\vec{u}) = U_{\alpha i}\vec{f}^\alpha\otimes\vec{f}^i \tag{6}$$

where the repetition of indices implies summation.

By virtue of (6), the Eq. (4) can be written in the form

$$U_{\alpha i}(\vec{f}^i\cdot\vec{g}^{(k)}) = w_\alpha^{(k)} \tag{7}$$

where $w_\alpha^{(k)} = \vec{w}^{(k)}\cdot\vec{f}_\alpha$, and moreover, the following relationship holds:

$$(N_s A_v \vec{x})\cdot\vec{f}_\alpha = \vec{x}\cdot\vec{f}_\alpha - \frac{\vec{n}_a\cdot\vec{f}_\alpha}{\vec{n}_a\cdot\vec{n}_v}\vec{x}\cdot\vec{n}_v \tag{8}$$

where $\vec{n}_a$ and $\vec{n}_v$ are the unit vectors perpendicular to the diaphragm plane and relating to the viewing direction, respectively.

Equations (7) make two systems of linear equation (for $\alpha = 1$ and $\alpha = 2$) with the same determinant

$$[\vec{g}^{(1)}\cdot(\vec{g}^{(2)}\times\vec{g}^{(3)})][\vec{f}_1\cdot(\vec{f}_2\times\vec{f}_3)].$$

If the vectors $\vec{g}^{(k)}$ are not coplanar, i.e., $\vec{g}^{(1)}\cdot(\vec{g}^{(2)}\times\vec{g}^{(3)}) \neq 0$, then the solution of Eq. (7) exists in the form (see Appendix)

$$U_{\alpha i} = \frac{\vec{m}_\alpha\cdot\vec{f}_i}{\vec{g}^{(1)}\cdot(\vec{g}^{(2)}\times\vec{g}^{(3)})}, \qquad (\alpha = 1, 2;\ i = 1, 2, 3) \tag{9}$$

where $\vec{m}_\alpha = \frac{1}{2}\varepsilon_{ijk}w_\alpha^{(i)}(\vec{g}^{(j)}\times\vec{g}^{(k)})$, and ε_{ijk} denotes the completely antisymmetric Levi–Civita's symbol.

The contravariant components of the tensor $N_s(\vec{\nabla}\otimes\vec{u})$ can be obtained by raising indices α and i in Eq. (9).

3. Applications to interpretation of holographic interferograms

In the case of small deformations, the left semiprojection of the displacement-gradient tensor can be decomposed as follows [5]:

$$N_s(\vec{\nabla}\otimes\vec{u}) = \boldsymbol{\Gamma} + \Omega\boldsymbol{E} + \vec{\omega}\otimes\vec{n}_s \tag{10}$$

where $\boldsymbol{\Gamma}$ is the surface-strain tensor, $\Omega\boldsymbol{E}$ is the skew-symmetric surface tensor, $\boldsymbol{E}$ is the two-dimensional permutation tensor, Ω is the pivot-rotation angle around the vector $\vec{n}_s$, and $\vec{\omega}$ – the vector of inclination of $\vec{n}_s$.

By virtue of (6) we get

$$\Gamma_{\alpha\beta} = \frac{1}{2}(U_{\alpha\beta} + U_{\beta\alpha}), \tag{11}$$

$$\omega_\alpha = U_{\alpha 3}, \tag{12}$$

$$\Omega = \frac{1}{2\sqrt{a}}(U_{12}-U_{21}) \tag{13}$$

where $a = \vec{f}_1^2\vec{f}_2^2-(\vec{f}_1\cdot\vec{f}_2)^2$ denotes the determinant of the metric tensor of the object surface.

Finally, the surface-rotation vector, defined in [4] by

$$\vec{\omega}_s = \omega\mathscr{E}\vec{n}_s+\Omega\vec{n}_s \tag{14}$$

($\mathscr{E}$ – three-dimensional permutation operator), generates the surface-rotation tensor

$$\boldsymbol{R}_s = \mathscr{E}\vec{\omega}_s. \tag{15}$$

This tensor takes the following matrix form:

$$\boldsymbol{R}_s = \begin{bmatrix} 0 & \frac{1}{2}(U_{12}-U_{21}) & -U_{13} \\ -\frac{1}{2}(U_{12}-U_{21}) & 0 & -U_{23} \\ U_{13} & U_{23} & 0 \end{bmatrix} \tag{16}$$

4. Final remarks and conclusions

The quantities determining the elements of the left semiprojection of the displacement-gradient tensor can be divided into three groups. The first group consists of the quantities $\vec{n}_a$, $\vec{n}_v$, $\vec{g}^{(k)}$, L and L' which are determined by geometry of holographic arrangement and by measuring conditions. The $\vec{x}$-vectors form the second group of the quantities which can be directly determined from measurements of fringe visibility [4]. The quantity L_v and the normal $\vec{n}_s$ of the object surface belong to that group because they can be evaluated also by means of the fringe-visibility method [6], though some other methods, e.g., holographic contouring methods, can be used, too. The vector $\vec{n}_s$ is necessary to define the vectorial base $\{\vec{f}^i\}$ composing the third group of quantities which, in general, can be chosen arbitrarily. In practice, however, the base of vectors $\vec{f}^\alpha$ should be chosen to simplify the calculations.

It should be emphasized that the obtained form (9) of the elements of the left semiprojection of the displacement-gradiednt tensor and the forms (11) and (16) of the strain and rotation tensors can be easily algorithmized for computer calculations.

Appendix

For a fixed α, where $\alpha = 1$ or 2, Eq. (7) turns into the following system of equations:

$$\begin{aligned} U_{\alpha1}(\vec{f}^1\cdot\vec{g}^{(1)})+U_{\alpha2}(\vec{f}^2\cdot\vec{g}^{(1)})+U_{\alpha3}(\vec{f}^3\cdot\vec{g}^{(1)}) &= w_\alpha^{(1)}, \\ U_{\alpha1}(\vec{f}^1\cdot\vec{g}^{(2)})+U_{\alpha2}(\vec{f}^2\cdot\vec{g}^{(2)})+U_{\alpha3}(\vec{f}^3\cdot\vec{g}^{(2)}) &= w_\alpha^{(2)}, \\ U_{\alpha1}(\vec{f}^1\cdot\vec{g}^{(3)})+U_{\alpha2}(\vec{f}^2\cdot\vec{g}^{(3)})+U_{\alpha3}(\vec{f}^3\cdot\vec{g}^{(3)}) &= w_\alpha^{(3)}. \end{aligned} \tag{A1}$$

In order to solve this linear system the Cramer formulae can be used. The main determinant D_α of the system (A1) takes the form

$$D_\alpha = \begin{vmatrix} \vec{f}^1\cdot\vec{g}^{(1)} & \vec{f}^2\cdot\vec{g}^{(1)} & \vec{f}^3\cdot\vec{g}^{(1)} \\ \vec{f}^1\cdot\vec{g}^{(2)} & \vec{f}^2\cdot\vec{g}^{(2)} & \vec{f}^3\cdot\vec{g}^{(2)} \\ \vec{f}^1\cdot\vec{g}^{(3)} & \vec{f}^2\cdot\vec{g}^{(3)} & \vec{f}^3\cdot\vec{g}^{(3)} \end{vmatrix}, \tag{A2}$$

or more compactly [7]

$$D_\alpha = [\vec{f}^1\cdot(\vec{f}^2\times\vec{f}^3)][\vec{g}^{(1)}\cdot(\vec{g}^{(2)}\times\vec{g}^{(3)})]. \tag{A3}$$

The determinant for the unknown element $U_{\alpha 1}$ has the form

$$D_{\alpha 1} = \begin{vmatrix} w_\alpha^{(1)} & \vec{f}^2\cdot\vec{g}^{(1)} & \vec{f}^3\cdot\vec{g}^{(1)} \\ w_\alpha^{(2)} & \vec{f}^2\cdot\vec{g}^{(2)} & \vec{f}^3\cdot\vec{g}^{(2)} \\ w_\alpha^{(3)} & \vec{f}^2\cdot\vec{g}^{(3)} & \vec{f}^3\cdot\vec{g}^{(3)} \end{vmatrix}. \tag{A4}$$

By virtue of the Laplace theorem, the determinant $D_{\alpha 1}$ can be decomposed as follows:

$$D_{\alpha 1} = w_\alpha^{(1)}\begin{vmatrix} \vec{f}^2\cdot\vec{g}^{(2)} & \vec{f}^3\cdot\vec{g}^{(2)} \\ \vec{f}^2\cdot\vec{g}^{(3)} & \vec{f}^3\cdot\vec{g}^{(3)} \end{vmatrix} - w_\alpha^{(2)}\begin{vmatrix} \vec{f}^2\cdot\vec{g}^{(1)} & \vec{f}^3\cdot\vec{g}^{(1)} \\ \vec{f}^2\cdot\vec{g}^{(3)} & \vec{f}^3\cdot\vec{g}^{(3)} \end{vmatrix} + w_\alpha^{(3)}\begin{vmatrix} \vec{f}^2\cdot\vec{g}^{(1)} & \vec{f}^3\cdot\vec{g}^{(1)} \\ \vec{f}^2\cdot\vec{g}^{(2)} & \vec{f}^3\cdot\vec{g}^{(2)} \end{vmatrix}. \tag{A5}$$

The above determinant can take a reduced form by using the well-known identity [7]

$$\begin{vmatrix} \vec{a}\cdot\vec{c} & \vec{b}\cdot\vec{c} \\ \vec{a}\cdot\vec{d} & \vec{b}\cdot\vec{d} \end{vmatrix} = (\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) \tag{A6}$$

where $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ are arbitrary vectors. Then

$$D_{\alpha 1} = \vec{m}_\alpha(\vec{f}^2\times\vec{f}^3) \tag{A7}$$

where

$$\vec{m}_\alpha = w_\alpha^{(1)}(\vec{g}^{(2)}\times\vec{g}^{(3)}) + w_\alpha^{(2)}(\vec{g}^{(3)}\times\vec{g}^{(1)}) + w_\alpha^{(3)}(\vec{g}^{(1)}\times\vec{g}^{(2)}).$$

The base vector normal to the object surface is defined by

$$\vec{f}^3 = \vec{f}_3 = \frac{\vec{f}_1\times\vec{f}_2}{|\vec{f}_1\times\vec{f}_2|} \tag{A8}$$

and contravariant base vectors $\vec{f}^i (i = 1, 2, 3)$ – by the orthogonality relations:

$$\vec{f}^i\cdot\vec{f}_j = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases} \tag{A9}$$

Then we have

$$\vec{f}^2\times\vec{f}^3 = \frac{1}{|\vec{f}_1\times\vec{f}_2|}\vec{f}^2\times(\vec{f}_1\times\vec{f}_2). \tag{A10}$$

Substituting the identity

$$\vec{f}^2 \times (\vec{f}_1 \times \vec{f}_2) = (\vec{f}^2 \cdot \vec{f}_2) \vec{f}_1 - (\vec{f}^2 \cdot \vec{f}_1) \vec{f}_2 \tag{A11}$$

into Eq. (A10) and using the orthogonality relations (A9) we get

$$\vec{f}^2 \times \vec{f}^3 = \frac{1}{|\vec{f}_1 \times \vec{f}_2|} \vec{f}_1. \tag{A12}$$

By virtue of (A12) and (A9), the mixed vectorial product takes the following form:

$$\vec{f}^1 (\vec{f}^2 \times \vec{f}^3) = \frac{1}{|\vec{f}_1 \times \vec{f}_2|}. \tag{A13}$$

Finally, substituting Eqs. (A12) and (A13) into Eqs. (A7) and (A3), respectively, we find the searched component of the left semiprojection of the displacement-gradient tensor

$$U_{\alpha 1} = \frac{D_{\alpha 1}}{D_\alpha} = \frac{\vec{m}_\alpha \cdot \vec{f}_1}{\vec{g}^{(1)} \cdot (\vec{g}^{(2)} \times \vec{g}^{(3)})}. \tag{A14}$$

The other components of that tensor are formed in the same way. Consequently, we get the uniform expression as in Eq. (9).

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Полезное выражение левой полупроекции тензора градиента смещения в методе контраста полосы и её применение в количественной интерпретации голографических интерферограмм

В работе получено полезное выражение левой полупроекции градиента смещения в случае малых деформаций твёрдого тела для метода контраста полосы в голографической интерферометрии. Вид этого выражения удобен для вычислений на ЭВМ тензоров деформаций и вращения непрозрачных объектов исследованных методом голографической интерферометрии.