Optica Applicata, Vol. XLI, No. 4, 2011

A study of propagation of cosh-squared-Gaussian beam through fractional Fourier transform systems

SOMAYE SADAT HASHEMI[1*], SAEED GHAVAMI SABOURI[2], MAHMOOD SOLTANOLKOTABI[3]

[1]Falavarjan Branch, Islamic Azad University, Isfahan, Iran

[2]Department of Physics, University of Isfahan, Isfahan, Iran

[3]Quantum Optics Group, Department of Physics, University of Isfahan, Isfahan, Iran

[*]Corresponding author: hashemi.somaye@gmail.com

In this paper, we consider properties of cosh-squared-Gaussian beam passing through ideal and apertured fractional Fourier transforms (FRFT) systems. We use Collins integral formula and the fact that a hard aperture function can be expanded into a finite sum of complex Gaussian functions. These studies indicate that the normalized intensity distributions with FRFT order are periodic. The variation period is 2 and is independent of the impact of aperture.

Keywords: cosh-squared-Gaussian, fractional Fourier transforms (FRFT).

1. Introduction

The Fourier transform is one of the most important mathematical tools that are used in physical optics, optical information processing, linear system theory, and some other areas.

The fractional Fourier transform (FRFT) is regarded as a generalization of the conventional Fourier transform. There were some reports about the relation between FRFT and quantum mechanics in the 1980s [1, 2], but those did not gain much attention. MENDLOVIC and OZAKTAS had introduced FRFT into optics [3, 4], then LOHMANN studied the relation between the image rotation in the optical system and the FRFT [5]. He introduced two optical setups for performing a fractional Fourier transform.

The application of the FRFT in other areas such as signal processing, beam shaping and image encryption, has gained more attention [6–11]. Moreover, the propagations of laser beams through FRFT systems have been widely investigated [12–21].

The optical field distributions on FRFT plane can be derived from Collins integral formula [22]. Although the lenses in FRFT systems are infinite, one can use in real life apertured lenses or other limited optical instruments. Thus, it is very important and necessary to consider the hard aperture FRFT systems [23–26].

There are some reports about beams passing through apertured FRFT systems [27–32], and also the effect of FRFT order and aperture size on the intensity distribution for two optical setups of Lohmann [27–32]. In these reports, they did not point out the variation period of intensity with FRFT order. However, CHEN *et al*. have shown the effect of the variation period of intensity on the FRFT order [33]. These authors have used the type I Lohmann system to achieve the FRFT of cosh-squared-Gaussian (CSG) beam, and they have studied the properties of CSG beam passing through the ideal and aperture FRFT systems. The characteristics of cosh-Gaussian beam have also been widely studied [21, 34–36].

In this paper, we have investigated the properties of CSG beam passing through the ideal and aperture type II Lohmann system, and we have studied the intensity distribution of CSG beam on FRFT plan. To do this, we have used two different methods, analytical formula and Collins diffraction integral formula. We have implemented these methods for both Lohmann systems (types I and II) and then have compared the results.

The paper is organized as follows: the theoretical analyses of CSG beam passing through ideal and apertured FRFT systems are given in Section 2. The numerical comparisons using the analytical formulae and the diffraction integral formulae are given in Section 3. Finally, our conclusion is given in Section 4.

2. Field distribution calculation for ideal and analytical cases

Let us consider the case of an infinite size of lenses in FRFT systems and Lohmann systems as illustrated in Fig. 1. We have used type II Lohmann system, as shown in Fig. 1**b**, where f_s is the standard focal length, p is the FRFT order, $\phi = p\pi/2$, d is the distance between the input ($z = 0$) and output ($z = d$) planes.

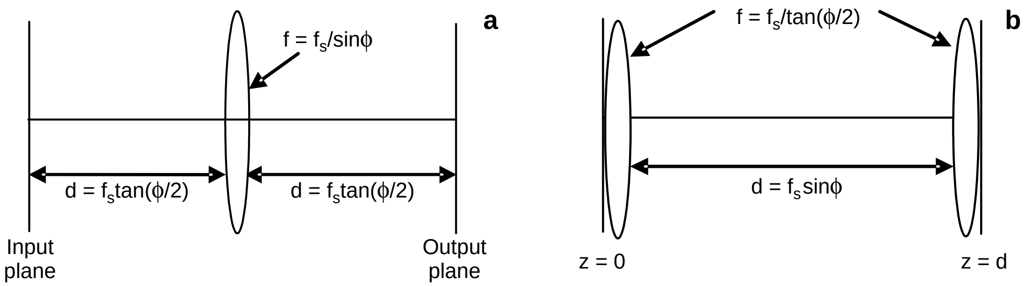

Fig. 1. Lohmann optical systems: type I (**a**), and type II (**b**).

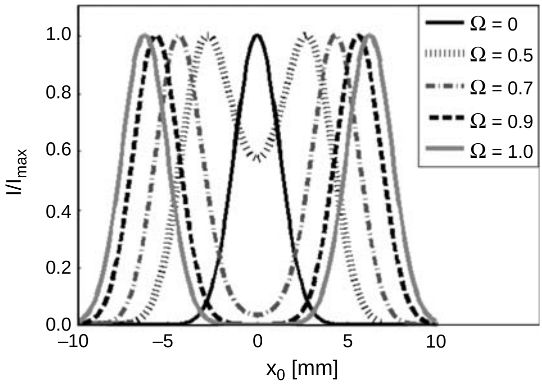

Fig. 2. Normalized intensity for different Ω's on input plane (w_0 = 2.5 mm).

It is known that the optical field distribution of the one-dimensional cosh-squared--Gaussian beams on the input plane is characterized by [33, 36]:

$$E_0(x_0) = \exp\left(-\frac{x_0^2}{w_0^2}\right)\cosh^2(\Omega x_0) \tag{1}$$

where w_0 represents the beam waist of Gaussian beam, Ω is the parameter associated with the cosh part, and x_0 is the transversal position on the plane $z = 0$. By changing the value of Ω, one gets different optical field distributions (Fig. 2). For $\Omega = 0$, Eq. (1) denotes the usual Gaussian beam.

To achieve the optical field distributions on FRFT plane we have use the Collins integral formula [22]:

$$E(x) = \sqrt{-\frac{i}{\lambda B}}\int_{-\infty}^{\infty} E_0(x_0)\exp\left[-i\frac{\pi}{\lambda B}\left(Ax_0^2 + Dx^2 - 2xx_0\right)\right]\mathrm{d}x_0 \tag{2}$$

The constant phase in Collins formula, which has no influence on the output intensity distribution has been omitted. A, B, and D are the elements of the system transfer matrix.

If the lenses of type II Lohmann system are infinite, the transfer matrix from the plane at $z = 0$ to plane $z = d$ becomes:

$$M = \begin{bmatrix} 1-\dfrac{d}{f} & d \\ \dfrac{d}{f^2}-\dfrac{2}{f} & 1-\dfrac{d}{f} \end{bmatrix} = \begin{bmatrix} \cos\phi & f_s\sin\phi \\ -\dfrac{1}{f_s}\sin\phi & \cos\phi \end{bmatrix} \tag{3}$$

By substituting relations (3) and (1) into relation (2), and performing the integration we obtain the optical field distribution on FRFT plane

$$E_{\text{ideal}}(x) = \sqrt{-\frac{i\pi w_0^2}{4(\lambda f_s \sin\phi + i\pi w_0^2 \cos\phi)}} \times$$
$$\times \exp\left[-\frac{(\pi x w_0)^2}{\lambda f_s \sin\phi(\lambda f_s \sin\phi + i\pi w_0^2 \cos\phi)} - \frac{i\pi x^2 \cos\phi}{\lambda f_s \sin\phi}\right] \times$$
$$\times \left[1 + \exp\left(\frac{\Omega^2 w_0^2 \lambda f_s \sin\phi}{\lambda f_s \sin\phi + i\pi w_0^2 \cos\phi}\right) \cosh\left(\frac{2\pi i w_0^2 \Omega x}{\lambda f_s \sin\phi + i\pi w_0^2 \cos\phi}\right)\right] \tag{4}$$

From relation (4) we see that the optical field distribution on FRFT plane in addition to the beam parameters depends on the system parameters such as the standard focal length f_s and FRFT order p.

It is worth mentioning that the transfer matrixes for types I and II Lohmann systems, in the case of an infinite size of lenses, are equal. In other words, the ideal optical field distribution for both Lohmann systems are identical. In this case, we have obtained the same distribution of the ideal field on FRFT plane for both Lohmann systems. Our results are for type I Lohmann system the same as given in Ref. [33].

Usually, the lens in FRFT system is finite and an aperture is to be added in the calculation. We consider two apertures, one in front of the input lens and the other on the output lens. The second aperture only truncates the output field distribution. According to the Collins diffraction integral formula the approximate analytical expression for the output field distribution of a CSG in the FRFT plane is derived as follows:

$$E_{\text{apertured}}(x) = \sqrt{-\frac{i}{\lambda B}} \int_{-a}^{a} E_0(x_0) \exp\left[-i\frac{\pi}{\lambda B}\left(Ax_0^2 + Dx^2 - 2xx_0\right)\right] dx_0 \tag{5}$$

where a denotes the half-width of the lens aperture, defined by the hard aperture function as:

$$t(x) = \begin{cases} 1, & |x| \le a \\ 0, & |x| > a \end{cases} \tag{6}$$

In this case, the relation (5) becomes

$$E_{\text{apertured}}(x) = \sqrt{-\frac{i}{\lambda B}} \int_{-\infty}^{\infty} t(x_0) E_0(x_0) \exp\left[-i\frac{\pi}{\lambda B}\left(Ax_0^2 + Dx^2 - 2xx_0\right)\right] dx_0 \tag{7}$$

In order to calculate the integral we should select a form of hard aperture function introduced in many references [23–26]. We use one given in Ref. [24]:

$$t(x) = \sum_{n=1}^{N} A_n \exp\left(-\frac{B_n}{a^2} x^2\right) \tag{8}$$

where A_n and B_n are the expansion and Gaussian coefficients, respectively, which can be obtained directly from computation of relations [25, 26].

By substituting relations (1), (3), and (8) into relation (7), and performing tedious integration:

$$\int_{-\infty}^{\infty} x^{2n} \exp\left(-\alpha x^2 - 2\beta x\right) \mathrm{d}x = \left(-\frac{\beta}{\alpha}\right)^{2n} \sqrt{\frac{\pi}{\alpha}} \exp\left(\frac{\beta^2}{\alpha}\right) \sum_{l=0}^{n} \frac{(2n)!}{l!(2n-2l)!} \left(\frac{\alpha}{4\beta^2}\right)^{l} \tag{9}$$

an approximate analytical expression for the output field distribution in the FRFT plane is obtained:

$$E(x) = \sum_{n=1}^{N} A_n \sqrt{-\frac{i\pi w_0^2}{4\xi_n}} \exp\left[-\frac{(\pi x w_0)^2}{\lambda f_s \xi_n \sin\phi} - \frac{i\pi x^2 \cos\phi}{\lambda f_s \sin\phi}\right] \times \left[1 + \exp\left(\frac{\Omega^2 w_0^2 \lambda f_s \sin\phi}{\xi_n}\right) \cosh\left(\frac{2\pi i w_0^2 \Omega x}{\xi_n}\right)\right] \tag{10}$$

where

$$\xi_n = \lambda f_s \sin\phi + i\pi w_0^2 \cos\phi + \frac{B_n w_0^2 \lambda f_s \sin\phi}{a^2} \tag{11}$$

$$\delta = \frac{a}{w_0} \tag{12}$$

Relations (10) to (12) are the general expressions, which are valid within the paraxial approximation. Then, apart from the standard focal length f_s and FRFT order p, the intensity distributions on FRFT plane depend on the truncation parameter δ as well.

We see that whenever $a/w_0 \to \infty$, Eq. (10) reduces to Eq. (4). Also, for $\Omega = 0$ Eqs. (4) and (10) reduce to the optical field distributions of Gaussian beam passing through ideal and apertured FRFT systems, respectively, as one expects.

Although Eqs. (10) to (12) are the approximate analytical expressions, they provide a more convenient method for studying the propagation characteristics of a flattened Gaussian beam through the two types of apertured FRFT systems than those using the diffraction integral formula directly.

3. Numerical and analytical analyses

In order to compare our result given by expression (10) with those given by the diffraction integral formula (5), we have performed numerical calculations. We have particularly paid attention to the truncation parameter and FRFT order of the normalized intensity distributions. We have also compared our result with that of type I Lohmann system [33]. In these numerical calculations, we have used $\lambda = 1.06$ μm, $f_s = 1000$ mm, $w_0 = 2.5$ mm, $\Omega = 0.6$, $N = 10$, $\delta = 0.7$, with A_n and B_n being taken from

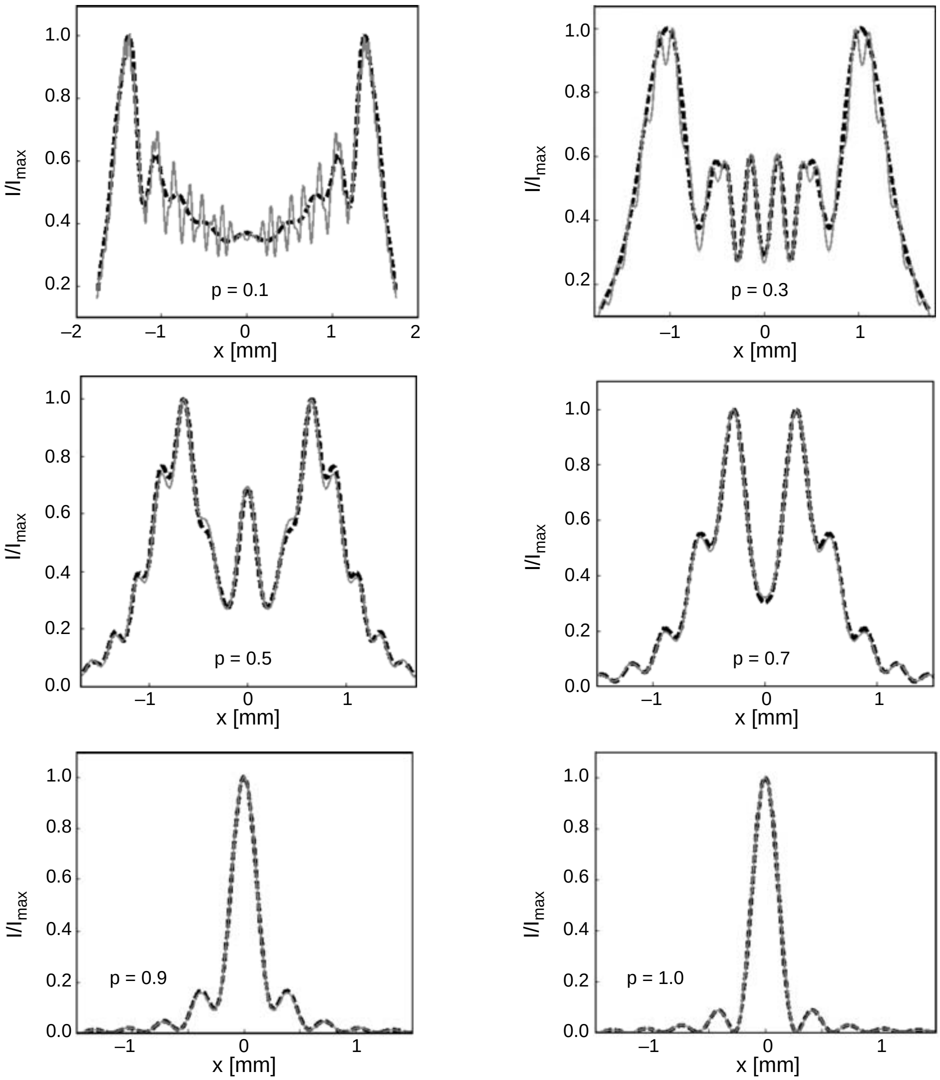

Fig. 3. Normalized intensity distributions of CSG beam on FRFT plane. The dotted lines represent the case of using the formula (10), and the solid lines denote the case of using the diffraction integral formula (5) for different FRFT order values.

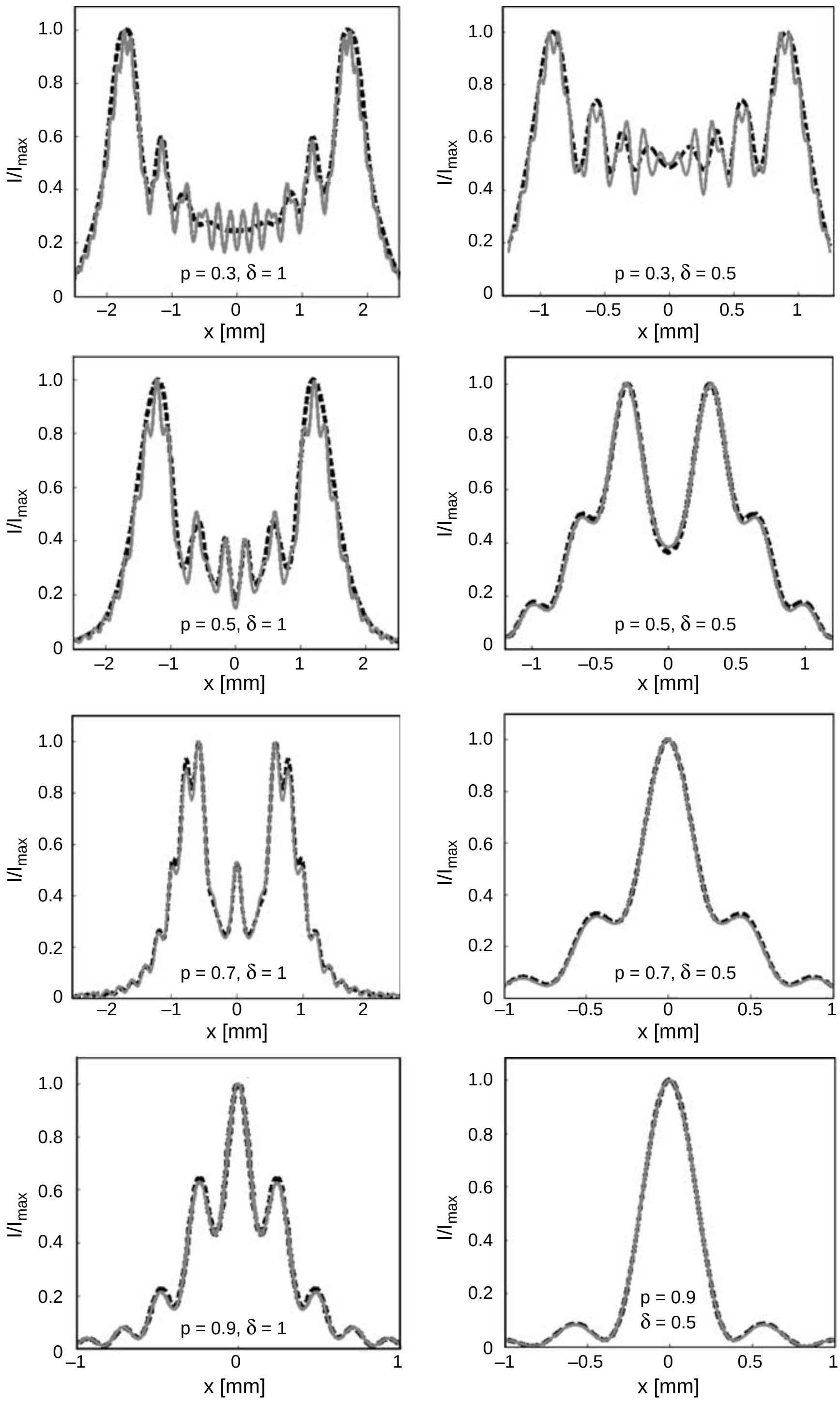

Fig. 4. The same as Fig. 3,but for $\delta = 0.5$ and $\delta = 1$.

Ref. [25, 26]. Figure 3 shows the normalized intensity distributions of CSG beam on FRFT plane.

From these figures, we reach a very good agreement between two approaches, *i.e.*, numerical and analytical calculations, especially for $p > 0.9$. It is worth mentioning that numerical calculation of approximate analytical formula was much faster than the numerical integral calculation.

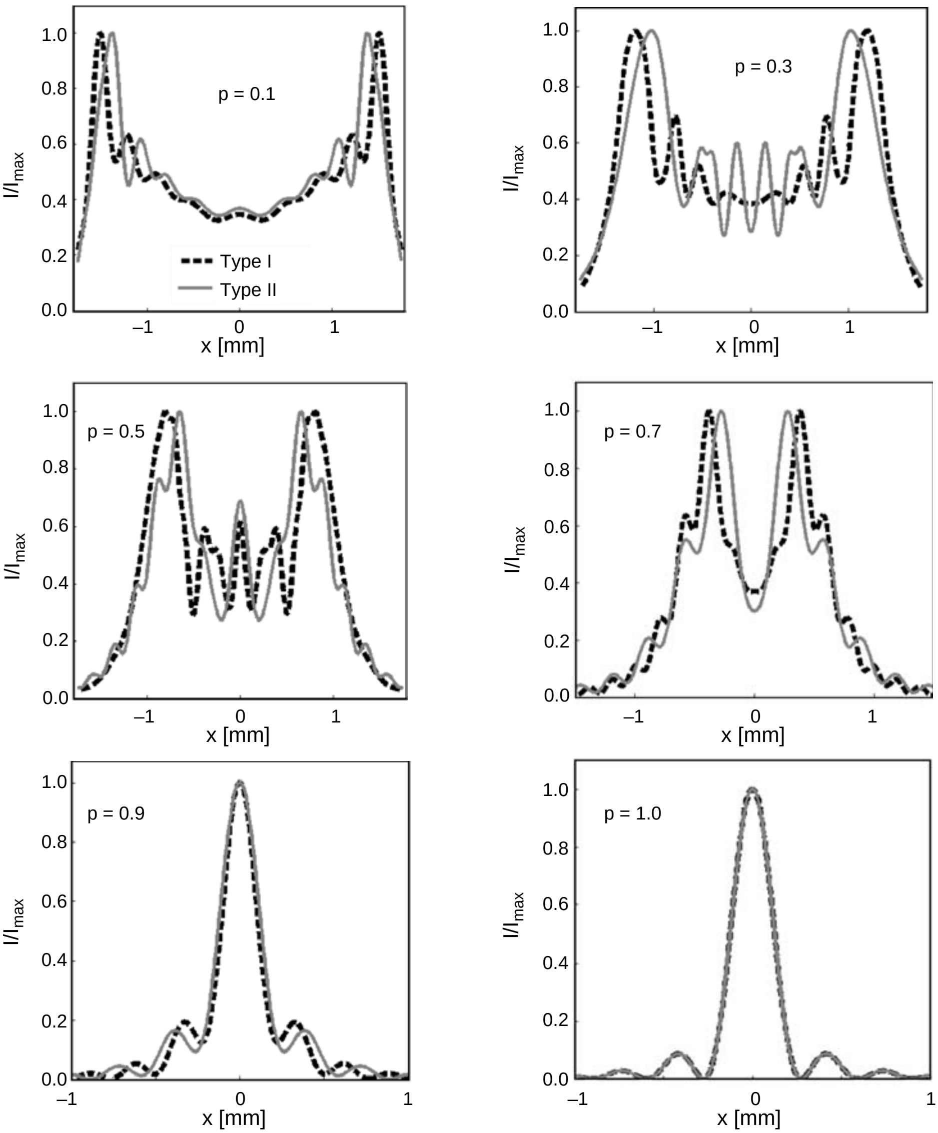

Fig. 5. Normalized intensity distributions of CSG beam on FRFT plane by using the analytical formula for type I and II Lohmann systems.

Next, we have examined the effect of δ on the coincidence of diffraction integral and analytical simulation. We have compared the results for $\delta = 0.5$ and $\delta = 1$ and $p = 0.3$, 0.5, 0.7 and 0.9 (Fig. 4).

We see that for the same value of p and for smaller values of δ the coincidence of figures is very good. But, for higher values of δ (and fixed p) the agreement is poor.

Next, we consider the variations of normalized intensity distributions of CSG beam on FRFT plane by using the analytical formula for type I and II Lohmann systems, by considering that the two transfer matrixes of these systems are equal in an ideal case. We can seen from Fig. 5, when p is near 1 ($p \geq 0.7$), that the simulation results for both Lohmann systems show good coincidence, especially on the Fourier transform ($p = 1$) plane.

It is worth mentioning that similar result for flattened Gaussian beams was reported [27]. The variations of normalized intensity distributions of CSG beam on FRFT plane by using the analytical formula for type I and II Lohmann systems, for $p = 1$ and $\delta = 0.5$, 0.7 and 1 are shown in Fig. 6.

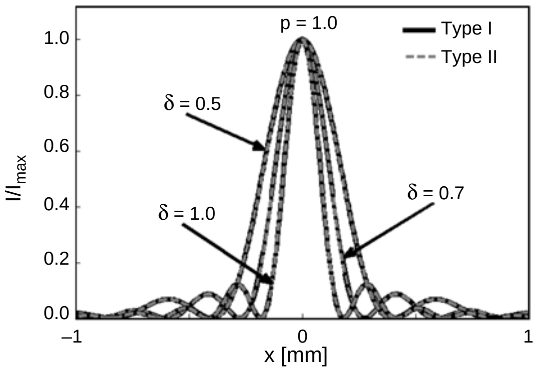

Fig. 6. Normalized intensity distributions of CSG beam on FRFT plane by using the analytical formula for type I and II Lohmann systems.

Now, we consider the variations of normalized intensity distributions of CSG beam for various p and δ, and compare them with ideal case. Because the intensity distributions are symmetric about the vertical-axis, so we have only drawn the normalized intensity distribution on the right of vertical axis in the Fig. 7.

In this figure, we find for $\delta > 3$, that the normalized intensity distributions are similar to the ideal case and this is especially seen for $\delta \geq 8$. We found that the variation period of the normalized intensity distributions versus p, is 2 for all the values of δ. In this system, the variation period of the normalized intensity distributions does not depend on δ, however, in type I Lohmann system the variation period of the normalized intensity distributions does depends on δ [33].

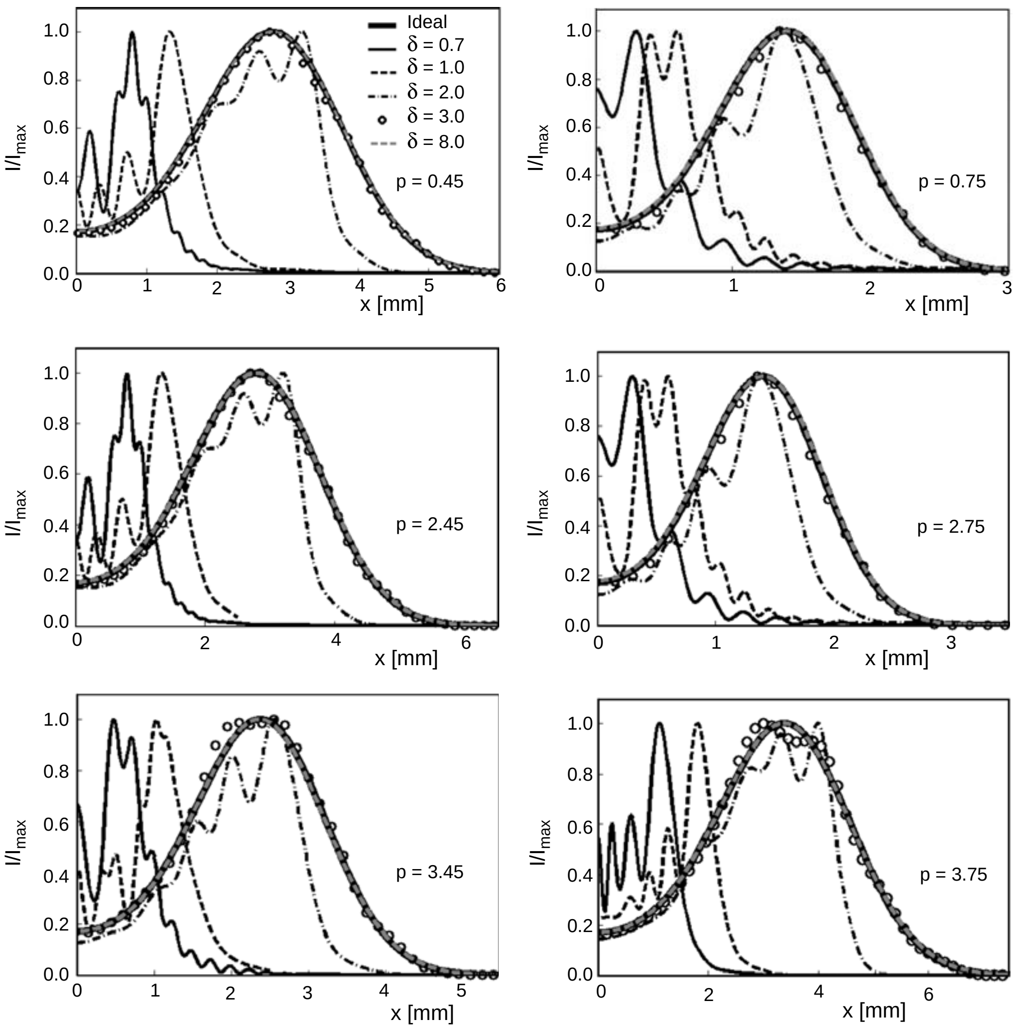

Fig. 7. Variations of normalized intensity distributions with FRFT order after a CSG beam passing through different FRFT systems.

Furthermore, the variations of normalized intensity distributions with Ω after the CSG beams passing through different FRFT systems are presented in Fig. 8. It is shown that the normalized intensity distributions on FRFT plane strongly depend on the initial beam parameters Ω in addition to FRFT order p and δ.

4. Conclusions

Based on the Collins integral formula and the fact that a hard aperture function can be expanded into a finite sum of complex Gaussian functions, the propagation properties

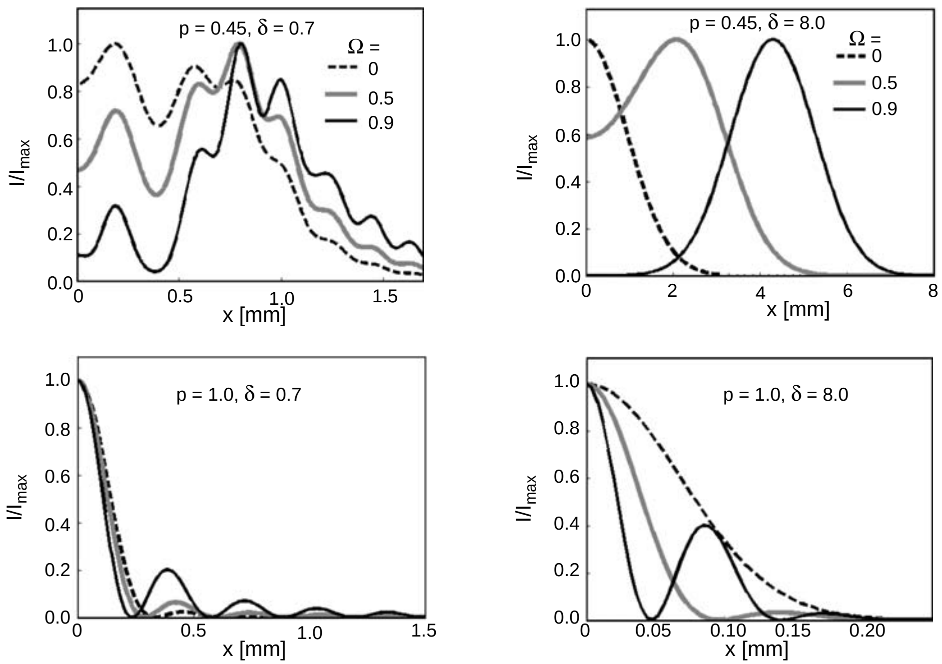

Fig. 8. Variations of normalized intensity distributions as a function of Ω.

of CSG beam passing through ideal and apertured type II Lohmann system on FRFT plane, have been studied and simulated.

By comparing the results obtained by analytical formula and diffraction integral we have reached the following conclusions. First, when p is near to one and any other odd number, the normalized intensity distributions obtained by using the approximate analytical and the numerical integral formulas coincide exactly. Moreover, when p is near to 1 ($p \geq 0.7$), the simulation results for type I and II Lohmann systems highly coincide, especially on the Fourier transform ($p = 1$) plane.

Second, for $\delta < 3$, the intensity distribution is very match dependent on the value of δ but for $\delta \geq 8$, this dependence is removed. Also, from Fig. 4 we see that the smaller δ causes the better match of the two different methods (analytical and the numerical integral formulas). Contrary to what has been reported in Ref. [33], when $\delta < 6$,the aperture has a great impact on the normalized intensity distributions, and the variation period of normalized intensity distributions with FRFT order is 4; on the other hand, when $\delta > 6$, the impact of aperture can be ignored, and the variation period of normalized intensity distributions with FRFT order is 2, as shown in Fig. 7. We have not seen any dependency of intensity distributions of the FRFT plane on the value of δ, which may be due to the existence of two apertures instead of one. Also, Figure 8 shows that the normalized intensity distributions on FRFT plane strongly depend on the initial beam parameter Ω.

References

[1] NAMIAS V., *The fractional order Fourier transform and its application in quantum mechanics*, Journal of Applied Mathematics **25**(3), 1980, pp. 241–265.

[2] MCBRIDE A.C., KERR F.H., *On namias's fractional Fourier transforms*, Journal of Applied Mathematics **39**(2), 1987, pp. 159–175.

[3] MENDLOVIC D., OZAKTAS H.M., *Fractional Fourier transforms and their optical implementation: I*, Journal of the Optical Society of America A **10**(9),1993, pp. 1875–1881.

[4] OZAKTAS H.M., MENDLOVIC D., *Fractional Fourier transforms and their optical implementation: II*, Journal of the Optical Society of America A **10**(12), 1993, pp. 2522–2531.

[5] LOHMANN A.W., *Image rotation, Wigner rotation, and the fractional Fourier transform*, Journal of the Optical Society of America A **10**(10), 1993, pp. 2181–2187.

[6] WANG X., ZHAO D., *Image encryption based on anamorphic fractional Fourier transform and three-step phase-shifting interferometry*, Optics Communications **268**(2), 2006, pp. 240–244.

[7] JIN W., MA L., YAN C., *Real color fractional Fourier transform holograms*, Optics Communications **259**(2), 2006, pp. 513–516.

[8] WANG X., ZHOU J., *Scaled fractional Fourier transform and optical systems*, Optics Communications **147**(4–6), 1998, pp. 341–348.

[9] SAHIN A., OZAKTAS H.M., MENDLOVIC D., *Optical implementation of the two-dimensional fractional Fourier transform with different orders in the two dimensions*, Optics Communications **120**(3–4), 1995, pp. 134–138.

[10] LÜ B., KONG F., ZHANG B., *Optical systems expressed in terms of fractional Fourier transforms*, Optics Communications **137**(1–3), 1997, pp. 13–16.

[11] CAI L.Z., YANG X.L., *Observation of fractional Fourier transforms of continuously variable orders with a scale invariant input*, Optics Communications **201**(4–6), 2002, pp. 319–323.

[12] DU X., ZHAO D., *Elliptical cosh-Gaussian beams*, Optics Communications **265**(2), 2006, pp. 418–424.

[13] CAI Y., LIN Q., *Fractional Fourier transform for elliptical Gaussian beams*, Optics Communications **217**(1–6), 2003, pp. 7–13.

[14] WANG F., CAI Y., *Experimental observation of fractional Fourier transform for a partially coherent optical beam with Gaussian statistics*, Journal of the Optical Society of America A **24**(7), 2007, pp. 1937–1944

[15] ZHENG C., *Fractional Fourier transform for partially coherent off-axis Gaussian Schell-model beam*, Journal of the Optical Society of America A **23**(9), 2006, pp.2161–2165.

[16] MEI Z., ZHAO D., *Propagation of Laguerre–Gaussian and elegant Laguerre–Gaussian beams in apertured fractional Hankel transform systems*, Journal of the Optical Society of America A **21**(12), 2004, pp. 2375–2381.

[17] CAI Y., LIN Q., *Properties of a flattened Gaussian beam in the fractional Fourier transform plane*, Journal of Optics A: Pure and Applied Optics **5**(3), 2003, pp. 272–275.

[18] CAI Y., LIN Q., *Transformation and spectrum properties of partially coherent beams in the fractional Fourier transform plane*, Journal of the Optical Society of America A **20**(8), 2003, pp. 1528–1536.

[19] LIN Q., CAI Y., *Fractional Fourier transform for partially coherent Gaussian–Schell model beams*, Optics Letters **27**(19), 2002, pp. 1672–1674.

[20] CAI Y., GE D., LIN Q., *Fractional Fourier transform for partially coherent and partially polarized Gaussian–Schell model beams*, Journal of Optics A: Pure and Applied Optics **5**(5), 2003, pp. 453–459.

[21] DU X., ZHAO D., *Fractional Fourier transform of off-axial elliptical cosh-Gaussian beams*, Optik **119**(8), 2008, pp. 379–382.

[22] COLLINS S.A., *Lens-system diffraction integral written in terms of matrix optics*, Journal of the Optical Society of America **60**(9), 1970, pp. 1168–1177.

[23] MEI Z., ZHAO D., *Models for an aperture function and corresponding analytical propagation equations of a flattened Gaussian beam through an annular apertured optical system*, Journal of Modern Optics **53**(3), 2006, pp. 313–321.

[24] MAO H., ZHAO D., *Different models for a hard-aperture function and corresponding approximate analytical propagation equations of a Gaussian beam through an apertured optical system,* Journal of the Optical Society of America A **22**(4), 2005, pp. 647–653.

[25] WEN J.J., BREAZEALE M.A., *A diffraction beam field expressed as the superposition of Gaussian beams*, Journal of the Acoustical Society of America **83**(5), 1988, pp. 1752–1756.

[26] WEN J.J., BREAZEALE M.A., *Computer Optimization of the Gaussian Beam Description of an Ultrasonic Field*, Computational Acoustics, Elsevier, Amsterdam, 1990.

[27] ZHAO D., MAO H., SHEN M., LIU H., JING F., ZHU Q., WEI X., *Propagation of flattened Gaussian beams in apertured fractional Fourier transforming systems,* Journal of Optics A: Pure and Applied Optics **6**(1), 2004, pp. 148–154.

[28] ZHAO D., MAO H., LIU H., WANG S., JING F., WEI X., *Propagation of Hermite–cosh-Gaussian beams in apertured fractional Fourier transforming systems*, Optics Communications **236**(4–6), 2004, pp. 225–235.

[29] ZHAO D., MAO H., LIU H., JING F., ZHU Q., WEI X., *Propagation of Hermite–Gaussian beams in apertured fractional Fourier transforming systems,* Optik **114**(11), 2003, pp. 504–508.

[30] JIANG H., ZHAO D., *Propagation characteristics of laser beams with amplitude modulations and phase fluctuations in apertured fractional Fourier transforming system*, Optics Communications **264**(1), 2006, pp. 18–24.

[31] DU X., ZHAO D., *Fractional Fourier transforms of elliptical Hermite–cosh-Gaussian beams*, Physics Letters A **366**(3), 2007, pp. 271–275.

[32] DU X., ZHAO D., *Off-axial elliptical Hermite–cosh-Gaussian beams*, Optics Communications **268**(2), 2006, pp. 282–288.

[33] CHEN S., ZHANG T., FENG X., *Propagation properties of cosh-squared-Gaussian beam through fractional Fourier transform systems*, Optics Communications **282**(6), 2009, pp. 1083–1087.

[34] ZHANG E., JI X., LÜ B., *Changes in the spectrum of diffracted pulsed cosh-Gaussian beams propagating through atmospheric turbulence*, Journal of Optics A: Pure and Applied Optics **9**(10), 2007, pp. 951–957.

[35] DU X., ZHAO D.,*Propagation of elliptical cosh-Gaussian beams in misaligned optical system*, Optics and Laser Technology **40**(1), 2008, pp. 194–200.

[36] YU S., GUO H., FU X., HU W., *Propagation properties of elegant Hermite–cosh-Gaussian laser beams,* Optics Communications **204**(1–6), 2002, pp. 59–66.

Received April 30, 2011
in revised form June 22, 2011