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Birefringence and dispersion of polarization in a single-mode fibre considered as stochastic processes

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If we consider random disturbances on an ensemble of single-mode fibres or planar devices, we must also treat the polarization dispersion, birefringence and other fibre parameters as random processes with their statistical characteristics. This work contains an explicit calculation of statistical parameters — average, variance and autocorrelation of random birefringence and polarization dispersion in single-mode media under random disturbances, both internal and external.

1. Introduction

In the recent years, an increasing interest in integrated optics theory and technology is observed. In particular, the contemporary technologies leading to achievement of high parameters of integrated optics create a necessity of carrying out intensive theoretical examinations taking account of advanced methods to enable description of subtle effects. The transition from a simple transmission of information through the intensity modulation method to the methods of modulation of phase, amplitude, frequency and polarization of the light widens significantly the possibilities not only for the optical communication systems but also for the elements of integrated optics. Many applications are based just on exploitation of the possibilities offered by the polarized light propagation. The present paper is devoted to the fibres but the whole apparatus may be easily adopted to the needs of planar optics as well as to examinations of effects observed in the elements of integrated optics.

The starting point is a description of the light wave propagation in the single-mode fibre — in the ideal case this wave propagates in the form of two linearly polarized orthogonal eigenmodes HE_{11} having the same propagation constant [1].

When starting with an ideal case we have to take account of different factors affecting the propagation in a fibre. They can be generally classified into external and internal ones [2]. The external (environmental) factors usually include: temperature, applied external forces, deformations, bendings, torsion, external electric and magnetic fields, while the internal factors emerging during production and exploitation are: deviation from the due geometry of the fibre, cracking, and internal stresses.

All this leads to a perturbation of the polarization state in the fibre, changing the properties of the latter (or even of the whole optical system) to a high degree.

There are obviously some methods of controlling the polarization in a fibre by producing artificially a strong birefringence [3], [4], but also in this case many factors influence the effective state of polarization.

Therefore, the examination of the polarization state in the single-mode fibres is an important task. In the recent years, many experimentalists devoted their attention to this problem, and also some theoretical works appeared [5]–[15]. They are based mainly on two models of phenomena. The first one, call it classic, characterizes the polarization state and its changes along the fibre due to couplings [6]–[16], but the results of this method are strongly limited by the assumptions of weak couplings [6]–[7] and the existence of "coupling centres" [14] which make a complete description of the real fibres impossible. The other approach consists in representing the fibre as a linear medium described totally by the transition matrix [8]. This, however, is of no use so far as the description of the polarization state inside the fibre is concerned.

In the present work, based on the first of the above models, the fibre parameters are examined taking account of the random effects. Each of the factors affecting the birefringence may be, and for a rigorous analysis should be, considered separately (being additionally charged with some "arbitrariness" in the sense of randomness). Thus, each of them should be considered as a random process. For convenience, we restrict ourselves to the case of three factors affecting the parameters of the fibre (via their contribution to the tensor of the dielectric permeability). This makes a demonstration of the typical effects possible and consequently the transition to the case of many factors becomes easier. The present approach based partly on the original method of solving the stochastic differential equations for the so-called polarization coefficients (in the sense of Jones vectors formalism [17]) described for the first time in [15] goes even further, *i.e.*, by showing the possibilities of analysis of the influence of particular factors (such as: bendings, stresses and external electric and magnetic fields) on the parameters of polarization states of fibres some corrections are indicated which should be introduced to the method and its interpretation. The method presented may be also extended to include the planar single-mode waveguides and the elements of the integrated optics working in the single-mode regime.

In the first part, starting with Maxwell equations for the eigenmodes in the fibre (being affected by some random external and internal factors), a stochastic equation of the first order describing the random coupling between the two linearly polarized eigenmodes is presented. Its solution in the sense of determining its main value [18] offers possibility of a detailed analysis of the effects of the random birefringence and the dispersion of polarization.

The second part is devoted to a detailed analysis of the random birefringence as a random process where such characteristics are determined as mean values, variance, and autocorrelation. We show in what way these magnitudes depend on the input random parameters. The effects of the mutual correlation of the input random tensors of the dielectric permittivity (for the particular factors affecting the birefringence) may be taken into consideration both when calculating the variance and the autocorrelation of this process.

In the third part, a similar characteristic of the process of random dispersion of polarization together with its average, autocorrelation and variance is discussed.

These magnitudes are important for the parameters of the optical transmission of the fibre since they influence the coherence matrix and the microscope parameters of polarization. However, the last topic will be the subject of further examinations. Here, a set of fibres of a certain length z the parameters of which may be changed in a random way not only along the fibre but also in time is subject of our calculations.

The fibres are considered to be lossless, which is justified by the fact that in the course of the last years a tremendous progress has been made in reducing the losses in fibres from about 20 dB/km in 1972 to 0.2 dB/km in 1987 [19], and lastly to 0.1 dB/km [20].

2. Intermode couplings in single-mode fibre in the presence of random perturbation

Let us assume that all the effects of random perturbations affecting the fibre are taken into account by representing the dielectric permittivity of the material in the form

$$\varepsilon(x,y,z,t) = \varepsilon_0(x,y)+\Delta\varepsilon_1(x,y,z,t)+\Delta\varepsilon_2(x,y,z,t)+\Delta\varepsilon_3(x,y,z,t) \tag{1}$$

where $\Delta\varepsilon_i$ ($i \in \{1, 2, 3\}$) describe the random contributions from single components (for instance, temperature, bending, stress, internal fields, and so on) and are dependent, in general, on the spatial coordinates (x, y, z) and the time t, while $\varepsilon_0(x, y)$ is a tensor of dielectric permittivity of an ideal waveguide.

All the above magnitudes are tensors of (1, 1) type.

We restrict our considerations to the monochromatic fields having in mind the possibility of obtaining the full image of the light pulse by a Fourier decomposition.

Let (E_1, H_1) and (E_2, H_2) be two linearly polarized eigenmodes HE_{11} propagating in an ideal uniform fibre, *i.e.*:

$$E_i = e_i(x,y)\exp(i\,\omega-\beta\, z), \tag{2}$$

$$i \in \{1,\ 2\}$$

$$H_i = e_i(x,y)\exp(i\,\omega-\beta\, z), \tag{3}$$

where e_i and h_i denote the eigenvectors, respectively.

When neglecting small losses caused by the waves propagating backwards as well as outwards, each wave $(E(\omega),\ H(\omega))$ may be written in the form

$$E(\omega) = c_1(z,t)\,E_1+c_2(z,t)\,E_2, \tag{4}$$

$$H(\omega) = c_1(z,t)\,H_1+c_2(z,t)\,H_2. \tag{5}$$

This is the field distribution in the base created by the eigenmodes (E_1, H_1) and (E_2, H_2) or, using the Jones formalism (the so-called Jones vector of the wave [17]), in the form

$$\begin{pmatrix} c_1(z,t) \\ c_2(z,t) \end{pmatrix} = C(z,t) \tag{6}$$

The magnitudes c_1 and c_2 are complex weights for the particular modes creating the field $E(\omega)$, $H(\omega)$. They depend both on the argument z and the time. In accordance with the former discussion these are random magnitudes (dependent on the tensor ε).

The Maxwell equations for the field $E(\omega)$, $H(\omega)$ may be written as follows:

$$\nabla x(c_1 H_1 + c_2 H_2) = \frac{\partial}{\partial t}\{\varepsilon(c_1 E_1 + c_2 E)\}, \tag{7}$$

$$\nabla x(c_1 E_1 + c_2 H_2) = -\mu_0 \frac{\partial}{\partial t}(c_1 H_1 + c_2 H_2) \tag{8}$$

(for the sake of convenience, the dependence on arguments has been omitted).

Since the symbol ∇ is a differential operator, then

$$\begin{aligned} & c_1 \nabla \times H_1 + (\nabla c_1) \times H_1 + c_2 \nabla \times H_2 + (\nabla c_2) \times H_2 \\ & = \varepsilon_0 \left(\frac{\partial}{\partial t} c_1\right) E_1 + \varepsilon_0 c_1 \frac{\partial}{\partial t} E_1 + \left(\frac{\partial}{\partial t} \sum_{i=1}^{3} \Delta\varepsilon\right) c_1 E_1 \\ & + \sum_{i=1}^{3} \Delta\varepsilon \left(\frac{\partial}{\partial t} c_1\right) E_1 + \sum_{i=1}^{3} \Delta\varepsilon\, c_i \frac{\partial}{\partial t} E_1 + \varepsilon_0 \left[\left(\frac{\partial}{\partial t} c_2\right) E_2 + c_1 \frac{\partial}{\partial t} E_2\right] \\ & + \left(\frac{\partial}{\partial t} \sum_{i=1} \Delta\varepsilon_i\right) c_2 E_2 + \left(\sum_{i=1}^{3} \Delta\varepsilon_i\right) \left\{\left(\frac{\partial}{\partial t} c_1\right) E_2 + c_2 \frac{\partial}{\partial t} E_2\right\}. \end{aligned} \tag{9}$$

Taking advantage of the fact that the fields (E_1, H_1) satisfy the Maxwell equations for an ideal fibre (expression (1) simplifies with (6), while (3) with (10)) and from the fact that the changes in light waves are significantly greater than the random losses (expressions (7), (12) are put equal to zero), we get

$$\begin{aligned} & \nabla c_1 \times H_1 + \nabla c_2 \times H_2 \\ & = i\omega \left(\sum_{i=1}^{3} \Delta\varepsilon_i\right) \left\{c_1 E_1 + c_2 E_2 + \varepsilon \left(E_1 \frac{\partial}{\partial t} c_1 + E_2 \frac{\partial}{t} c_2\right)\right\}. \end{aligned} \tag{10}$$

Treating the second equation in a similar way, we obtain

$$\nabla c_1 \times E_1 + \nabla c_2 \times E_2 = -\mu_0 \left(H_1 \frac{\partial}{\partial t} c_1 + H_2 \frac{\partial}{\partial t} c_2\right), \tag{11}$$

in which only c_1 and c_2 appear.

Multiplying scalarly Eq. (10) by $-E_1^*$, while (11) by H_1^* and adding them together, we obtain

$$\frac{\partial}{\partial t} c_1 + a_{11} \frac{\partial}{\partial t} c_1 + a_{12} \frac{\partial}{\partial t} c_1 = -i(k_{11} c_1 + k_{12} c_2). \tag{12}$$

Similarly, by multiplying Eq. (10) by $-E_2^*$, and (11) by H_2^*, and adding them together, we obtain

$$\frac{\partial}{\partial t}c_2 + a_{21}\frac{\partial}{\partial t}c_2 + a_{22}\frac{\partial}{\partial t}c_2 = -i\,(k_{21}c_1 + k_{22}c_2) \tag{13}$$

where:

$$a_{ij} = \int\int_{-\infty}^{-\infty} (\mu_0 H_i^* \times H_j + E_i^* \varepsilon E_j)\,dx\,dy, \tag{14}$$

$$k_{ij} = \omega \int\int_{-\infty}^{+\infty} E_i^* \left(\sum_{i=1}^{3} \Delta\varepsilon_i\right) E_j\,dx\,dy. \tag{15}$$

Equations (12) and (13) describe the propagation and coupling of two linearly polarized HE_{11} modes in a single mode fibre perturbed randomly.

From the mathematical point of view, this is a set of partial differential equations of first order and hyperbolic type with random coefficients. This creates the starting point to the analysis of effects of light wave polarization.

In order to solve this set of equations, we take advantage of the fact that the single-mode waveguides satisfy the condition of weak coupling [16]. In accordance with this

$$E_i \simeq \frac{\omega\mu_0}{\beta} H_i \times \hat{e}_z, \quad i \in \{1,\ 2\} \tag{16}$$

where $\hat{e}_z$ is a versor of $0z$ axis.

On the basis of this, we may write [15]

$$a_{11} \simeq \beta/\omega, \tag{17a}$$
$$a_{22} \simeq \beta/\omega, \tag{17b}$$
$$a_{12} \simeq a_{21} = 0. \tag{17c}$$

The energy conservation for the fibre provides self-conjugation of the dielectric permittivity tensor, hence it follows that the random tensor

$$\sum_{i=1}^{3} \Delta\varepsilon_i = \varepsilon - \varepsilon_0 \tag{18}$$

is also self-conjugated (since ε and ε_0 are self-conjugated). This fact is of essential importance since from Eq. (15)

$$k_{ij}(z,t) = \omega \int\int_{-\infty}^{+\infty} E_i^* \sum_{i=1}^{3} \Delta\varepsilon_i(x,y,z,t)\,E_j\,dx\,dy \tag{19}$$

it follows that the tensor in the matrix form $K = (k_{ij})$ is self-conjugated. Moreover, from Eq. (19) we have that all its components behave in a similar way with respect to (z,t) – it may, thus, be written in the form

$$K = k(z,t)\begin{pmatrix} d_i & d_3 e^{ir} \\ d_3 e^{-ir} & d_2 \end{pmatrix}. \tag{20}$$

The random process $k(z,t)$ describes the changes along the fibre and the time-dependence of K, while the random magnitudes d_1, d_2, d_3, r describe the perturbations of the fibre in its cross-section chosen randomly from the ensemble of fibres of the length z.

We introduce the notations:
average $k_0 := \langle k(z,t) \rangle$,
autocorrelation of this process $R_k(z,t)$,
averages: d_1, d_2, d_3, r, respectively,
$d_{0i} := \langle d_i \rangle \quad i \in \{1, 2, 3\}$,
$r_0 := \langle r \rangle$,
variances: $\sigma_1^2 := \sigma_{d_1}^2$, $\sigma_2^2 := \sigma_{d_2}^2$, $\sigma_3^2 := \sigma_{d_3}^2$.

Substituting expressions (17 a,b,c) and (20) to Eqs. (12) and (13), we obtain solution of these equations in the sense of mean value [18], [22]

$$C(z,t) = \{D_- e^{-i\varphi_-(z,t)} + D_+ e^{-i\varphi_+(z,t)}\} C(0, t-az) \tag{21}$$

where:

$$\varphi_{\pm}(z,t) = \omega S \int_0^z k(x, ax+t-az)dx, \tag{22}$$

$$S_{\pm} = 1/2\{d_1+d_2 \pm \sqrt{(d_1-d_2)^2+4d_3^2}\}, \tag{23}$$

$$D_{\pm} = \frac{1}{S_+ - S_-} \begin{bmatrix} \mp(S_{\mp}-d_1) & \pm d_3 e^{ir} \\ \pm d_3 e^{-ir} & \pm(S_{\pm}-d_1) \end{bmatrix} \tag{24}$$

($C(0,t)$ is the input state of polarization).

From this equation, it follows that the light wave propagates in the form of two modes of different phases. This is caused by the perturbations of fibre parameters (changing in cross-section, along the fibre, and in time).

3. Stochastic characteristic of the process $\Delta\varphi$

From formula (22)

$$\Delta\varphi = \omega(S_+ - S_-)\int_0^z k(x, ax+t-az)dx \tag{25}$$

where: $k(z,t)$ is a stochastic process of average over the ensemble $\langle k(z,t)\rangle = k_0$ and autocorrelation $R_k(a,b) := \langle k(z+a, t+b)\rangle$, we conclude that also $\Delta\varphi$ is a stochastic process. The magnitude $\Delta\varphi$ describes a random difference of phases between two elliptically polarized eigenmodes of the fibre taken over the ensemble of fibres of the length z. The average of this magnitude is

$$\langle\Delta\varphi\rangle = \omega\langle(S_+ - S_-)\int_0^z k(x, ax+t-az)dx\rangle. \tag{26}$$

From the previous assumptions (concerning independence of S and k) we have

$$\langle\Delta\varphi\rangle = \omega\langle(S_+ - S_-)\rangle\langle\int_0^z k(x, ax+t-az)dz\rangle = \omega\langle(S_+ - S_-)\rangle k_0 z \tag{27}$$

where

$$\langle(S_+ - S_-)\rangle = \frac{1}{(2\pi)^{3/2}\sigma_1\sigma_2\sigma_3}\iiint_{-\infty}^{+\infty}\sqrt{(x-y)+4(z+d_{30})}$$

$$\times \exp 1/2[-(x/\sigma_1)^2-(y/\sigma_2)^2-(z/\sigma_3)]dxdydz,$$

the random variance of the phase dispersion

$$\sigma_{\Delta\varphi}^2 = \langle(\Delta\varphi)^2\rangle - \langle\Delta\varphi\rangle^2 = \omega^2 \langle(S_+ - S_-)^2$$

$$\times\int_0^z\int_0^z \langle k(x, ax+t-az)\, k(ay, ay+t-az)\rangle dxdy - \omega^2 k_0^2 z^2 \langle(S_+ - S_-)^2\rangle$$

$$= \omega^2\langle(S_+ - S_-)^2\rangle\left\{\int_0^z\int_0^z\langle k(y+x-y, ay+t+a(x-y)-az)k(y, ay+t-az)\rangle dxdy\right.$$

$$\left. - k_0^2 z^2\right\} = \omega^2\langle(S_+ - S_-)^2\rangle\left\{\int_0^z\int_0^z R_k(z, t+a(x-y))dxdy - k_0^2 z^2\right\}, \quad (28)$$

and autocorrelation of this magnitude

$$R_{\Delta\varphi}(z, t+\tau) = \langle\Delta\varphi(z, t+\tau)\Delta\varphi(z, t)\rangle$$

$$= \omega^2\langle(S_+ - S_-)^2\rangle\int_0^z\int_0^z\langle k(x, ax+t+\tau-z)\, k(y, ay+t-z)\rangle dxdy$$

$$= \omega^2\langle(S_+ - S_-)^2\rangle\int_0^z\int_0^z R_k(z, t+a(x-y)+\tau)dxdy. \quad (29)$$

Before starting the physical discussion of these results, it is helpful to consider whether the magnitude $R_{\Delta\varphi}(z, t+\tau)$ is antisymmetric with respect to τ, *i.e.* whether $R_{\Delta\varphi}(z, t+\tau) = -R_{\Delta\varphi}(z, t-\tau)$, in other words, whether from the stationarity of the process k follows the stationarity of $\Delta\varphi$. If k is stationary and real, then

$$R_{\Delta\varphi}(z, t-\Delta t) = \omega^2\langle(S_+ - S_-)^2\rangle\int_0^z\int_0^z R_k(z, t+a(x-y)-\Delta t)dxdx,$$

by changing the variables: $x \to -x$, $y \to -y$,

$$= \omega^2\langle(S_+ - S_-)^2\rangle\int_0^{-z}\int_0^{-z} R_k(z, t-a(x-y)-\Delta t)dxdy,$$

$$= \omega^2\langle(S_+ - S_-)^2\rangle\int_{-z}^{0}\int_{-z}^{0} R_k(z, t+a(x-y)+\Delta t)dxdy,$$

$$= \omega^2\langle(S_+ - S_-)^2\rangle\int_0^z\int_0^z R_k(z, t+a(x-y)+\Delta t)dxdy. \quad (30)$$

The last but one equality follows from the evenness of the function R_k with respect to $(a(x-y)+\Delta t)$ for the stationary processes, while the last one from the formula for changing the variables.

Thus we have

$$R_{\Delta\varphi}(z, t-\Delta t) = R_{\Delta\varphi}(z, t+\Delta t), \tag{31}$$

in other words, if k is a stationary process, then $\Delta\varphi$ has also this property.

Given the average $\langle\Delta\varphi\rangle$, we may define also some other useful parameters characterizing the fibre:

– macroscopic birefringence $\Delta\beta = \frac{\langle\Delta\varphi\rangle}{z}$,

– birefringence of the mode B (the magnitude describing the possibility of a fibre preserving its polarization state)

$$B = \frac{\Delta\beta\ \lambda}{2\pi}, \tag{32}$$

and

– beat length [2]

$$L_p = \frac{2\pi}{\Delta\beta}. \tag{33}$$

Typical values of B lie within the interval $10^{-5}-10^{-7}$ for the fibres and within $10^{-3}-10^{-4}$ for the fibres preserving the polarization [23].

Now, let us start the analysis of the particular factors influencing the value of birefringence. As we know, many factors may be accounted for, such as temperature, deformation, magnetooptic effects, and others [24], [25]. All of them cause changes in the tensor of dielectric permittivity, besides each of them taken separately may be "charged" with some random factors contributing to formula (1).

The tensor (1) may be treated in a broader sense by taking account of the influence of the external effects (for instance, electric, magnetic and accoustic fields) being of essential importance. Thus, this method may find application not only to the fibres or single-mode planar structures but also to the analysis of effects taking place in different optoelectronic devices; this, however, exceeds significantly the frame of this work. The topic is worth closer consideration, since the technology of production approaches the limits of possibilities offered by the materials used providing the fibres and elements of top parameters in which the random factors start to play an essential role. Returning to the analysis of factors affecting the birefringence, we may rewrite (26) in the form

$$\langle\Delta\varphi\rangle = \omega\langle(S_+ - S_-)\int_0^z k(x, ax+t-az)dx\rangle. \tag{34}$$

In the more general case the magnitudes $(S_+ - S_-)$ and $k(x, t)$ are not necessarily independent. However, this case is very difficult to treat because of the necessity of introducing a four-dimensional density function [18], [26], although from the physical circumstances it follows that a wide class of fibres possess this property. Then,

$$\langle\Delta\varphi\rangle = \omega\langle(S_+ - S_-)\int_0^z \langle k(x, ax+t-az)\rangle dx. \tag{35}$$

The average $k_0 := \langle k(x,t) \rangle$ is affected by many factors. We restricted ourselves to considering only three of them (formula (1)); these might be photoelastic, magneto-optic and temperature effects. Then, in accordance with (19), we have

$$k_0 = \frac{1}{\langle d_{ij} \rangle} \omega \int\int_{-\infty}^{+\infty} \langle E_i^* \Delta\varepsilon E_j \rangle dx dy$$

$$= \frac{\omega}{\langle d_{ij} \rangle} \int\int_{-\infty}^{+\infty} \langle E_i^* (\Delta\varepsilon_1 + \Delta\varepsilon_2 + \Delta\varepsilon_3) E_j \rangle dx dy \tag{36}$$

where: $d_{11} = d_1$, $d_{12} = d_3$, $d_{21} = d_3$, $d_{22} = d_2$.

This formula provides a starting point to a more detailed analysis of the influence of the particular external factors on the birefringence. If we want to know in what way the whole ensemble of factors affects $\langle \Delta\varphi \rangle$ (great number of random components independent of both the time and the variable z), then taking advantage of the central limiting theorem [26] we may assume that $k(z,t)$ is a stationary random process of normal distribution the average k_0 of which may be determined from the physical conditions. In the case, however, when the analysis concerns several factors of strong infleunce the above formula may be applied.

Also, the magnitude $\langle (S_+ - S_-) \rangle$ may be calculated, in general, as a three-dimensional density (the product of three densities $g(d_1, d_2, d_3) = g(d_1) \times g(d_2) \times g(d_3)$, in the case when d_1, d_2, d_3 are independent. Obviously, $g(d_i)$ may be of another distribution than the normal one [27].

Besides, in the general case we must take account of the mutual correlation of the processes ε_1, ε_2, ε_3. This will have influence on both the variance of the phase dispersion $\sigma^2_{\Delta\varphi}$ and autocorrelation $R_{\Delta\varphi}$. It has significant importance when considering the nonmonochromatic waves, which affects both coherence and polarization. This will be the subject of further examinations.

4. Random characteristics of the polarization dispersion

Two elliptically coupled eigenvalues of the fibre (formula (21)) have, in general, different transition phases. This difference may be described by the formula

$$\Delta\tau = \tau_+ - \tau_- = \frac{\partial}{\partial\omega} \Delta\varphi. \tag{37}$$

In the general case, the phase dispersion $\Delta\varphi$, beside the linear dependence on the frequency ω depends also on ω in a hidden way (via the chromatic dispersion resulting in dependence of $\Delta\varepsilon$ in ω).

In the first approximation, taking account of Eqs. (25) and (37), we have

$$\Delta\tau(z,\ t) = (S_+ - S_-) \int_0^2 k(x, ax + t - az) dz. \tag{38}$$

The above formula describes the random difference of the transition times for two elliptically polarized modes at the moment t chosen randomly. Since k is a random process, so is $\Delta\tau$. It has the following characteristics:

– average over the ensemble of fibres of the length z

$$\langle\Delta\tau\rangle = \langle(S_+ - S_-)\rangle z \langle k\rangle = \langle(S_+ - S_-)\rangle z k_0 \tag{39}$$

(when $S_\pm$ and k are independent),

– and variance

$$\begin{aligned}
\sigma^2_{\Delta\tau} &= \langle(\Delta\tau)^2\rangle - \langle\Delta\tau\rangle^2 = \langle(S_+ - S_-)^2\rangle \left\{ [\int_0^z k(x, ax+t-az)dc]^2 - z^2 k_0^2 \right\} \\
&= \langle(S_+ - S_-)\rangle \left\{ \int_0^z\int_0^z \langle k(x, ax+t-az) k(y, ay+t-az)\rangle dx dy - z^2 k_0^2 \right\} \\
&= \langle(S_+ - S_-)^2\rangle \left\{ \int_0^z\int_0^z \langle k(y+(x-y), ay+a(x-y)+t-az) \right. \\
&\left. \times\, k(y, ay+t-az)\rangle dx dy - z^2 k_0^2 \right\} \\
&= \langle(S_+ - S_-)^2\rangle \left\{ \int_0^z\int_0^z R(x-y, a(x-y)) dx dy - z^2 k_0^2 \right\}.
\end{aligned} \tag{40}$$

Autocorrelation of this process is

$$\begin{aligned}
R_{\Delta\tau}(u) &= \langle\Delta\tau(z, t+u)\Delta\tau(z, t)\rangle \\
&= \langle(S_+ - S_-)^2\rangle \int_0^z\int_0^z \langle k(x, ax+t+u-az)\, k(y, ay+t-az)\rangle dx dy \\
&= \langle(S_+ - S_-)^2\rangle \int_0^z\int_0^z \langle k(y+(x-y), ay+a(x-y)+u+t-az) \\
&\times k(y, ay+t-az)\rangle dx dy = \langle(S_+ - S_-)^2\rangle \int_0^z\int_0^z R_k(a(x-y)+u) dx dy.
\end{aligned} \tag{41}$$

From the considerations similar to those which led to the formula (31) we have

$$R_{\Delta\tau}(-u) = R_{\Delta\tau}(u).$$

Hence, from the fact that $\Delta\tau$ is real, we obtain that when only k is a random process stationary in a broader sense with respect to the time, then so is $\Delta\tau$, as well.

In the case when the random perturbations are very small, of variances close to zero, we obtain $\langle\Delta\tau\rangle = 2d_{30} z k_0$ (the case of an ideal single mode fibre) for the birefringence caused by internal d_{30} and external k_0 factors, and $\sigma^2_{\Delta\tau} = 0$, and $\langle\Delta\tau\rangle$ described above is an effective description of polarization.

5. Concluding remarks

In the present paper, the phase differences in a fibre exhibiting both birefringence and phase dispersion as random processes have been presented. Their characteristic parameters such as: average over the ensemble of fibres of the length z, the variances and autocorrelations have been given.

The method applied above and proposed partly in [15] allows us to obtain the solution in the sense of the mean value, but has the shortcoming that the parameters of a concrete fibre may deviate from the said main value. In spite of this, the method is very useful for examinations of ensembles of fibres. For the particular processes (ergodicity of process k) describing the distribution of perturbations along the fibre and in time, the ensemble of fibres may be substituted by the fragments of the length z taken from one long fibre, but the existence of ergodicity may be decided on the basis of further examinations and testing.

The solution of the equation describing the state of polarization in the fibre subject to the random perturbation of parameters (random variability in time and space) renders it possible to examine the phase dispersion in a birefringent fibre and the dispersion of polarization as certain random processes. Particularly interesting is the fact that the variance and autocorrelation of both the processes depend immediately upon the autocorrelation of the random process in the form of tensor K describing the random perturbations of the fibre. The particular factors influencing the birefringence such as temperature, internal stress, tension of the fibre, bendings, different other deformations as well as electric magnetic and acoustic fields [24], [25] contribute to the tensor K as a combination of processes which are not necessarily independent.

Their parameters (distributions, averages, variances, autocorrelations, cross-correlations, density spectra) should be determined from the further examinations or experiments.

The autocorrelations and cross-correlations of those processes contribute to the autocorrelation of the process k. They influence the fibre parameters, polarization (polarization degree) by affecting the coherence matrix, but this will be the subject of further examinations. In addition to the fibres, the presented formalism may be applied also to planar optics and to description of random phenomena in elements of integrated optics.

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