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Photon statistics in non-linear optical processes: a simplified approach

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As the light propagates through non-linear media its statistical properties undergo a significant change. It is manifested in the intensity distribution of the tramsmitted light and its moments. The fluctuation behaviour can be explicitly characterized by a few of the statistical parameters like the intensity distribution function, its first and second order moments and the associated bunching behaviour. We have discussed a simplified mathematical tool to study qualitatively the fluctuation phenomena associated with high power laser interaction with non-linear condensed matter media. To illustrate the formalism, we have adopted a model system and evaluated the various relevant statistical parameters of interest by assigning four different intensity distributions, namely exponential, Gaussian, log-normal and Dirac-delta, to the incident laser light. The formalism has been further extended to include photon counting distribution and its moments.

1. Introduction

The advent of laser has contributed enormously to the field of non-linear optics. This is because laser sources provide highly intense, monochromatic and coherent radiations spread over a frequency range starting from ultraviolet to far infrared region of the electromagnetic spectrum. Moreover, the optical properties of the laser radiation like the frequency and polarization are easily tunable. When such radiation propagates through condensed matter medium a variety of interesting physical phenomena occur depending on the experimental conditions. At sufficiently high value of incident intensity, the susceptibility χ of the medium becomes intensity dependent. This gives rise to non-linear changes in the macroscopic parameters of the medium such as the dielectric constant ε, refractive index n and molecular polarizability β. For the first order in incident light intensity I_0, the real part of susceptibility χ makes a non-linear contribution to the refractive index n and dielectric constant ε as

$$n = n_0 + n_2 \langle I_0 \rangle, \tag{1}$$

and correspondingly

$$\varepsilon = \varepsilon_0 + \varepsilon_2 \langle I_0 \rangle \tag{2}$$

where n_0 and ε_0 are the values of n and ε when I_0 is very small and n_2 and ε_2 are the first order non-linear coefficients. The angular brackets represent ensemble averages. The still higher order non-linear coefficients make smaller contributions so we will not consider them in our present discussion. Using Lorenz–Lerentz relation the refractive index of the medium can be written as

$$\frac{n^2-1}{n^2+1} = \frac{4\pi}{3}\varrho\langle\beta\rangle \tag{3}$$

where ϱ is molecular number density. From Eq. (3) it is readily inferred that changes in refractive index can arise from changes in ϱ, β or both. Different physical mechanisms are responsible for inducing changes in ϱ and β. Correspondingly, they give rise to different observable non-linear effects. These have been discussed and classified in details by SVELTO [1]. When non-linear interaction between incident laser radiation and condensed matter medium occurs, the intensities of the incident (I_0) and transmitted light (I) often exhibit non-linear dependence. For example, in two photon absorption media, the transmitted intensity at a distance z inside the media is given by [2]

$$I(z) = \frac{I_0}{(1+BzI_0)} \tag{4}$$

where B is the non-linear absorption coefficient. In optical bistable media, I_0 and I are related in the form [3]

$$I_0 = \sum_{n=0}^{3} a_n I^n \tag{5}$$

where the coefficients are characteristic parameters of the medium and the interaction process under the experimental condition. For thermal lensing media [4]

$$I = \frac{I_0}{(1+AI_0)}. \tag{6}$$

Here, A is the parameter that describes the thermal lensing effect. There are many more systems like this. In the examples cited above the incident and transmitted intensities are non-linearly related. Because of this the statistical behaviour of the transmitted light will not be the same as that of the incident light. Analysis of the fluctuation properties of transmitted light plays an important role in the overall understanding of light and condensed matter interaction.

The objective of this paper is to demonstrate how this can be achieved through simple mathematical calculations. The principle involved uses elementary statistics and the entire fluctuation properties can be explicitly derived without going into rigorous mathematics. This has been further extended to include the evaluation of photon counting distribution of the transmitted light and the associated moments. The procedure will be very useful for students and researchers who lack enough training in probability theory and statistical methods in physics. This apart from fluctuation study is of prime importance in many engineering problems dealing with signal processing. Keeping this in mind, this presentation has been tailor made to be elaborate, self-contained and exhaustive.

2. Model system and statistics of incident light

In order to simplify the problem we will be discussing it within the framework of a concieved model. Later on we will attach some physical meaning to the model itself. Let us consider a physical system where incident light of intensity I_0 enters the non-linear medium on one side and emerges out with intensity I on the other side. The intervening non-linear medium will be treated like a black box and we will not take into account the physical processes taking place inside the box. Let us also assume that I_0 and I are related as

$$I = I_0^P, \text{ for } P \text{ real.} \tag{7}$$

This is our model non-linear equation of the state between I_0 and I. We will assign different probability distributions $P(I_0)$ to the incident light and investigate its evolution as it traverses through the non-linear medium following Eq. (7). The distribution $P(I_0)$ will be assigned the following forms

$$P(I_0) = \frac{1}{\langle I_0 \rangle} e^{-\frac{I_0}{\langle I_0 \rangle}} : \text{exponential} \tag{8}$$

$$P(I_0) = \frac{2}{\sqrt{\pi \langle I_0 \rangle}} e^{-\frac{I_0^2}{\langle I_0 \rangle^2}} : \text{Gaussian} \tag{9}$$

$$P(I_0) = \frac{1}{\sqrt{2\pi \gamma I_0}} e^{-\frac{(\ln I - \langle \ln I_0 \rangle)^2}{2\gamma^2}} : \text{log-normal} \tag{10}$$

where:

$$\gamma^2 = \langle (\ln I_0)^2 \rangle - \langle \ln I_0 \rangle^2,$$

$$P(I_0) = \delta(I_0 - \langle I_0 \rangle) : \text{Dirac-delta.} \tag{11}$$

The k-th moment of the incident intensity is defined as

$$\langle I_0^k \rangle = \int_0^\infty I_0^k P(I_0) dI_0. \tag{12}$$

The degree of fluctuations associated with a particular light is quantitatively characterized by its bunching parameter defined as [5], [6]

$$\langle (\Delta I_0)^2 \rangle = \frac{\langle (I_0 - \langle I_0 \rangle)^2 \rangle}{\langle I_0 \rangle^2} \tag{13}$$

A smaller value of $\langle (\Delta I_0)^2 \rangle$ implies less fluctuations associated with the particular light concerned. For the $P(I_0)$ distributions considered in Eqs. (8)–(11), we have

$$\langle(\Delta I_0)^2\rangle = \begin{cases} 1 & \text{:exponential} \\ \left(\dfrac{\pi}{2}-1\right) & \text{:Gaussian} \\ \dfrac{\gamma^2}{\langle I_0\rangle^2} & \text{:log-normal} \\ 0 & \text{:Dirac-delta} \end{cases} \tag{14}$$

From Equation (14) it is observed that for an intensity stabilized source characterized by a Dirac-delta distribution there are no fluctuations associated with it. On the other hand, fluctuations are very pronounced in light exhibiting exponential intensity distribution. For Gaussian light it is intermediate.

3. Probability conservation

Let us consider an ensemble of stochatic variables $\{X(t)\}$. Although the instantaneous values of $X(t)$ will randomly depend on time t, the results of single-shot measurements of X can be described by a distribution function $P(X)$. This distribution function $P(X)$ describes the probability of obtaining a value X at time t. Let us assume the process to be stationary and ergodic. This will make the statistical behaviour of the ensemble independent of time translation and the time and ensemble averages will be identical. This can be put in the mathematical form

$$\langle F(X)\rangle = \int_{-\infty}^{\infty} F(X)P(X)dX = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} F(X(t))dt. \tag{15}$$

In the above equation we are evaluating the average of a certain parameter $F(X)$. It can be alternatively written as

$$\langle F(X)\rangle = \int_{-\infty}^{\infty} FP(F)dF. \tag{16}$$

Combining Equations (15) and (16) and using simple variable transformation it can be readily proved that

$$P(F)dF = P(X)dX. \tag{17}$$

This expression is often called the probability conservation rule. This implies that if $F(X)$ is expressible explicitly as a function of X and vice versa, it is possible to determine one of the distribution functions in Eq. (17) provided the other one is known a priori. More discussion on this can be found elsewhere [7].

4. Statistics of transmitted light

The fluctuation behaviour of the transmitted light can be instantly determined through the use of Eqs. (7) and (17). Identifying $X = I_0$ and $F(X) = I$ we have from Eq. (17)

$$P(I)dI = P(I_0)dI_0 \tag{18}$$

and from Eq. (7) it can be readily shown that

$$\frac{dI_0}{dI} = nI^{n-1} \text{ for } n = 1/P. \tag{19}$$

Substituting the appropriate distributions from Eqs. (8)–(11) successively in Eq. (18), we can show

$$P(I) = \frac{nI^{n-1}}{\langle I_0\rangle} e^{-\frac{I^n}{\langle I_0\rangle}} \text{ : exponential} \tag{20}$$

$$P(I) = \frac{2nI^{n-1}}{\sqrt{\pi}\langle I_0\rangle} e^{-\frac{I^{2n}}{\langle I_0\rangle^2}} \text{ : Gaussian} \tag{21}$$

$$P(I) = \frac{n}{\sqrt{2\pi}\gamma I} e^{-\frac{(n\ln I - \langle I_0\rangle)^2}{2\gamma^2}} \text{ : log-normal} \tag{22}$$

$$P(I) = \delta(I_0 - \langle I_0\rangle) \text{ : Dirac-delta} \tag{23}$$

Equations (20) through (23) describe the intensity distribution of the transmitted light corresponding to the intensity distributions of the incident light given by Eqs. (8)–(11). The first and second order moments of the transmitted light can be evaluated by substituting appropriate $P(I)$ distributions in Eq. (12). The results are

$$\langle I\rangle = \begin{cases} \langle I_0\rangle^P \Gamma(P+1) & \text{:exponential} \\ \dfrac{\langle I_0\rangle^P}{\sqrt{\pi}}\Gamma(P+1) & \text{:Gaussian} \\ \dfrac{1}{2}e^{\left((P+1)\langle \ln I_0\rangle + \frac{\gamma^2(P+1)^2}{2}\right)}(1+\operatorname{erf}(\varphi)) & \text{:log-normal} \\ \langle I_0\rangle & \text{:Dirac-delta} \end{cases} \tag{24}$$

and

$$\langle I^2\rangle = \begin{cases} \langle I_0\rangle^{2P} \Gamma(2P+1) & \text{:exponential} \\ \dfrac{\langle I_0\rangle^{2P}}{\sqrt{\pi}}\Gamma(P+1/2) & \text{:Gaussian} \\ \dfrac{1}{2}e^{\left(\langle \ln I_0\rangle(2P+1) + \frac{\gamma^2(2P+1)^2}{2}\right)}\left(1+\operatorname{erf}\left(\varphi + \dfrac{\gamma P}{\sqrt{2}}\right)\right) & \text{:log-normal} \\ \langle I_0\rangle^2 & \text{:Dirac-delta} \end{cases} \tag{25}$$

In the expressions above $\Gamma(x)$ is normal gamma function and $\operatorname{erf}(x)$ represents error function defined as [8]

$$\operatorname{erf}(x) = \frac{2}{\pi}\int_0^x e^{-x^2}dx \tag{26}$$

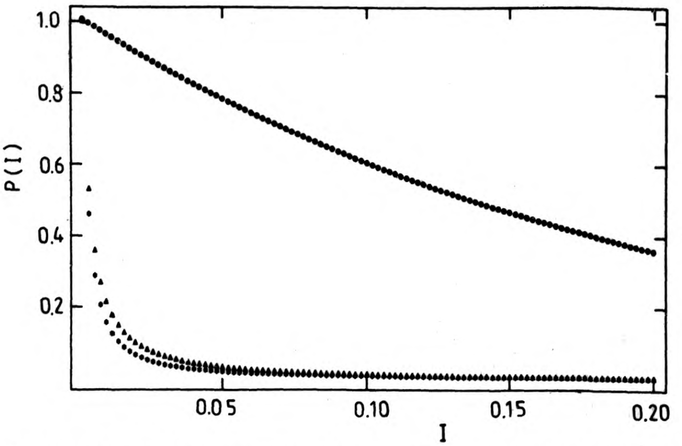

Fig. 1. Normalized probability distribution of the intensity of transmitted light $P(I)$ for incident intensity distribution $P(I_0)$ being exponential (Eq. (8)). Circle is for $P = 1$, triangle for $P = 3$ and star for $P = 5$

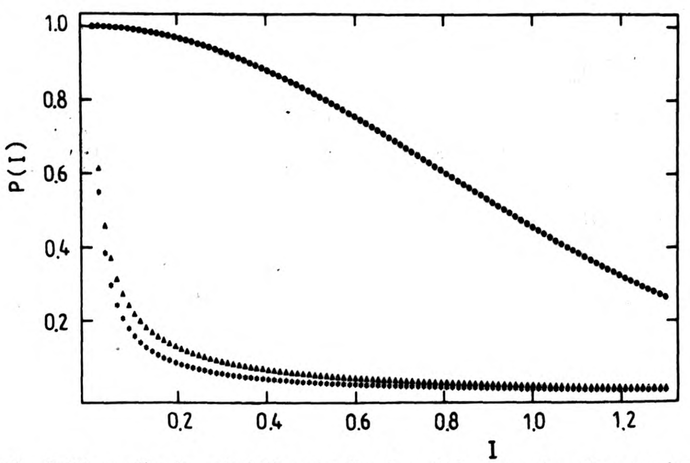

Fig. 2. Normalized probability distribution of the intensity of transmiteted light $P(I)$ for incident intensity distribution $P(I_0)$ being Gaussian (Eq. (9)). Circle is for $P = 1$, triangle for $P = 3$ and star for $P = 5$

and φ is defined as

$$\varphi = \langle \ln I_0 \rangle + (1+P)\gamma/2.$$

We can determine the bunching parameters from Eqs. (24) and (25). These come out as

$$\langle (\Delta I)^2 \rangle = \begin{cases} \dfrac{2P!}{P!P!} - 1 & \text{:exponential} \\ \dfrac{\Gamma(P+1/2)}{(\Gamma((P+1)/2))^2} - 1 & \text{:Gaussian} \\ 2e^{-(\langle \ln I_0 \rangle - \gamma^2(P^2 - P - 1/2))} \dfrac{(1+\mathrm{erf}(\varphi + P\gamma/\sqrt{2})}{(1+\mathrm{erf}(\varphi))^2} & \\ 0 & \text{:Dirac-delta} \end{cases} \tag{27}$$

The $P(I)$ distributions have been plotted as a function of P in Figs. 1–3 corresponding to different $P(I_0)$ distributions. It is obvious in these figures that $P = 1$ plots directly correspond to $P(I_0)$ distributions. The bunching parameter has been similarly plotted in Fig. 4. For making these plots we have made use of the normalized $\langle I_0 \rangle$ values given in [9].

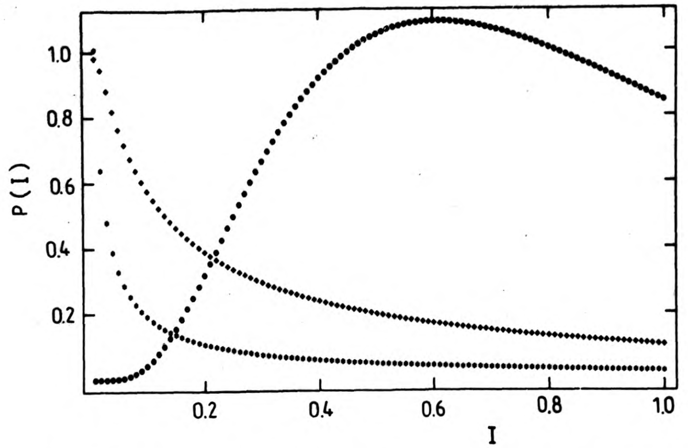

Fig. 3. Normalized probability distribution of the intensity of transmitted light $P(I)$ for incident intensity distribution $P(I_0)$ being log-normal (Eq. (10)). Circle is for $P = 1$, triangle for $P = 3$ and star for $P = 5$

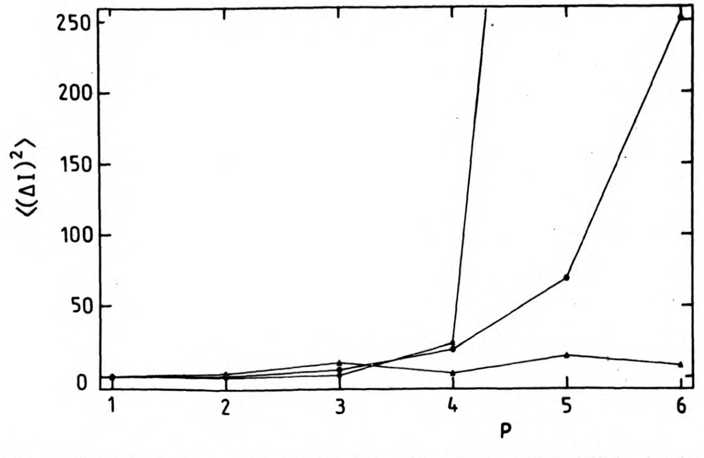

Fig. 4. Bunching parameter of the intensity of transmitted light for incident intensity distribution $P(I_0)$ being exponential (circle), Gaussian (triangle) and log-normal (star) (Eq. (27)). The solid line is for guiding the eye. For the Gaussian case the values have been multiplied by a factor 5 to get a better resolution on the graph

5. Discussion

We have shown that the statistical properties of the transmitted light can be readily determined through the probability conservation relation. However, this necessitates an explicit relation between I_0 and I and knowing the $P(I_0)$ distributions beforehand. In many experimental situations these requirements are within our control. For example, for a laser source, the $P(I_0)$ distribution can be controlled by operating the laser either below, at or above threshold of oscillation by changing the pump parameter (b) [5]. When b assumes large negative value, the $P(I_0)$ of the output intensity exhibits exponential distribution as given by Eq. (8). As the pump parameter approaches large positive value, the intensity distribution is given by a Dirac-delta function (Eq. (11)) and the source shows complete intensity stabilization. When $b = 0$, the intensity distribution of the laser light is described by a Gaussian (Eq. (9)). The log-normal distribution is observed in the physical problem of laser propagation through atmosphere [10].

On the other hand, expressions relating the incident and transmitted intensities have been worked out for many non-linear optical systems. These include processes like optical bistability, two photon absorption, intensity dependent absorption, thermal lensing, non-exponential relaxation, etc. Some of these have been explicitly mentioned in the introduction. For all these physical problems, the fluctuation properties of the transmitted light have been extensively studied using the formalism we have described here [11]–[15].

A second approach to get an insight into the fluctuation properties of light is to evaluate its photon counting statistics. It is rather straightforward to transform the intensity distributions $P(I)$ to corresponding photon distribution $P(m)$. $C(x)$ is the characteristic function defined as

$$C(x) = \int_0^\infty e^{ixI} P(I) dI. \tag{28}$$

Since we know the $P(I)$ distributions exactly, the $C(x)$ functions can be evaluated explicitly from Eq. (28). Next we define a generating function $G(s)$ as

$$G(s) = C(x = i\alpha s) \tag{29}$$

The photon counting distribution $P(m)$ and its k-th moment can be determined from [6]

$$P(m) = \left[(-1)^m \frac{d^m}{ds^m} G(s) \right]_{s=1}, \tag{30}$$

and

$$\langle m^k \rangle = \left[(-1)^k \frac{d^k}{ds^k} G(s) \right]_{s=0}. \tag{31}$$

Hence, it is possible to determine the entire statistical properties of the transmitted light either through its intensity distribution $P(I)$ or through its photon distribution $P(m)$. Of course, physically both yield the same information.

From Figure 4 it is clearly observed that except for the situation $P(I_0) = \delta(I_0 - \langle I_0 \rangle)$, the fluctuations grow as the light propagates through the non-linear media. This implies that the medium acts as an amplifier of fluctuations. However, if the fluctuations are absent in the incident light (Dirac-delta distribution), the transmitted light too will exhibit no fluctuations (zero bunching).

Finally, some comments about the model Eq. (7), $I = I_0^P$. If we substitute the intensity I_0 by the corresponding electromagnetic field E_0 and set $P = 2$, we can rewrite the equation as

$$I_0 = |E_0|^2. \tag{32}$$

In this situation the non-linear medium in our model will represent a square law detector (a photomultiplier, for example). So, the problem we have discussed reduces to the physical problem of determination of the intensity distribution $P(I_0)$ provided $P(E_0)$ are known and are given by Eqs. (8)–(11). The model equation can also be compared to equation of the state of optical bistability given by Eq. (5).

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Статистики фотонов в нелинейных оптических процессах: упрощенный подход

Статистические свойства света подвергаются значительным изменениям по мере распространения в нелинейной оптической среде. Они проявляются в распределении напряжения проходящего света, а также в его моментах. Флуктуации легко можно охарактеризовать при помощи нескольких статистических параметров, как функции распределения напряжения его первый и второй момент, а также сопутствующий момент группировки. Проведена дискуссия упрощенных математических середин для качественного исследования явления флуктуаций, связанных с воздействием лазеров большой мощности с нелинейной сгущенной материальной серединой. Для иллюстрации формализма принята модельная система, а также проведена оценка соответствующих статистических параметров посредством рассуждения четырех разных распределений напряжения, а именно: экспоненциального, Гауса, лог-нормального, а также лапсасиана Дирака. Формализм был дальше растянут на случай распределений счислений фотонов и их моментов.