*Optica Applicata, Vol. XIX, No. 1, 1989*

# Optical dispersive bistability in media of forced anisotropy

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In the paper, results concerning optical dispersive bistaility in media of forced anisotropy (due to electro- and elasto-optical effects, for instance) are presented. Radiation transmitted through a nonlinear Fabry-Perot cavity is shown to reveal bistability of intensity, of both the total field and each of its Cartesian components, as well as bistable states of polarization. If the electrical vector of the incident wave is neither parallel nor normal to the direction of optical axis the transmitted light possesses the elliptical polarization. That elliptically polarized radiation changes bistably not only the value of proportion of the polarization ellipse semiaxes but also orientation of its major semiaxis.

## 1. Formulation of problem and simplifying assumptions

In the paper a Fabry-Perot resonator, bounded by two parallel mirrors extending to infinity and orthogonal to the vector $e_z$, is examined. It is filled with medium which, in absence of any forcing factor, appears to be isotropic and to have nonlinear third-order electric susceptibility. Influence of the forced anisotropy on the light transmitted through a cavity is studied under the assumption that external factors change only linear susceptibility leaving the nonlinear tensor unchanged. The linear susceptibility is described by the following tensor:

$$\varepsilon_{jk} = \delta_{jk}(\varepsilon+\delta_{1k}\Delta\varepsilon) \tag{1}$$

where $\Delta\varepsilon$ is caused by the external factors.

The intensity reflectivities $R = 1-T$ of the mirrors differ for waves of different velocities of propagation. It is assumed that:

1. There is a plane monochromatic wave incident from the outside of one of the mirrors

$$E_0 = [\cos\alpha, \sin\alpha, 0] \;\; E_0 \exp[i(\omega t-k_0 z)] \tag{2}$$

where $k_0^2 = \omega^2/c^2$.

2. Waves propagating inside and outside a cavity are independent of variables $x$ and $y$.

3. Wave inside a cavity is composed of two components of slowly varying amplitudes and phase functions both propagating in mutually opposite directions.

4. Components of the nonlinear polarization vector proportional to $\exp[i\omega t]$ are the only ones which are taken into account.

## 2. Nonlinear polarization

When incident field approaches a suitable big value, a nonzero nonlinear polarization vector appears in the cavity. It has the following Cartesian components [1]:

$$P_y^{\mathrm{NL}} = \chi(E_x E_x E_x + E_y E_y E_x),$$

$$P_y^{\mathrm{NL}} = \chi(E_x E_x E_y + E_y E_y E_y),$$

$$P_z^{\mathrm{NL}} = 0$$

where $\chi$ denotes one of the elements of the nonlinear tensor

$$\chi = \chi_{xxxx} = \chi_{yyyy} = \chi_{zzzz}.$$

Having represented real functions as sums of complex functions and their complex conjugates:

$$E_x = \frac{1}{2}(\tilde{E}_x + \tilde{E}_x^*),$$
$$E_y = \frac{1}{2}(\tilde{E}_y + \tilde{E}_y^*), \tag{4}$$

the components of the nonlinear polarization vector $P^{\mathrm{NL}}$ can be rewritten as follows:

$$P_x^{\mathrm{NL}} = \frac{\chi}{4}\{(3\,|\tilde{E}_x|^2 + |\tilde{E}_y|^2)\,\tilde{E}_x + \tilde{E}_y\,\tilde{E}_y\,\tilde{E}_x^*\},$$
$$P_y^{\mathrm{NL}} = \frac{\chi}{4}\{(3\,|\tilde{E}_x|^2 + 2\,|\tilde{E}_x|^2)\,\tilde{E}_y + \tilde{E}_x\,\tilde{E}_x\,\tilde{E}_y^*\}. \tag{5}$$

## 3. Differential equations describing the field inside a Fabry-Perot cavity

The Maxwell equations comprising both the tensor (1) and the nonlinear polarization vector (5) imply the following equations for wave field of the frequency $\omega$ inside the cavity:

$$\frac{d^2}{dz^2}\tilde{E}_x + \mu_0\,\varepsilon_0\,\varepsilon_1\omega^2\,\tilde{E}_x = -\frac{\mu_0\,\chi\,\omega^2}{4}[(3\,|\tilde{E}_x|^2 + 2\,|\tilde{E}_y|^2)\,E_x + \tilde{E}_y\,\tilde{E}_y\,\tilde{E}_x^*],$$
$$\frac{d^2}{dz^2}\tilde{E}_y + \mu_0\,\varepsilon_0\,\varepsilon_2\,\omega^2\,\tilde{E}_y = -\frac{\chi\omega^2\,\mu_0}{4}[(2\,|\tilde{E}_x|^2 + 3\,|\tilde{E}_y|^2\,\tilde{E}_y + \tilde{E}_x\,\tilde{E}_x\,\tilde{E}_y^*] \tag{6}$$

where: $\varepsilon_0$ — electric permittivity of vacuum, $\varepsilon_1 = \varepsilon_{11}/\varepsilon_0$ and $\varepsilon_2 = \varepsilon_{22}/\varepsilon_0$ — relative electric permittivities for the directions $0x$ and $0y$.

In general, $\varepsilon_1$ and $\varepsilon_2$ may be complex numbers:

$$\varepsilon_j = \varepsilon_j^{(r)} - i\varepsilon_j^{(i)}, \quad (j = 1, 2). \tag{7}$$

It is assumed [2] that the wave field inside the Fabry-Perot cavity consists of two waves propagating in two opposite directions:

$$\tilde{E}_j = \xi_F^{(j)} \exp[i(\Phi_F^{(j)} - k_j z)] + \xi_B^{(j)} \exp[i(\Phi_B^{(j)} + k_j z)], \qquad (j = 1, 2)$$

where: $j = 1 = x$, $j = 2 = y$ and $\xi_F^{(j)}$, $\xi_B^{(j)}$, $\Phi_F^{(j)}$, $\Phi_B^{(j)}$ are slowly varying functions of the variable $z$, i.e., such that

$$\left|\frac{d^2}{dz^2}\xi_\alpha^{(j)}\right| \ll 2k_j \left|\frac{d}{dz}\xi_\alpha^{(j)}\right|,$$

$$\left|\frac{d^2}{dz^2}\Phi_\alpha^{(j)}\right| \ll 2k_j \left|\frac{d}{dz}\Phi_\alpha^{(j)}\right|, \tag{8a}$$

$$\alpha = F, B, \qquad k_j^2 = k_0^2 \varepsilon_j^{(r)}. \tag{8b}$$

The assumption (8a) leads to the following system of differential equations:

$$\frac{d}{dz}\xi_F^{(j)} = -\varrho_j \xi_F^{(j)},$$

$$\frac{d}{dz}\xi_B^{(j)} = \varrho_j \xi_B^{(j)},$$

$$\frac{d}{dz}\Phi_F^{(j)} = -\gamma_j [3|\xi_F^{(j)}|^2 + 6|\xi_B^{(j)}|^2 + 2|\xi_F^{(3-j)}|^2 + 2|\xi_B^{(3-j)}|^2],$$

$$\frac{d}{dz}\Phi_B^{(j)} = +\gamma_j [6|\xi_F^{(j)}|^2 + 3|\xi_B^{(j)}|^2 + 2|\xi_F^{(3-j)}|^2 + 2|\xi_B^{(3-j)}|^2] \tag{9}$$

where:

$$\varrho_j = \frac{\varepsilon_0 \mu_0 \omega^2 \varepsilon_j^{(i)}}{2k_j},$$

$$\gamma_j = \frac{\varepsilon_0 \mu_0 \omega^2 \chi}{8k_j}, \qquad j = 1, 2. \tag{9a}$$

The rapidly varying components of the nonlinear polarization vector are eliminated, but it causes loss of symmetry of Maxwell equations. Hence, it must be assumed that

$$\left|\frac{d}{dz}\Phi_F^{(j)}\right|, \quad \left|\frac{d}{dz}\Phi_B^{(j)}\right| \ll |k_1 - k_2|. \tag{10}$$

Equations (9) imply that the waves propagating in the direction of positive $z$-coordinates are damped when $z$ increases, while the backward waves are damped when $z$ decreases:

$$\xi_F^{(j)}(z) = \xi_F^{(j)}(0) \exp[-\varrho_j z],$$

$$\xi_B^{(j)}(z) = \xi_B^{(j)}(0) \exp[\varrho_j z], \qquad j = 1, 2. \tag{11}$$

Equations (9) allow us to define:

$$\Delta\varphi_j(z) = [\Phi_F^{(j)}(z) - \Phi_B^{(j)}(z) + \Phi_B^{(j)}(0) - \Phi_F^{(j)}(0)]$$

$$= -\gamma_j \int_0^z \{9[|\xi_F^{(j)}(z)|^2 + |\xi_B^{(j)}(z)|^2] + 4[|\xi_F^{(3-j)}(z)|^2 + |\xi_B^{(3-j)}|^2]\}\, dz, \quad j = 1, 2. \quad (12)$$

## 4. Boundary conditions and solutions of the equations for the wave field inside a cavity

When light is incident from the direction of negative $z$-coordinate the continuity conditions for the Cartesian components of the electric vector $\boldsymbol{E}$ within the space limited by boundary mirrors (contained in the planes $z = 0$ and $z = L$) give the formulas:

$$\sqrt{T_j}E_{0j} + \sqrt{R_j}\,\xi_B^{(j)}(0)\exp[i\Phi_B^{(j)}(0)] = \xi_F^{(j)}(0)\exp[i\Phi_F^{(j)}(0)],$$

$$\xi_B^{(j)}(L)\exp[i(\Phi_B^{(j)}(L) + k_jL)] = \sqrt{R_j}\,\xi_F^{(j)}(L)\times\exp[i(\Phi_F^{(j)}(L) - k_jL)], \quad j = 1, 2, \quad (13)$$

$$E_{0j} = E_0(\delta_{1j}\cos\alpha + \delta_{2j}\sin\alpha), \quad j = 1, 2. \quad (13a)$$

The Cartesian components $E_x^{tr}$ and $E_y^{tr}$ of wave field transmitted through the cave and the intensities $I_x^{tr}$ and $I_y^{tr}$ of planes $x$- and $y$-polarizations are stated as follows:

$$E_x^{tr} = \sqrt{T_1}\,\xi_F^{(1)}(L)\exp[i(\Phi_F^{(1)}(L) - k_1L)],$$

$$E_y^{tr} = \sqrt{T_2}\,\xi_F^{(2)}(L)\exp[i(\Phi_F^{(2)}(L) - k_2L)], \quad (14)$$

$$I_x^{tr} = \frac{1}{2}\sqrt{\frac{\varepsilon_0}{\mu_0}}\,T_1|\xi_F^{(1)}(L)|^2,$$

$$I_y^{tr} = \frac{1}{2}\sqrt{\frac{\varepsilon_0}{\mu_0}}\,T_2|\xi_F^{(2)}(L)|^2. \quad (15)$$

Defining

$$\eta_j = \exp[-\varrho_jL], \quad j = 1, 2, \quad (16)$$

and taking account of the fact that the functions: $\xi_F^{(j)}$, $\xi_B^{(j)}$, $\Phi_F^{(j)}$, $\Phi_B^{(j)}$ ($j = 1, 2$) are real one derives from (11) and (13) the following equations:

$$\xi_F^{(j)}(L) = \frac{\eta_j\sqrt{T_j}E_{0j}}{\left\{(1-\eta_j^2R_j)^2 + 4\eta_j^2R_j\sin^2\left[\frac{\Delta\varphi_j}{2} - k_jL\right]\right\}^{1/2}}, \quad (17a)$$

$$\exp[i\Phi_F^{(j)}(0)] = \frac{\eta_j\sqrt{T_j}E_{0j}}{\xi_F^{(j)}(0)\{1 - R_j\eta_j^2\exp[i(\Delta\varphi_j - 2k_jL)]\}}, \quad (17b)$$

$$\xi_B^{(j)}(0) = \frac{\eta_j^2 \sqrt{R_j T_j} E_{0j}}{\left\{(1-R_j \eta_j^2)^2 + 4\eta_j^2 R_j \sin^2\left[\frac{\Delta\varphi_j}{2} - k_j L\right]\right\}^{1/2}}, \tag{17c}$$

$$\exp[i\Phi_B^{(j)}(0)] = \frac{\eta_j^2 \sqrt{T_j R_j} E_{0j}}{\xi_B^{(j)}(0)\{\exp[i(2k_j L - \Delta\varphi_j)] - R_j \eta_j^2\}} \tag{17d}$$

where $\Delta\varphi_j = \Delta\varphi_j(L)$ are defined by (12) and with the use of Eqs. (11), (15) and (16) can be described as follows:

$$\begin{aligned} \Delta\varphi_1 &= -\gamma_1\{9\Gamma_1 I_x^{tr} + 4\Gamma_2 I_y^{tr}\}, \\ \Delta\varphi_2 &= -\gamma_2\{4\Gamma_1 I_x^{tr} + 9\Gamma_2 I_y^{tr}\} \end{aligned} \tag{18}$$

where

$$\begin{aligned} \Gamma_j &= \sqrt{\frac{\mu_0}{\varepsilon_0}\frac{(1-\eta_j^2)}{\varrho_j T_j \eta_j^2}}(1+\eta_j^2 R_j), \quad \text{for } \varepsilon_j^{(i)} \neq 0, \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad (j = 1, 2) \\ \Gamma_j &= 2\sqrt{\frac{\mu_0}{\varepsilon_0}}(1+R_j)\frac{L}{T_j}, \quad \text{for } \varepsilon_j^{(i)} = 0. \end{aligned} \tag{18a}$$

Conditions determining $I_x^{tr}$ and $I_y^{tr}$ can be also deduced from (15), (17) and (18):

$$\begin{aligned} I_x^{tr} &= \frac{\eta_1^2 T_1^2 I_0 \cos^2\alpha}{(1-\eta_1^2 R_1)^2 + 4\eta_1^2 R_1 \sin^2\left[\frac{\gamma_1}{2}(9\Gamma_1 I_x^{tr} + 4\Gamma_2 I_y^{tr}) + k_1 L\right]}, \\ I_y^{tr} &= \frac{I_0 \eta_2^2 T_2^2 \sin^2\alpha}{(1-\eta_2^2 R_2)^2 + 4\eta_2^2 R_2 \sin^2\left[\frac{\gamma_2}{2}(9\Gamma_2 I_y^{tr} + 4\Gamma_1 I_x^{tr}) + k_2 L\right]}. \end{aligned} \tag{19}$$

Expressions (19) indicate a bistable dependence of each of the intensities $I_x^{tr}$ and $I_y^{tr}$ upon the intensity $I_0$ of the incident light. These two intensities are mutually dependent. However, the mutual dependence disappears when the electric vector $\boldsymbol{E}_0$ of the incident wave is oriented at either the angle $\alpha = 0$ or $\alpha = \pi/2$ with respect to the axis $0x$. In those cases, the classical formulas describing bistability are obtained [3]–[6] and the transmitted light is polarized linearly in either the plane $y = 0$ (for $\alpha = 0$) or $x = 0$ (for $\alpha = \pi/2$).

When $0 < \alpha < \pi/2$, both the intensities $I_x^{tr}$ and $I_y^{tr}$ are different from zero and there exists the following dependence between them:

$$\begin{aligned} &\frac{I_x^{tr}}{\eta_1^2 T_1^2 \cos^2\alpha}\left\{(1-\eta_1^2 R_1)^2 + 4\eta_1^2 R_1 \sin^2\left[\frac{\gamma_1}{2}(9\Gamma_1 I_x^{tr} + 4\Gamma_2 I_y^{tr}) + k_1 L\right]\right\} \\ &= \frac{I_y^{tr}}{\eta_2^2 T_2^2 \sin^2\alpha}\left\{(1-\eta_2^2 R_2)^2 + 4\eta^2 R_2 \sin^2\left[\frac{\gamma_2}{2}(4\Gamma_1 I_x^{tr} + 9\Gamma_2 I_y^{tr}) + k_2 L\right]\right\} = I_0. \end{aligned} \tag{20}$$

## 5. Polarization state of transmitted wave field

The state of polarization of the wave field can be determined by means of a matrix of coherence [7]. The matrix of coherence $\check{M}$ of the transmitted light takes the form

$$M = \begin{bmatrix} M_{xx} & M_{xy} \\ M_{yx} & M_{yy} \end{bmatrix} =$$

$$= \begin{bmatrix} |\xi_F^{(1)}(L)|^2 T_1, & \xi_F^{(1)} \xi_F^{(2)} \sqrt{T_1 T_2} \exp\{i[\Phi_F^{(1)} - \Phi_F^{(2)} + (k_2 - k_1) L]\} \\ \xi_F^{(1)} \xi_F^{(2)} \sqrt{T_1 T_2} \exp[i(\Phi_F^{(2)} - \Phi_F^{(1)}) + i(k_1 - k_2) L], & |\xi_F^{(2)}(L)|^2 T^2 \end{bmatrix}.$$

The determinant of the matrix $\check{M}$ is equal to zero. The parameter of polarization $P$ is given by the expression [7]

$$P = \sqrt{1 - \frac{|\check{M}|}{(M_{xx} + M_{yy})^2}} \tag{22}$$

where $|\check{M}|$ denotes the determinant of matrix $\check{M}$.

In our case, the parameter of polarization $P = 1$, which means the full polarization of transmitted light. It is an elliptic polarization. The ratio of the ellipse semiaxes [7] may be expressed by $\tan\theta = \pm a/b$. The tangent can be determined taking account of the following expression:

$$\sin 2\theta = \frac{i(M_{yx} - M_{xy})}{M_{xx} + M_{yy}} = \frac{2\sqrt{I_x^{\mathrm{tr}} I_y^{\mathrm{tr}}} \sin[\Phi_F^{(1)}(L) - \Phi_F^{(2)}(L) + (k_2 - k_1) L]}{I_x^{\mathrm{tr}} + I_y^{\mathrm{tr}}}. \tag{23}$$

Numerical calculations based on formula (23) together with (17) and (9) show (see Sect. 6) bistable changes of the ratio $b/a$. The angle $\psi$ between the major semiaxis of the polarization ellipse and the axis $0x$ is given by the formula [7]

$$\tan 2\psi = \frac{M_{xy} + M_{yx}}{M_{xx} - M_{yy}} = \frac{2\sqrt{I_x^{\mathrm{tr}} I_y^{\mathrm{tr}}} \cos[\Phi_F^{(1)}(L) - \Phi_F^{(2)}(L) + (k_2 - k_1) L]}{I_x^{\mathrm{tr}} - I_y^{\mathrm{tr}}}. \tag{24}$$

That quantity can be also estimated by means of numerical analysis which is presented in Sect. 6, and shows bistable changes of the angle $\psi$.

## 6. Interpretation of received results

Numerical analysis of received formulas has been carried out on a computer compatible with IBM PC/XT. For the calculations the following assumptions have been made: The material constants are those characteristic for GaAs [8], [9], the width of the Fabry-Perot cavity $L = 0.005$ m, the angle between the plane of polarization of the incident light and the 0x-axis $\alpha = \pi/6$ and, if assumed to be constant, the forced difference of relative permittivities $\Delta\varepsilon = 0.0001$. The medium has been assumed not to reveal any damping.

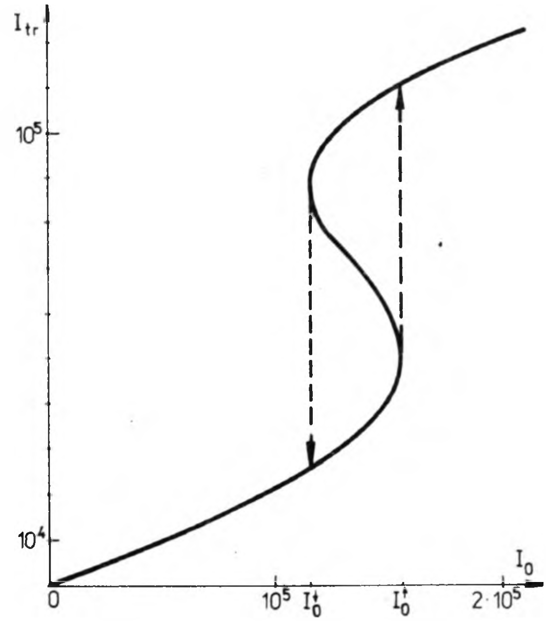


Fig. 1. Bistability of the intensity $I_{tr}$ of the total transmitted

The relation between $I_{tr} = I_x^{tr} + I_y^{tr}$ and $I_0$ is shown in Fig. 1. Bistability of $I_{tr}$ as a function of $I_0$, i.e., $I_{tr} = f(I_0)$ is incontestable. Figures 2**a** and 2**b** show similar dependences for $I_x^{tr}$ and $I_y^{tr}$. Figure 3 being an illustration of the formula (20), shows the bistable relation between $I_y^{tr}$ and $I_x^{tr}$. Figure 4 presents the proportion of polarization ellipse semiaxes $\tan\theta$ as a function of the incident intensity $I_0$. Figure

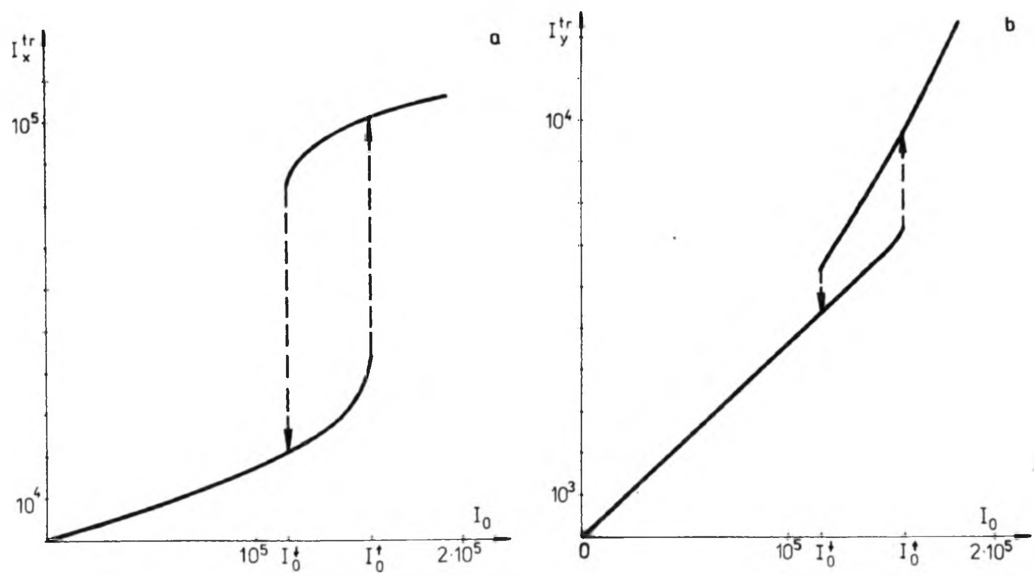


Fig. 2. Bistability of the intensities $I_x^{tr}$ (**a**) and $I_y^{tr}$ (**b**) of the Cartesian components of the transmitted field

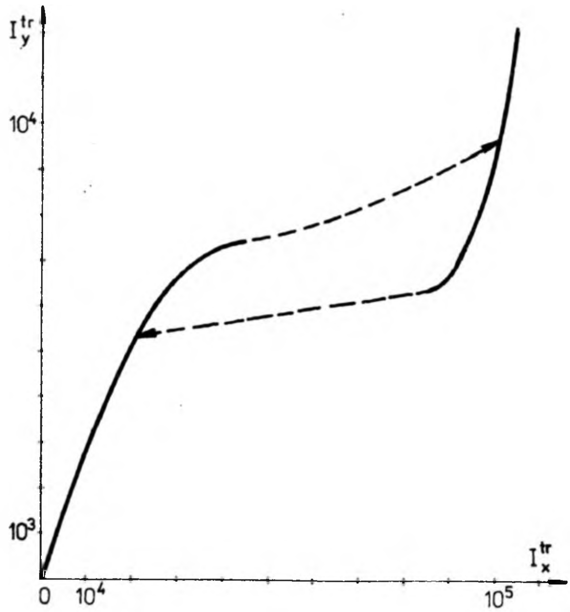


Fig. 3. Bistable relation between the intensities $I_x^{tr}$ and $I_y^{tr}$

5 illustrates the bistability of orientation angle $\psi$ between the major semiaxis of the polarization ellipse and the axis $0x$.

As Figures **6a–c** show, it is possible to modify the parameters of bistability hysteresis, i.e., the width of the hysteresis cycles $\Delta I_0 = I_{0\uparrow} - I_{0\downarrow}$ (Fig. **6a**), the value of the upward bistable jump $\Delta I_{tr}^{(\uparrow)}$ (Fig. **6b**) and downward bistable jump $\Delta I_{tr}^{(\downarrow)}$ (Fig. **6c**) – as functions of $\Delta\varepsilon$.

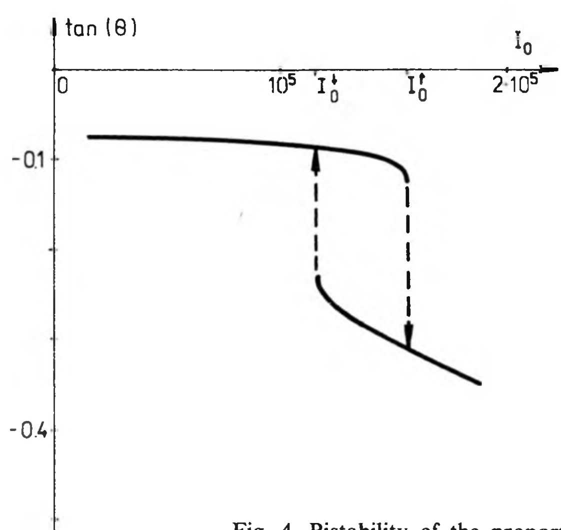


Fig. 4. Bistability of the proportion of the polarization ellipse semiaxes $\tan\theta$

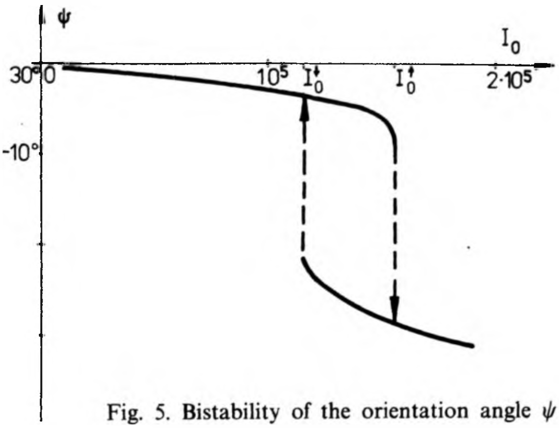


Fig. 5. Bistability of the orientation angle $\psi$ between the major semiaxes and the axis $0x$

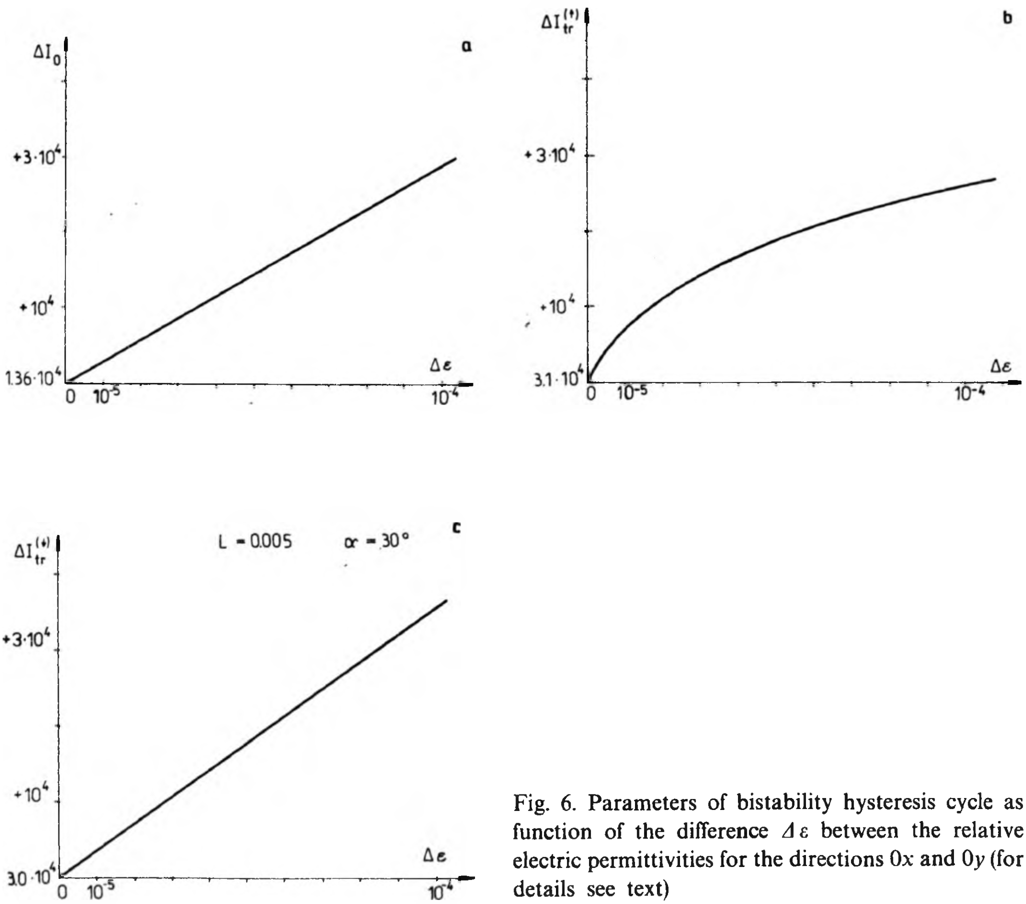


Fig. 6. Parameters of bistability hysteresis cycle as function of the difference $\Delta\varepsilon$ between the relative electric permittivities for the directions $0x$ and $0y$ (for details see text)

## 7. Summary and conclusions

The results presented above are indicative of possibilities of interfering in a course of intensity hysteresis cycles by means of electro- and elastooptical effects.

It has been pointed out that an anisotropic medium located in a Fabry-Perot cavity may exhibit the bistability of polarization of transmitted light (Figs. 4 and 5) provided that the incident intensity reaches a suitably high value.

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*Received October 3, 1988*

### Оптическая дисперсионная бистабильность в средах с вынужденной анизотропией

Представлены результаты, относящиеся к дисперсионной бистабильности в средах, обладающих вынуженной анизотропией (вызванной, например, электро- или упругооптическим эффектом). Было показано, что свет, прошедший через нелинейный резонатор Фабри-Перо, обладает бистабильностью интенсивности и поляризации. В случае, когда электрический вектор падающего поля не является параллельным или перпендикулярным к оптической оси, прошедший свет обладает эллиптической поляризацией. Этот эллиптически поляризованный цвет изменяет бистабильно не только отношение малой и большой полуосей поляризационного эллипсиса, но и пространственную ориетацию его большой полуоси.