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Compacting of images of periodic objects by amplitude-matched spatial filtering

The method of the elementary patterns compaction in the images of periodic objects is presented. This effect is based on spatial filtering of the object Fourier spectrum with the use of the spatial periodic filter.

1. Introduction

The method presented here can be applied to 2-D objects composed of the identical patterns periodically disposed in a plane lattice (fig. 1). The Fourier spectra of such objects

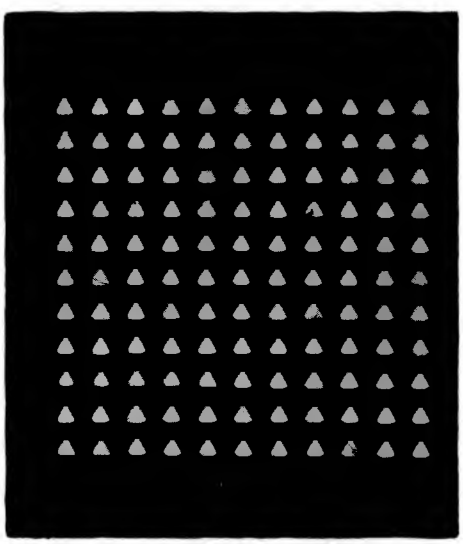

Fig. 1. Object: $d_x = d_y = d$ image in coherent illumination

are discrete. They are composed of the lumineus points corresponding to various object spatial frequencies, disposed in a reciprocal lattice. Images are compacted by putting a mask, e.g. amplitude periodic filter. In this way a part of the object discrete Fourier spectrum is eliminated. In the image plane we obtain reconstruction of the object with greater number of elementary patterns. This method makes some generalization of D. Post's results, who produces the compacted raster diffraction gratings by selecting some orders of their spectrum [1, 2].

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2. General theory

The transmittance $t = t(x_0, y_0)$ of the object composed of the periodic patterns can be written as fo lows:

$$t(x_0, y_0) = t_c * \sum_{m=-\infty}^{m=+\infty} \sum_{n=-\infty}^{n=+\infty} \delta(x_0 - m d_{x_0}, y_0 - n d_{y_0}), \quad (1)$$

where $t_c = t_c(x_0, y_0)$ – the amplitude transmittance of an elementary pattern, limited to $x_0 \in \mu(A_{x_0})$, $y_0 \in \mu(A_{y_0})$, and d_{x_0}, d_{y_0} – the lattice constants, $*$ – convolution symbol. The elementary pattern dimensions are limited so that the following relation holds: $\mu(A_{x_0}) < d_{x_0}$, $\mu(A_{y_0}) < d_{y_0}$; $\mu(A)$ – measure of the bonded support.

The light field (fig. 2) of the object Fourier spectrum [1] in the focal plane x, y of the lens

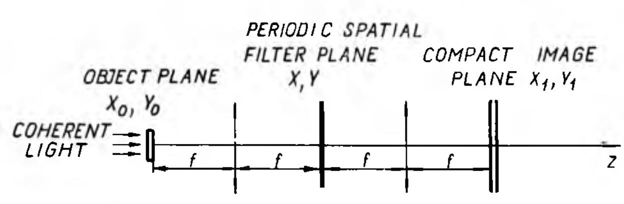

Fig. 2. Optical arragement

L_1 with focal length f_1 can be presented as:

$$\hat{U}(x, y) \propto F\{t(x_0, y_0)\} = T_c(x, y) \sum_{m=-\infty}^{m=+\infty} \sum_{n=-\infty}^{n=+\infty} \delta(x - m d_x, y - n d_y), \quad (2)$$

where:

$F\{\ \}$ ≡ Fourier transformation operation;

$F\{t(x_0, y_0)\} = \hat{F}(x, y) = \hat{\hat{F}}(f_x, f_y)$ – Fourier transform of the object transmittance $t(x_0, y_0)$;

$F\{t_c(x_0, y_0)\} = \hat{T}_c(x_0, y_0) = \hat{\hat{T}}_c(f_x, f_y)$ – Fourier transform of the object elementary pattern $t_c(x_0, y_0)$,

and

$f_x = \frac{x}{\lambda f_1}$, $f_y = \frac{y}{\lambda f_1}$ – corresponding spatial frequencies;

$d_x = \frac{\lambda f_1}{d_{x_0}}$, $d_y = \frac{\lambda f_1}{d_{y_0}}$ – constants of the reciprocal lattice to the initial lattice d_{x_0}, d_{y_0}.

For simplicity it is assumed that the object is illuminated by a plane wave propagating in the direction of z axis:

$$\hat{U} = U_0 \exp(ikz),\ k = \frac{2\pi}{\lambda},\ U_0 = 1.$$

Matched periodic spatial filter has the transmittance:

$$t_f(x, y) \propto \sum_{k=-\infty}^{k=+\infty} \sum_{l=-\infty}^{l=+\infty} \delta(x - kD_x,\ y - lD_y). \quad (3)$$

Let $D_x/d_x = a$, and $D_y/d_y = b$, where a and b are the natural numbers.

After inserting the filter in the Fourier plane (fig. 3, 4) the light field becomes:

$$U_f \propto F\{t\} t_f$$

$$= T_c(x, y) \sum_{m=-\infty}^{m=+\infty} \sum_{n=-\infty}^{n=+\infty} \delta(x - md_x,\ y - nd_y) \times$$

$$\times \sum_{k=-\infty}^{k=+\infty} \sum_{n=-\infty}^{n=+\infty} \delta(x - kD_x,\ y - lD_y)$$

$$= T_c(x, y) \sum_{k=-\infty}^{k=+\infty} \sum_{l=-\infty}^{l=+\infty} \delta(x - kD_x,\ y - lD_y) \quad (4)$$

for $m = k$, $n = l$.

The information about the elementary patterns t_c can be preserved. Behind the filter, the Fourier spectrum of the elementary pattern of the object $T_c = F\{t_c\}$ is discretisated thinner: $T_c(md_x, nd_y)$ is replaced by $T_c(kD_x, lD_y)$. For the full reconstruction of the function t_c the following condition must be fulfilled: from the sampling theorem [3] it follows that function $t_c = t_c(x_0, y_0)$ describing the object elementary pattern will be reconstructed with preservation

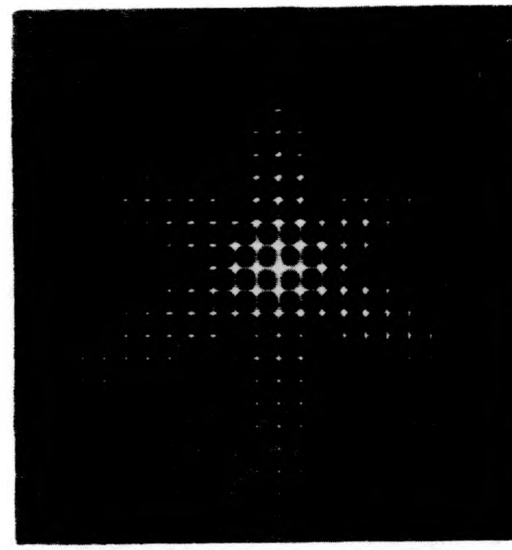

Fig. 3. Fourier image of finite periodic object in focal plane of lens L_1

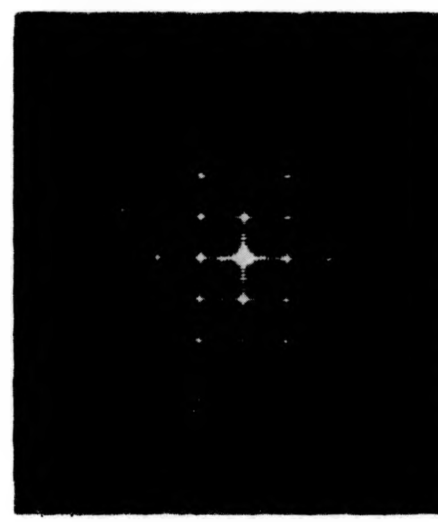

Fig. 4. Fourier image of finite periodic object filtered through periodic spatial filter

of full information, if the new sampling constants D_x, D_y of its Fourier transform $T_c(x, y)$ obey the relations:

$$D_x \leqslant \frac{\lambda f_1}{\mu(A_{x_0})},\ D_y \leqslant \frac{\lambda f_1}{\mu(A_{y_0})}. \quad (5)$$

The natural numbers a and b are: $a = D_x d_{x_0}(\lambda f_1)^{-1}$, $b = D_y\, d_{y_0}(\lambda f_1)^{-1}$. Then from eq. (5) we have:

$$1 \leqslant a \leqslant w,\ 1 \leqslant b \leqslant u \quad (6)$$

where

$$w = \frac{d_{x_0}}{\mu(A_{x_0})} \quad \text{and} \quad u = \frac{d_{y_0}}{\mu(A_{y_0})}.$$

The numbers a and b serve as a measure of image compaction at the optical system output.

In the focal plane of the second lens L_2 with the focal length f_2 we get reconstruction of the adequately compacted images (fig. 5):

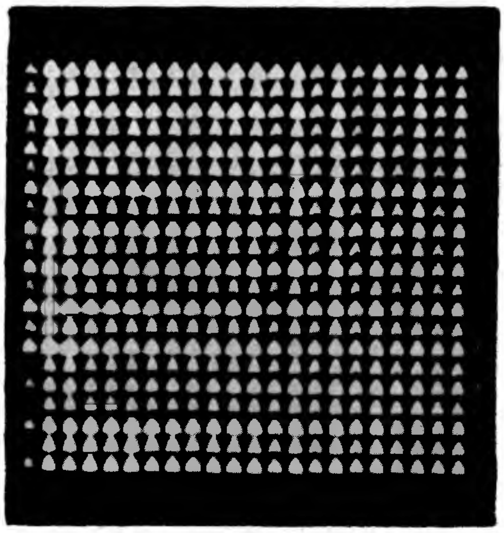

Fig. 5. Compacted image formed by periodic spatial filtering: $a = b = 2$

$$\hat{U}_{im} \propto t_c(x_1, y_1) * \sum_{p=-\infty}^{p=+\infty} \sum_{q=-\infty}^{q=+\infty} \delta(x_1 - p D_{x_1}, y_1 - q D_{y_1}), \qquad (7)$$

$$I_{im} \propto t_c^2(x_1, y_1) * \sum_{p=-\infty}^{p=+\infty} \sum_{q=-\infty}^{q=+\infty} \delta(x_1 - p D_{x_1}, y_1 - q D_{y_1}), \qquad (8)$$

where

$$D_{x_1} = \frac{f_2}{f_1} \frac{d_{x_0}}{a}, \quad D_{y_1} = \frac{f_2}{f_1} \frac{d_{y_0}}{b}.$$

For simplicity we assume $f_1 = f_2$, and magnification $M = 1$:

$$D_{x_1} = \frac{d_{x_0}}{a}, \quad D_{y_1} = \frac{d_{y_0}}{b}$$

and

$$t_c = t_c(x_1, y_1) = t_c(x_0, y_0). \qquad (9)$$

The resulting image will be adequately compacted. For $a = w$, and $b = u$, $D_{x_1} = \mu(A_{x_0})$, $D_{y_1} = \mu(A_{y_0})$, the compacted lattice D_{x_1}, D_{y_1} is completely filled in.

3. Experimental

The matched filters eliminating part of the discrete object Fourier spectrum can be performed in two ways:

1. In the black absorbing screen the transparent, preferably the circular, sampling elements are disposed. Their radius r_0 must be large enough to transmit a given fragment of the object discrete spectrum without diffraction. Because real objects have limited finite dimensions of the aperture their Fourier spectra do not consist of the sharp points. The points are enlarged [4]. The Fourier spectrum of the finite periodic object composed of $N \times M$ element of transmittance $t(x_0, y_0)$:

$$t(x_0, y_0) = t_c(x_0, y_0) * \sum_{m=-M}^{m=+M} \sum_{n=-N}^{n=+N} \delta(x_0 - m d_{x_0}, y_0 - n d_{y_0}), \qquad (10)$$

is

$$F\{t(x_0, y_0)\} = \frac{\sin\left[\left(M + \frac{1}{2}\right) \frac{2\pi x}{\lambda f_1} d_{x_0}\right]}{\sin \frac{\pi x}{\lambda f_1} d_{x_0}} \times \frac{\sin\left[\left(N + \frac{1}{2}\right) \frac{2\pi y}{\lambda f_1} d_{y_0}\right]}{\sin \frac{\pi y}{\lambda f_1} d_{y_0}} F(t_c). \qquad (11)$$

The function of the type

$$\frac{\sin\left(M + \frac{1}{2}\right) a}{\sin a}$$

describes smear of the object discrete Fourier spectrum.

2. Construction of such filters is rather difficult, but they can be produced on the other ground. The plane square, rectangular, or hexagonal lattice can be described in radial coordinates: r, φ [5]. For example the plane rectangular lattice can be described by the equation $m^2 D_x^2 + n^2 D_y^2 = h R_0^2$, where h, m, n — the corresponding integers, $(m^2 v_y^2 + n^2 v_x^2) h^{-1} = R_0^2 v_x v_y (D_x D_y)^{-1}$, R_0 — constant of radial lattice, $D_x v_x = D_y v_y$, and $v_x \; v_y^{-1}$ — smallest rational number.

Thus one can make a radial filter eliminating periodically the adequate fragments of the discrete object spectrum (Fresnel periodic object Fourier spectrum). Such a filter is composed of the concentric circles with radius $R = \sqrt{h} R_0$. The slit dimension must be fitted to the actual smear of the Fourier spectrum points for a real object. It resembles the Fresnel zone plate [5].

Fig. 5 presents experimental results of the patterns compaction in the images of the periodic objects. Thus the number $a = b = 2$, and the filter lattice constants are $D_x = D_y = 2d$, $d_{x_1} = d_{y_1} = d/2$. Compacted image shown in fig. 5 is a little defected due to the faults in the spatial filter used by the author.

Appendix

Compacting filtration can be realised by the use of a single lens as well. The filter is then located in its focal plane. In such case eqs. (2), (5) must include the phase factor (see cf., [6]).

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Упаковка изображений периодических предметов методом амплитудного пространственного подобранного фильтрования

В статье представлен метод уплотнения элементарных узоров в изображении плоского предмета. Предмет, являющийся чёрным экраном с элементарными узорами, которые размещены периодически, помещается при входе когерентной оптической системы, предназначенной для образования изображений. При выходе получается сгущённое изображение благодаря помещению в первой фокусной плоскости соответствующей маски. Этот эффект опирается на использовании подобранного периодического фильтрования пространственных частот.

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