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Off-axis paraxial interpretation of holography

The analysis of a holographic recording and reconstruction of a plane transparency is carried out for an arbitrary located and oriented object, i.e. for arbitrarily great angles between object and reference beams, but in the paraxial approximation. Generalized imaging relations for the central point of the object, and the relations for the image magnification are obtained. The conditions for the tilt of the image and for removing the curvature of the image field are given.

1. Introduction

Holographic reconstruction is usually discussed for an object like a plane transparency, and for the total paraxial space [1]. The total paraxial space is defined by assuming the narrowness of not only four (object, reference, reconstruction, and image) beams separately, but also all these beams together. However, this approach idealizes very inadequately the real experimental conditions when angles between separate beam are rather great.

The imaging equation was derived by NEUMANN [2] for a point object under the assumption that the angles are great. This work is unfortunately not as known as the paper [3] which does not describe this general case but is cited more often.

A purpose of this work is to generalize the holographic reconstruction for the plane transparency placed in the beam of paraxial rays, and for other also paraxial beams, the angles between the beams being arbitrary. For the sake of simplicity the calculations are performed for a two-dimensional space only.

2. A phase of a homocentric wave impinging very obliquely on a recording plane

Let a cylindrical wave (a line of intersection with a plane $x_h z$) be spread from a point source Q_0 (fig. 1) to an x_h-axis on which the phase of the wave is to be determined. Let the distance between the source Q_0 and the centre of coordinates O be denoted by $\boldsymbol{R}_Q$, and the angle between this position vector and the z-axis by α_h. The length of a ray from the point Q_0 to the point H on the x_h-axis (its distance from the origin O being x_h), is given by the formula

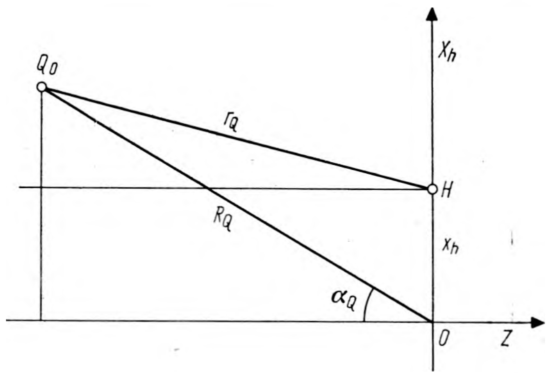

Fig. 1. The phase for the ray of the homocentric wave obliquely impinding on the plane

$$r_Q = \sqrt{[R_Q^2\cos^2\alpha_Q+(R_Q\sin\alpha_Q-x_h)^2]}, \tag{1}$$

which can be written as

$$r_Q = R_Q\sqrt{\left(1+\frac{x_h^2}{R_Q^2}-\frac{2x_h\sin\alpha_Q}{R_Q}\right)}.$$

For the sake of the analytic calculation we substitute this irrational expression in the paraxial space by the series, in which the terms of the order higher than 2 are omitted. With this substitution a segment of the circular line of intersection is changed in the parabolic one. In this case it will be

$$r_Q = R_Q\left(1+\frac{x_h^2}{2R_Q^2}-\frac{x_h}{R_Q}\sin\alpha_Q-\frac{x_h^2\sin^2\alpha_Q}{2R_Q^2}\ldots\right),$$

whence

$$r_Q \approx R_Q+\frac{\cos^2\alpha_Q}{2R_Q}x_h^2-x_h\sin\alpha_Q. \tag{2}$$

This expression is the same as in [2], while in [3] the term $\cos^2\alpha_Q$ was omitted and therefore the results are limited to small angles α_Q.

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The relation for the path difference (2) will be used for the phase term describing the reference and/or reconstruction wave.

3. The phase of a wave outgoing from an object point

Let a plane transparency be placed on the line denoted as the x-axis in the neighbourhood of the point Q_0. The normal to the x-axis makes with the position vector $\boldsymbol{R}_Q$ the angle σ (fig. 2). A cylindrical

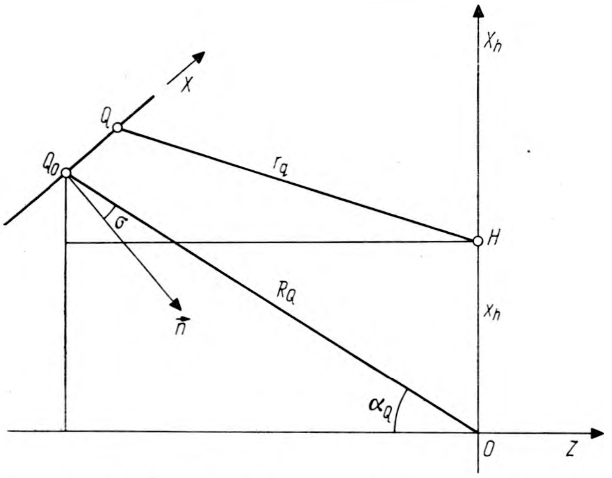

Fig. 2. The phase for the ray emerging from the object point

wave is emitted from a point Q, situated at the distance x from the central point Q_0 of the object, and a ray reaching the point H will have the whole path

$$r_q = \sqrt{\{[R_Q\cos a_Q - x\cos(\pi/2 - a_Q - \sigma]^2 + \\ + [R_Q\sin a_Q - x_h + x\sin\pi/2 - a_Q - \sigma]^2\}}, \tag{3}$$

whence, on the assumption that in the binominal expansion we limit ourselves only to the terms of the order lower than 3, we get

$$r_q \approx R_Q + \frac{\cos^2\sigma}{2R_Q}x^2 + \frac{\cos^2 a_Q}{2R_Q}x_h^2 - \\ - \frac{\cos a_Q \cos\sigma}{R_Q}x_h x - x\sin\sigma - x_h\sin a_Q, \tag{4}$$

where besides the terms contained in (2) there are also terms for the coordinate x, analogical to those in (2), and a mixed term.

4. Holographic recording

Let us suppose that a recording plate is placed in the x_h-axis and its centre is identical with the origin of the coordinates $x_h z$. In the negative halfspace of the z-axis transparency is placed, its centre S_0 being in the distance R_S from the origin of the coordinates, and the connecting line $S_0 O$ makes with the axis z the angle a_S. The normal to the transparency $\boldsymbol{n}$ and the connecting line $S_0 O$ make the angle σ. The axis in the transparency is denoted by x. Diffracted ray is emitted from a point S of the transparency and reaches the general point H of the transparency. The distance $SH = r_s$ is the path of the ray (fig. 3).

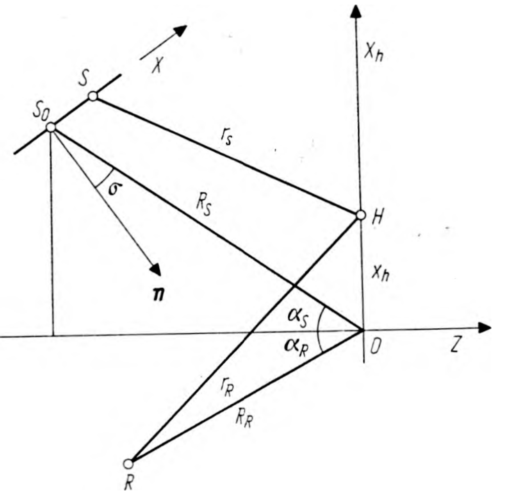

Fig. 3. The hologram recording

For the sake of simplicity let the transparency be illuminated by an equiphase wavefront. If constant terms are omitted the diffraction on the transparency is given by the relation

$$u_s(x_h) = \int u_{0s}(x)\exp(-ik_S r_s)dx, \tag{5}$$

where $u_s(x_h)$ is the amplitude of the diffraction pattern in the hologram plane, $u_{0s}(x)$ is the amplitude transmission of the transparency, and $k_S = 2\pi/\lambda_S$ is the wavenumber of the light wave when the hologram is recorded. As an ideal imaging, not influenced by the diffraction effects on the edges of the object and the hologram, is to be investigated, we shall take integral as unlimited.

The field (5) obtained by the diffraction of light on the object in the region of the recording medium interferes in this region with a reference wave

$$u_R(x_h) = A_R\exp(-ik_S r_R). \tag{6}$$

For the sake of simplicity the decrease of the amplitude with the distance is also neglected. At the same time the expression in relations (5) and (6) for the path length of the ray r (relations (2) and (4)) will be required for further calculation.

The interference between the fields (5) and (6) gives the intensity

$$I(x_h) = [u_s(x_h) + u_R(x_h)]^2 \\ = [u_s(x_h)]^2 + [u_R(x_h)]^2 + u_s(x_h)u_R^*(x_h) + u_s^*(x_h)u_R(x_h),$$

or by substitution of the relations (5) and (6) into the last formula

$$I(x_h) = [u_s(x_h)+u_R(x_h)]^2$$
$$= [u_s(x_h)]^2+A_R^2+A_R^*\int u_{0s}(x)\exp[-ik_S(r_s-r_R)]dx+$$
$$+A_R\int u_{0s}^*(x)\exp[-ik_S(r_R-r_s)]dx. \qquad (7)$$

The third term on the right side will express the so-called primary reconstructed beam, which will be important in the next analysis.

This intensity is recorded in the hologram plate and changes its amplitude transmittivity τ_H. If we confine ourselves only to the linear transmission, then

$$\tau_H = \bar{\tau}+\varkappa(\bar{I}-I),$$

where $\varkappa = (d\tau/dW)_{\bar{\tau}}\cdot t$, because $W = It$ for the stationary conditions during the exposure. Then for the amplitude transmittivity of the primary reconstructed beam we obtain

$$\tau_p(x, x_h) = \varkappa A_R^*\int u_{0s}(x)\exp[-ik_S(r_s-r_R)]dx. \qquad (8)$$

5. Image reconstruction

If the hologram is illuminated by a reconstruction wave, the diffraction of light arises on a structure of the amplitude transmittivity, and the diffracted wave forms an image beam (fig. 4).

The reconstruction wave is a simple cylindric wave again

$$u_C(x_h) = A_C\exp(-ik_I r_C), \qquad (9)$$

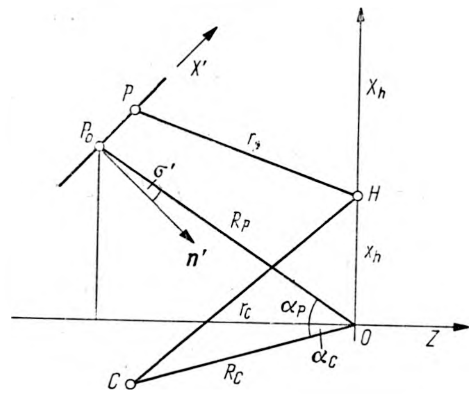

Fig. 4. The reconst uction of the primary wave

its wave-length need not coincide with the wave-length of light used for the recording process; the wave vector $k_I = 2\pi/\lambda_I$. At the same time the path length of the ray is similar to the reference wave given by the relation (2).

The diffraction of this wave on the hologram gives the reconstruction of the primary wave and this process is described by the relation

$$u_{dP}(x') = A_C\int \tau_p(x, x_h)\exp[-ik_I(r_C-r_d)]dx_h, \qquad (10)$$

where the path-length of the ray r_d is given by the relation (4). Inserting (8) into (10) we get

$$u_{dP}(x') = \varkappa A_C A_R^*\int\int u_{0s}(x)\exp[-ik_S(r_s-r_R)-$$
$$-ik_I(r_C-R_d)]dx_h dx, \qquad (11)$$

whence, by substitution of the relations (2) and (4) for the path-lengths of the rays we get

$$u_{dP}(x') = \varkappa A_C A_R^*\exp[-i(k_S R_S-k_S R_S+k_I R_C-k_I R_D)]\int u_{0s}(x)\exp\left[-i\pi\left(\frac{\cos^2\sigma}{\lambda_S R_S}x^2-\frac{\cos^2\sigma'}{\lambda_I R_D}x'^2\right)\right]\times$$
$$\times\exp\left[2\pi i\left(\frac{\sin\sigma}{\lambda_S}x-\frac{\sin\sigma'}{\lambda_I}x'\right)\right]dx\int\exp\left[-i\pi\left(\frac{\cos^2\alpha_S}{\lambda_S R_S}-\frac{\cos^2\alpha_R}{\lambda_S R_R}+\frac{\cos^2\alpha_C}{\lambda_I R_C}-\frac{\cos^2\alpha_D}{\lambda_I R_D}\right)x_h^2\right]\times \qquad (12)$$
$$\times\exp\left\{2\pi i\left[\left(\frac{\sin\alpha_S}{\lambda_S}-\frac{\sin\alpha_R}{\lambda_S}+\frac{\sin\alpha_C}{\lambda_I}-\frac{\sin\alpha_D}{\lambda_I}\right)x_h+\left(\frac{\cos\alpha_S\cos\sigma}{\lambda_S R_S}x-\frac{\cos\alpha_D\cos\sigma'}{\lambda_I R_D}x'\right)x_h\right]\right\}dx_h,$$

which is the final relation for the Fresnel diffraction. This allows to calculate the field emerging from the holographic record in the arbitrary place in the direction of the primary reconstructed beam. This general case, will be not analyzed, as we confine ourselves only to the conditions for the creation of the image.

6. The primary holographic image

From the relation (12) we obtain the image. We shall first determine its location, given by the distance and by the direction in which the centre of the image is situated. These two quantities are obtained from the conditions that the quadratic term of the phase equals zero, and that the first part of the linear term of the phase with the variable x_h is also equal to zero. Then we have

$$\frac{\cos^2\alpha_S}{\lambda_S R_{SR}}-\frac{\cos^2\alpha_R}{\lambda_S R_R}+\frac{\cos^2\alpha_C}{\lambda_I R_C}-\frac{\cos^2\alpha_D}{\lambda_I R_D}=0,$$
$$(13)$$
$$\frac{\sin\alpha_S}{\lambda_S}-\frac{\sin\alpha_R}{\lambda_S}+\frac{\sin\alpha_C}{\lambda_I}-\frac{\sin\alpha_D}{\lambda_I}=0,$$

whence the location of the image is given by

$$\frac{\cos^2\alpha_D}{R_D} \equiv \frac{\cos^2\alpha_P}{R_P} = \frac{\cos^2\alpha_C}{R_C} + \mu\left(\frac{\cos^2\alpha_S}{R_S} - \frac{\cos^2\alpha_R}{R_R}\right), \tag{14}$$

$$\sin\alpha_D \equiv \sin\alpha_P = \sin\alpha_C + \mu(\sin\alpha_S - \sin\alpha_R),$$

where $\mu = \lambda_I/\lambda_S$.

The conditions for the centre of the image (14) are identical with the relations, derived by NEUMANN [2] for the location of the point image.

On the assumption (13) the integral with the variable x_h in (12) is reduced to the expression

$$\int \exp\left[2\pi i\left(\frac{\cos\alpha_S\cos\sigma}{\lambda_S R_S}x - \frac{\cos\alpha_P\cos\sigma'}{\lambda_I R_P}x'\right)\right]x_h dx_h, \tag{15}$$

which gives the Dirac δ-function. Then the expression (15) will have the form

$$\frac{\lambda_S R_S}{\cos\alpha_S\cos\sigma}\delta\left(x - \frac{\cos\alpha_P\cos\sigma'}{\cos\alpha_S\cos\sigma}\frac{R_S}{\mu R_P}x'\right), \tag{16}$$

and the amplitude in the primary image will be described by he relation

$$u_P(x') = A_P\int u_{0s}(x)\exp\left[-i\pi\left(\frac{\cos^2\sigma}{\lambda_S R_S}x^2 - \frac{\cos^2\sigma'}{\lambda_I R_P}x'^2\right)\right]\exp\left[2\pi i\left(\frac{\sin\sigma}{\lambda_S}x - \frac{\sin\sigma'}{\lambda_I}x'\right)\right]\times$$
$$\times\,\delta\left(x - \frac{\cos\alpha_P\cos\sigma'}{\cos\alpha_S\cos\sigma}\frac{R_S}{\mu R_P}x'\right)dx. \tag{17}$$

By virtue of the well known formula

$$\int U(x)\delta(x-c)dx = U(c)$$

we obtain from (17)

$$u_P(x') = A_P u_{0s}\left(\frac{\cos\alpha_P\cos\sigma'}{\cos\alpha_S\cos\sigma}\frac{R_S}{\mu R_P}\right)x'\exp\left[-i\pi\left(\frac{1}{\lambda_S}\frac{\cos^2\alpha_P}{\cos^2\alpha_S}\frac{R_S}{\mu^2 R_P^2}\cos^2\sigma' - \frac{\cos^2\sigma'}{\lambda_I R_P}\right)x'^2\right]\times$$
$$\times\exp\left[2\pi i\left(\frac{1}{\lambda_S}\tan\sigma\frac{\cos\alpha_P}{\cos\alpha_S}\frac{R_S}{\mu R_P}\cos\sigma' - \frac{1}{\lambda_I}\sin\sigma'\right)x'\right]. \tag{18}$$

From (18) it follows, that the description contains still a quadratic and a linear phase; this means that the image is curved and tilted. To obtain a solely magnified image

$$u_p(x') = A_P u_{0s}\left(\frac{\cos\alpha_P}{\cos\alpha_S}\frac{\cos\sigma'}{\cos\sigma}\frac{R_S}{\mu R_P}x'\right), \tag{19}$$

we have to lay the quadratic and the linear phase equating zero. Then

$$\frac{\cos^2\sigma'}{R_P}\left(\frac{1}{\lambda_S}\frac{\cos^2\alpha_P}{\cos^2\alpha_S}\frac{R_S}{\mu^2 R_P} - \frac{1}{\lambda_I}\right) = 0, \tag{20}$$

from where we have the following conditions

1) $\sigma' = \pi/2$,

2) $R_P \to \infty$

3) $\dfrac{\cos^2\alpha_P}{\cos^2\alpha_S} = \mu\dfrac{R_P}{R_S}$.

The condition 1) assumes that the image lies on the line identical to the position vector, which does not occur in practice. The condition 2) describes the reconstruction in the Fraunhofer region, which is realized by the choice of the parameters after (14), and by usage ot the Fourier lens. The condition 3) for non-zero α_P, α_S and $\mu \neq 1$ generalizes the assumption that for $R_P = R_S$ the image is identical with the object.

If the linear terms are to be equal to zero, then

$$\tan\sigma' = \frac{\cos\alpha_P}{\cos\alpha_S}\frac{R_S}{R_P}\tan\sigma \tag{21}$$

holds, from where it follows that $\sigma' = \sigma$, partly on the assumption that $\sigma = 0$, and partly for $\alpha_P = \alpha_S$ and $R_P = R_S$. For general values of the given quantities the tilt of the image to the connecting line R_P can be calculated from (21).

From the argument of the amplitude of the image we obtain the formula for the magnification of the image

$$M = \mu\frac{\cos\alpha_S}{\cos\alpha_P}\frac{\cos\sigma}{\cos\sigma'}\frac{R_P}{R_S}, \tag{22}$$

which dependens on the ratio of wavelengths and on the distances of the object and image from the hologram, as well as on the off-axis angles and the angles of the tilt of the planes of the object and image.

For the sake of simplicity we assume that the transparency is illuminated by the equiphase front of the light wave. This case practically does not arise by the general tilt of the object plane, and it is more suitable to take the assumption, that the object plane is perpendicular to the position vector ($\sigma = 0$) only, or to place suitable random phase mask in front of the transparent. Then instead of the amplitude function of the transparent it will arise the product of the random phase function $f(x) = \exp i\varphi(x)$, and the amplitude function of the transparent $u_{0s}(x)$ and all the calculations will fit this case.

For the reconstructed conjugate wave the analysis is similar and it is not necessary to carry out it here. As it is well known the same formulae, but with the opposite sign in front of the coefficient μ, can be obtained, e.g. for the location of the image.

7. Conclusion

The paraxial analysis of the general off-axis case of the holographic recording and reconstruction leads to the more general formulae for the location of the centre of the image (14), where the relations for the distance include also the values of angles between the beams. Only for small angles between the beams the first relation (14) is restricted to the formula which is currently used.

The transversal magnification of the image, relation for the tilt of the image, and the condition for removing the curvature of the image were determined. All the relations are generalization of the simple relations which can be obtained for the total paraxial conditions.

Интерпретация внеосевой голографии в параксиальном приближении

Проведен анализ регистрации и голографической реконструкции плоского диапозитова для предметов, произвольно расположенных и ориентированных, т. е. для произвольно больших углов между пучками: предметным и отнесения, но с применением параксиального приближения. Получили обобщенные зависимости на отображение центральной точки предмета, а также зависимости для увеличения изображения. Приведено условие на наклон изображения, а также устранение кривизны поля в изображении.

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