Optica Applicata, Vol. XXI, No. 1, 1991

Gaussian beam propagation in nonlinear Kerr medium

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In order to investigate propagation of electromagnetic field in dielectric Kerr medium, the nonlinear Schroedinger equation is studied. The method of analysis of spherical data in the inverse scattering transform for arbitrary initial conditions is developed. A special case of Gaussian initial conditions is discussed in detail. Parameters of both the soliton and linear dispersive waves propagating in nonlinear medium are given explicitly in terms of entering Gaussian envelope.

1. Introduction

In this paper we study the nonlinear Schroedinger equation (NSE)

$$iu_t+u_{xx}+\chi|u|^2u = 0 \tag{1}$$

with certain condition

$$u_0(x) = u(x, 0)$$

where small x and t subscripts denote space and time derivatives, and χ describes nonlinearity of the medium and is assumed to be a positive real number [1].

The first complete solution of this equation was found by ZAKHAROV and SHABAT [1] by application of the inverse scattering transformation (IST). They found that solution of the NSE consists of nonlinear waves, called solitons, and linear waves vanishing at the infinity.

The NSE describes two important optical phenomena in nonlinear Kerr medium: the self-trapping of an electromagnetic beam (for review see, e.g., [2], [3] and the latest experimental work [4]) and the soliton formation during pulse propagation in Kerr-type waveguide (for review see, e.g., a clever book written by HASEGAWA [5]).

The phenomenon of self-trapping of a powerful electromagnetic (laser) beam may occur in materials whose refractive index increases with field intensity, but which are homogeneous in the absence of electromagnetic wave. For nonlinear Kerr medium the initial laser beam can produce its own dielectric waveguide and propagate in it without spreading [2]–[4]. In the case of pulse narrowing and soliton formation during propagation in Kerr-type waveguide the nonlinearity of the refractive index is used to compensate for the pulse broadening effect of dispersion in low-loss optical fibres [5]. These phenomena received a great deal of attention because of growing interest in optical bistability, laser mode-locking, logic elements like soliton couplers, soliton logic elements, coupled optical (nonlinear) fibres, etc.

This paper is organized as follows. Section 2 summarizes the main advantages of IST for NSE. In Section 3 we shortly present the method of spectrum analysis and the results obtained for Gaussian initial values.

2. Inverse scattering transform

The strategy of the IST is the same as that of any transform method, i.e., one defines a transform of the original problem into a space in which the time dependence is particularly simple. After determining the transformed data at a later time we invert our fransform to obtain the solution.

To use the IST one must fromulate the problem of obtaining an appropriate eigenvalue from the initial wave amplitudes. This step is called "direct scattering problem" and consist in determining the space-asymptotic behaviour of the eigenfunctions of our eigenvalue problem. In this manner one obtains a set of "scattering data". In other words, the eigenvalue problem provides an unique mapping of the (initial) wave amplitudes into a set of (initial) scattering data [6]. Then the "time evolution" of the scatering data in the "inverse (scattering) space" is given by the trivial ordinary differential equation of the first order. The wave amplitudes at the time t can be reconstructed using the inverse scattering equations (that is, by solving the Rieman problem [1] or Gelfand–Levitan–Martchenko equations [6]).

The IST decomposes the wave amplitude into "normal modes", which are linearly independent in scattering space. Each "normal mode" evolves in the scattering space independent of all other "normal modes". There are two kinds of such modes [7]. The first one, which has no linear analog is called "soliton". The solitons are the manifestation of a discrete bound-state spectrum of the eigenvalue problem and are localized, oscillatory, travelling waveforms. Modes of the second kind refer to the continuous part of this eigenvalue spectrum. Such modes are called "radiation" because in the presence of dispersion they propagate away from disturbance as radiation would [7].

The direct scattering problem associated with NSE is equivalent to a set of equations:

$$\begin{aligned} v_{1x} &= iq(x)\exp(2i\zeta x)v_2, \\ v_{2x} &= iq^*(x)\exp(-2i\zeta x)v_1 \end{aligned} \tag{2}$$

where $q(x)$ represents an "initial potential" for the direct scattering problem,

$$q(x) = (\chi/2)^{1/2}u_0(x), \tag{3}$$

ζ is an eigenvalue of investigated problem and $v = (v_1,v_2)^T$ is the eigenvector.

We assume that ζ is real and $q(x)$ and all its x-derivatives tend to zero as x tends to infinity. We define φ to be a solution of (2) which satisfies the boundary condition

$$\varphi\begin{pmatrix}1\\0\end{pmatrix} \text{ as } x-\infty. \tag{4}$$

Then in the limit $x+\infty$

$$\varphi\begin{pmatrix}a(\zeta)\\b(\zeta)\end{pmatrix}, \tag{5}$$

which defining "scattering coefficients" $a(\zeta)$ and $b(\zeta)$. It has been shown that both $a(\zeta)$ and $b(\zeta)$ can be analytically extended in the upper half of ζ-plane [6].

The zeros of $a(\zeta)$ determine the soliton part of eigenvalue spectrum and correspond to discrete eigenvalues (bound states in scattering problem). We designate a set of this zeros by $\{\zeta_k\}_{k=1}^{N}$ and assume N to be finite. These zeros are located in the upper half of ζ-plane $(\mathrm{Im}(\zeta) > 0)$ [6].

The continuous part of the eigenvalue spectrum is determined by a "reflection coefficient"

$$r(\sigma) = b(\sigma)/a(\sigma) \tag{6}$$

where $\sigma = \mathrm{Re}(\zeta)$. This coefficient specifies the part of spectral modes which are called "radiation". In the linear limit $(\chi \to 0)$, $r(\zeta)$ represents the linear Fourier transformation of the initial amplitude [7].

In order to reconstruct the scattering potential $q(x)$ at the time t we use SEGUR and ABLOWITZ [8] and SEGUR [9] results. They find that in the absence of soliton solutions (that is, when there is no discrete spectrum) the soliton has the form of decaying oscillations with decaying amplitude of $t^{-1/2}$. But when only one soliton is present, the solution behaves like soliton near the soliton, and like the decaying oscillations elsewhere.

Following ABLOWITZ and SEGUR [8] we can write for large values of t the wave amplitude in the absence of discrete spectrum as

$$q(x,t) = t^{-1/2}R(x/t)\exp[it\theta(x/t)] + \mathcal{O}(\ln t/t) \tag{7}$$

where:

$$R(x/t) = (1/4\pi)\ln\{1+|r(-x/4t)|^2\}^{1/2},$$

$$\theta(x/t) = (x/2t)^2 + \mathcal{O}(\ln t/t).$$

When only one discrete value $\zeta = \sigma + i\eta$ in spectrum is present, the main part of the asymptotic field has the form [9]

$$q(x,t) = 2\eta\exp(i\varphi)/\mathrm{ch}(\psi) + t^{-1/2}R(x/t) \tag{8}$$

$$\times\left[\frac{\exp(i\theta)(\sigma+x/4t+i\eta\,\mathrm{th}\,\psi)^2}{(\sigma+x/4t)^2+\eta^2} + \frac{\exp(2i\varphi-\mathrm{i}t\theta)^2\eta}{\mathrm{ch}^2(\psi\eta^2(\sigma+x/4t)^2)}\right] + \mathcal{O}(t^{-1/2})$$

where: $\varphi = -2[\sigma x + 2(\sigma^2-\eta^2)t] + \varphi_0$,

$$\psi = 2\eta(x+4\sigma t) + \psi_0$$

(ψ_0 and φ_0 are unknown initial position and phase). The first term in (8) represents the soliton solution. The second one is simply the "radiation", which is perturbed by soliton field near the soliton. The last term describes the interaction between nonlinear wave (that is, soliton) and the linear diffractive one (that is, "radiation").

3. Gaussian initial conditions

In order to find the solution of (2) we assume that $q_0(x)$ is real and that solution has the form:

$$v_1(x,\zeta) = \sum_{n=0}^{\infty} (i\zeta)^n f_n(x),$$
$$v_2(x,\zeta) = \sum_{n=0}^{\infty} (i\zeta)^n g_n(x). \tag{9}$$

Simple calculations give us the general recurrence formula for coefficients f_n and g_n. In the limit of $x = +\infty$ we find the "scattering coefficients" $a(\zeta)$ and $b(\zeta)$ in the form of the power series of ζ. These coefficients are expressed in terms of entering field envelope (see below). One can solve the equation $a(\zeta) = 0$ and find an exact form of $\zeta = \sigma + i\eta$. As we mentioned in the previous section, only for $\eta > 0$ the soliton solution is well defined. It has been shown [6] that $a(\zeta)$ is a continuous function of ξ in the vicinity of $\eta = 0$. The condition of η being positive is a condition of soliton existence in the medium considered.

In the analysis that follows we shall consider certain aspects of the propagation of Gaussian beams in a nonlinear Kerr medium. The problem we address is most closely related to a phenomenon of two-dimensional self-trapping [4]. In this case the time coordinate t plays a role of the longitudinal coordinate z, while x is the transversal dimension.

Let a Gaussian beam entering a Kerr medium have the form

$$u_0(x) = u_0 \exp[(-x/w_0)^2] \tag{10}$$

where u_0 and w_0 are real parameters describing the amplitude and the width.

In the limit of a small value of ζ, where only linear term is taken into account, the coefficient $a(\zeta)$ has purely imaginary zero whose approximate location is given by

$$\eta = -(\pi/2)^{1/2} \mathrm{ctan}(\beta_g)/(w_0\beta_g) \tag{11}$$

where $\beta_g = (\pi\chi/2)^{1/2} w_0 u_0$. Because $\sigma = 0$, one can see from the first term in (8) that Eq. (11) describes both the amplitude and the inverse width of the soliton.

The amplitude of "radiation", which represents the continuous part of spectrum, is fully characterized by module of the "reflection coefficient" $r(\sigma)$ (see (7) and (8))

$$|r(\sigma)| = [\mathrm{ctg}^2(\beta_g) + 2(\sigma w_0 \beta_g)^2]^{-1/2} \tag{12}$$

In order to analyse the problem let us notice that η in Eq. (11) changes the sign from negative to positive for the threshold value $\beta_{cr} = \pi/2$. It is convenient to investigate, instead of β_{cr}, the threshold Gaussian beam power I_{cr} in the form

$$I_{cr} = (\pi/2)^{3/2}(w_0\chi)^{-1} \tag{13}$$

for given χ and w_0,

where $I = \int_{-\infty}^{+\infty} u_0^2(x)dx = (2/\pi)^{1/2}(\chi w_0)^{-1}\beta_g^2$

Two cases are discussed below.

i) $I < I_{cr}$ (below threshold)

In nonlinear Kerr medium only decaying oscillations exist. Their envelope shape for a large value of z is described by (7). For increasing I their amplitude increases according to (12).

ii) $I > I_{cr}$ (above threshold)

When the input power is larger than I_{cr} there exists a single self-trapped channel (spatial soliton), described by the amplitude and the channel width (8). The amplitude and the inverse channel width increase with the input power intensity. In the background of the optical channel there occurs a linear dispersive wave (radiation). For a large value of z the amplitude of this radiation for the given σ ($\sigma = x/4z$) decreases with the increase of I.

4. Conclusions

We have used the IST to describe Gaussian beam propagation in nonlinear Kerr medium. We find that Gaussian initial conditions generate both discreate and continuous spectra, so the solution of the NSE contains both the soliton and the decaying oscillations. There exists some critical power intensity I_{cr} of the input Gaussian beam, below which only decaying oscillations propagate in nonlinear medium. Above I_{cr}, a one-soliton solution occurs as s spatial self-trapped channel, described by the amplitude and channel width. One can express these parameters by the product of the nonlinear refractive-index of the medium, the amplitude of the input Gaussian beam and the radius of the Gaussian beam at the spot. We expect this approach to apply to other nonlinear problems, as well.

Acknowledgments – The author is grateful for many helpful discussions with Dr W. Nasalski. This work was supported by Polish CPBP 02.13 Research Programme.

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Received September 20, 1990
in revised form February 5, 1991

Распространение гауссовского луча в нелинейной среде Керра

Для исследования распространения электромагнитного поля в диэлектрической среде Керра рассматривается нелинейное уравнение Шредингера. Развит метод анализа спектральных данных в обратной задаче рассеяния для любых начальных условий. В особенности рассматривается случай гауссовских краевых условий. Выражены параметры солитона и линейных дисперсных волн, распространяющихся в среде при помощи параметров, описывающих входную гауссовскую огибающую.

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