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Characterization of the refractive index in gradient-index elements

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The paper is focused on measurement techniques of gradient-index elements particularly useful for slightly inhomogeneous glasses from small laboratory melts. A construction of the measurement cuvette and dependence of the refractive index of an immersion liquid on its parameters is discussed. The dependence of the refractive index of α-bromonaphtalene on temperature for $\lambda = 0.6328$ μm is described. A measuring method of the refractive index profile of gradient-index elements is presented.

Keywords: GRIN lenses measurements, refractive index measurement, GRIN glass.

1. Introduction

In gradient-index (GRIN) technology the key issue is to measure a refractive index profile across an element in question. There are no references (to the best of the authors' knowledge) dealing with measurement techniques suitable for GRIN elements from small melts of glass, with some amount of inhomogeneity.

Non-destructive measurement techniques applied to cylindrical elements of high degree of symmetry have been more extensively treated in the scientific literature. This paper presents a method of measurement of refractive index for elements for which one cannot use a non-destructive method. The following sections of the paper discuss theory, a construction of the cuvette and measurement results of refractive index of α-bromonaphtalene, a popular immersion liquid, against temperature. The paper is concluded with comments and remarks on measurement of gradient-index elements.

2. The dependence of a refractive index of immersion liquid on parameters of the cuvette

Consider an arrangement of two coherent beams of light of wavelength λ with plane wave fronts which intersects at an angle θ. The beams interfere with each other and straight interference fringes are produced. The fringes make the angle equal to $\theta/2$ with direction of each of the beams. The spacing between fringes is Λ (Fig. 1).

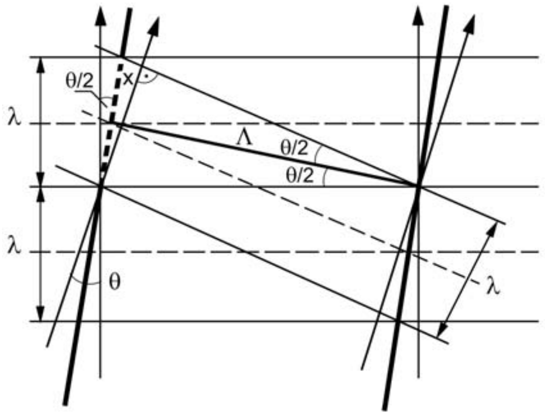


Fig. 1. Two plane-parallel beams of light of wavelength λ which intersect at the angle θ and the system of interference fringes. Thin solid line marks maximum amplitude of light and thin dashed line – minimum. Thick solid and dashed line marks places of interference fringes; Λ – the distance between fringes.

In Figure 1, if x is the length of a fragment of a fringe, marked with the thick dashed line, and λ is a wavelength, then:

$$\frac{\lambda}{x} = \cos\frac{\theta}{2} \tag{1}$$

If Λ is the distance between fringes then:

$$\frac{x/2}{\Lambda} = \tan\frac{\theta}{2} \tag{2}$$

In terms of the above expressions, the formula for the spacing Λ becomes:

$$\Lambda = \frac{\lambda}{2\sin\frac{\theta}{2}} \tag{3}$$

The distance between fringes depends on wavelength λ and the angle between both beams θ. In the above relation, λ is a wavelength in air.

Consider a cuvette which is built of two plane-parallel walls, and a prism of the known angle, between the walls (Fig. 2). If a beam of light falls perpendicularly to the front side of the cuvette, it crosses the immersion liquid of the refractive index n_b without refraction, provided the walls of the cuvette are made plane-parallel with high accuracy. The prism with the refractive index n_c is in optical contact with “rear” side of the cuvette. In Figure 2, for simplicity, the real thickness of the walls is neglected.

The beam of light refracts at the boundary between the immersion liquid and the prism and again when it leaves the prism (and the cuvette). If C is an angle of

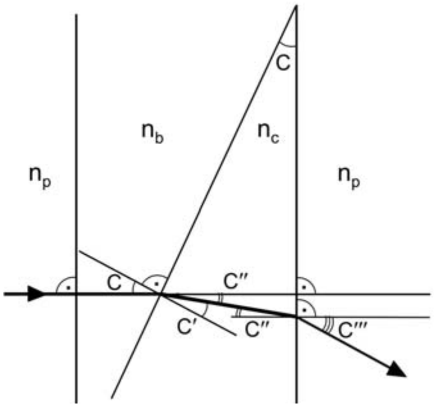


Fig. 2. A plane-wave propagated through the cuvette.

incidence of the beam of light at the surface of the prism, C' is the refraction angle in the prism, as marked in Fig. 2, then:

$$C = C' + C'' \tag{4}$$

The angle C is equal to the angle of the prism.

For both the immersion liquid–prism and the prism–air boundaries we can apply Snell's law:

$$n_b \sin C = n_c \sin C' \tag{5}$$

$$n_c \sin C'' = n_p \sin C''' \tag{6}$$

where angles C, C', C'', C''' are marked in Fig. 2, n_b is the refractive index of the immersion liquid, n_c – refractive index of the prism, and n_p – refractive index of air. From Eq. (6) we have:

$$n_c \sin(C - C') = n_p \sin C''' \tag{7}$$

$$n_c \left[\sin C \cos C' - \cos C \sin C' \right] = n_p \sin C''' \tag{8}$$

Substituting (5) in this equation:

$$n_c \sin C \cos C' - n_b \sin C \cos C = n_p \sin C''' \tag{9}$$

From Equation (5):

$$n_b^2 \sin^2 C = n_c^2 \sin^2 C' \tag{10}$$

$$n_b^2 \sin^2 C = n_c^2 \left[1 - \cos^2 C' \right] \tag{11}$$

So, we have:

$$n_c \cos C' \;=\; \sqrt{n_c^2 - n_b^2 \sin^2 C} \tag{12}$$

where sign "–" before the square root is not used (because $C' > 0$ and $C' < 90°$). Substituting the last equation in (9) we have:

$$\sin C \sqrt{n_c^2 - n_b^2 \sin^2 C} \; - n_b \sin C \cos C \;=\; n_p \sin C''' \tag{13}$$

When the cuvette with the prism is put in one arm of the Mach–Zehnder interferometer (which is arranged so that the interference field is homogeneous) [1, 2], as in Fig. 3, then an auxiliary lens images the "rear" side of the cuvette with background interference fringes (the one which is in optical contact with the prism, as in Fig. 2) on the CCD sensor. These fringes are produced through interference of the wave passing through the prism and the wave propagated in the second arm of the interferometer.

The angle between the beam which falls on the prism in the cuvette and the beam which leaves the prism is C''' (Fig. 2). In this case, interference fringes observed in the mirror Zw (Fig. 3) are inclined to the normal line to the wall of the cuvette at an angle $C'''/2$ (Fig. 4).

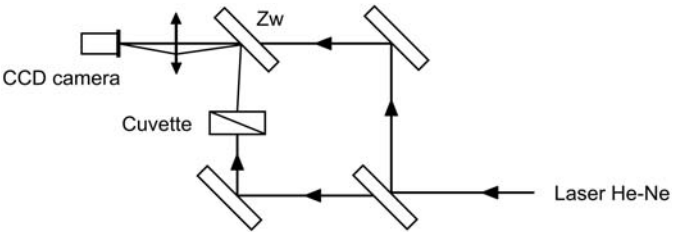


Fig. 3. The cuvette in the Mach–Zehnder interferometer.

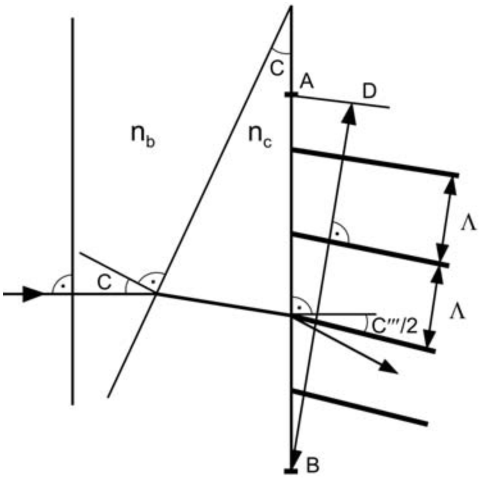


Fig. 4. Virtual interference fringes observed after refraction of the beam at the prism in the cuvette.

The distance between fringes Λ can be expressed as in formula (3):

$$\Lambda = \frac{\lambda}{2\sin\dfrac{C'''}{2}} \tag{14}$$

If k is the number of fringes in segment BD which is perpendicular to the fringes, d is the length of the scale AB, then, as in Fig. 4:

$$\frac{k\Lambda}{d} = \cos\frac{C'''}{2} \tag{15}$$

Joining both of these expressions we have:

$$\frac{k\lambda}{2d\sin\dfrac{C'''}{2}} = \cos\frac{C'''}{2} \tag{16}$$

$$\frac{k\lambda}{d} = 2\sin\frac{C'''}{2}\cos\frac{C'''}{2} \tag{17}$$

$$\frac{k\lambda}{d} = \sin C''' \tag{18}$$

We can substitute expression (18) into formula (13):

$$\sin C\sqrt{n_c^2 - n_b^2\sin^2 C} - n_b\sin C\cos C = n_p\frac{k\lambda}{d} \tag{19}$$

Assuming $n_c > n_b$, which means that the refractive index of the prism is greater than the refractive index of the immersion oil (with directions of beams as in Fig. 2) and the refractive index of air $n_p = 1$, and the indices of glass and immersion oil are measured in air we have:

$$\sqrt{n_c^2 - n_b^2\sin^2 C} - n_b\cos C = \frac{k\lambda}{d\sin C} \tag{20}$$

If the right-hand side of this expression we denote by A, we will get the expression for the index of the immersion oil n_b in the form:

$$A = \frac{k\lambda}{d\sin C} \tag{21}$$

$$\sqrt{n_c^2 - n_b^2\sin^2 C} - n_b\cos C = A \tag{22}$$

Formula (22) can be converted into a classic quadratic equation, the solution of which is as follows:

$$n_b = \pm\sqrt{n_c^2 - \left(\frac{k\lambda}{d}\right)^2} - \frac{k\lambda}{d}\operatorname{ctg}C \tag{23}$$

A negative n_b has not any physical context in this case. For $n_c > n_b$ (the refractive index of the prism is higher than the index of the immersion oil):

$$n_b = \sqrt{n_c^2 - \left(\frac{k\lambda}{d}\right)^2} - \frac{k\lambda}{d}\operatorname{ctg}C \tag{24}$$

And, in a similar way, we get for $n_c < n_b$:

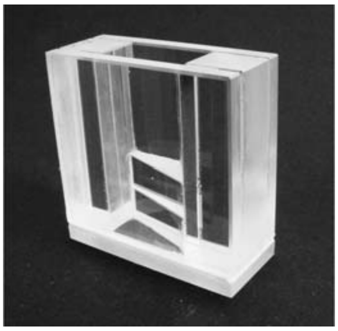
$$n_b = \sqrt{n_c^2 - \left(\frac{k\lambda}{d}\right)^2} + \frac{k\lambda}{d}\operatorname{ctg}C \tag{25}$$

Equations (24) and (25) express dependence of the refractive index of the immersion oil on the parameters of the prism, *i.e.*, the refractive index of the prism n_c, angle of the prism C, length of the scale d, the number of fringes on the scale k, the wavelength λ. It is valid on the strict assumption that the beam of light falls exactly perpendicularly upon the front side of the cuvette.

3. Construction of the cuvette

For measurements of a gradient-index distribution in glass and indices of immersion liquids a glass cuvette has been designed (Fig. 5).

It has been built of two plane-parallel plates of glass (BK7) of dimensions 50×55 mm and the thickness of 5 mm. These plates are polished with high accuracy, with the flatness of the surfaces being better than $\lambda/10$, and they are in optical contact



▲ Fig. 5. The cuvette for interference measurements of gradient-index materials.

Fig. 6. The homogenous dark interference field in the Mach–Zehnder interferometer with the empty cuvette.

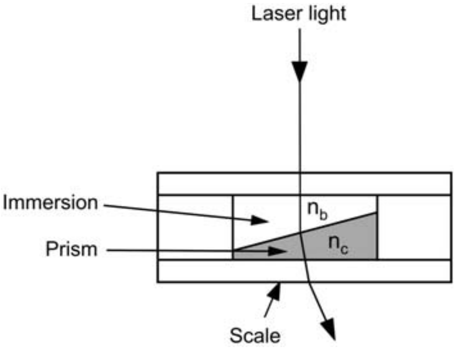


Fig. 7. A view of the cuvette from the top for $n_b < n_c$ (refractive index of immersion is smaller than index of prism).

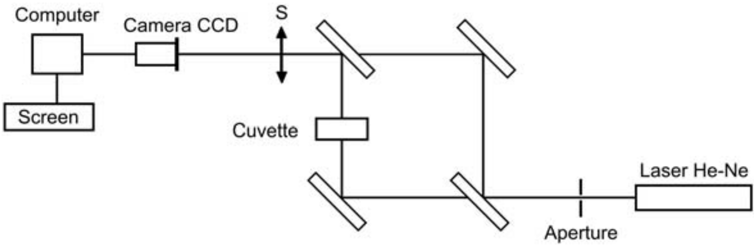


Fig. 8. The Mach–Zehnder interferometer and cuvette for observation of gradient-index elements.

with the side walls of the cuvette, *i.e.*, two cuboids with a 12×15 mm base and the height of 50 mm. The flatness of the side walls is the same as that of the "front" plane-parallel plates and their thickness is such that it is possible to obtain in a homogenous interference field, in the Mach–Zehnder interferometer, a homogenous color of light passing through two such plates – walls of the cuvette (Fig. 6).

The above setup allows us to measure polished slices cut from gradient-index elements with the diameter smaller than 25 mm or elements of this size and the thickness smaller than 12 mm (it is the interior width of the cuvette).

In the lower part of the cuvette there are two prisms mounted by means of optical contact with plane-parallel "front" plate of dimensions 20×10×(3–10) mm and with exactly measured angles (about 18°) – Fig. 7. The prisms are made of glass BK7 ($n_d = 1.51498$) and PSK3 ($n_d = 1.55168$). A beam of laser light which falls perpendicularly on the lower part of the cuvette passes through an immersion liquid and refracts at the prisms (Fig. 7) in the cuvette. Scales for measurements are placed at half height of the prisms.

When the cuvette is inserted in one of the arms of the Mach–Zehnder interferometer perpendicularly to the beam of light, like in Fig. 8, one can see, by means of the lens S, the scale of the cuvette and an image of the interference fringes oriented perpendicularly to the line of the scale (Fig. 9).

The number of interference fringes between extreme lines of the scale depends on the refractive index of the immersion liquid and the wavelength. The scale and

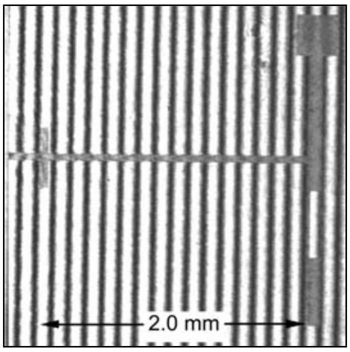


Fig. 9. Interference fringes at the end of the scale of the measurement prism. The immersion liquid is not α-bromonaphtalene here.

the fringes are recorded utilizing a computerized image acquisition system. The number of fringes is counted by a dedicated software. The refractive index of the immersion liquid can be calculated according to formulae (24) or (25).

In the cuvette there are two prisms of different indices of refraction. For counting the fringes we apply the prism that produces a smaller number of fringes. In addition, it is possible to estimate a fraction of the distance between fringes near the edges of the scale.

4. Measurement of the dependence of the refractive index of α-bromonaphtalene against temperature

The cuvette described above has been applied to measure the dependence of the refractive index of α-bromonaphtalene against temperature. The index can differ

Table. The measured number of fringes on the scale of the cuvette and the corresponding refractive index of α-bromonaphtalene against temperature (λ = 0.6328 μm).

Lower prism			Upper prism		
Temperature [°C]	Number of fringes	Refractive index of immersion oil	Temperature [°C]	Number of fringes	Refractive index of immersion oil
16.77	1455.4	1.65277	16.66	1068.7	1.65275
17.04	1453.1	1.65225	16.98	1067.6	1.65264
17.66	1451.0	1.65236	17.58	1064.7	1.65236
17.79	1450.0	1.65226	17.65	1063.5	1.65225
17.95	1449.9	1.65225	17.86	1062.5	1.65215
18.3	1447.8	1.65205	17.86	1063.7	1.65227
18.37	1446.7	1.65194	18.07	1062.5	1.65215
19.21	1443.6	1.65165	18.45	1061.2	1.65203
19.49	1441.4	1.65144	19.07	1058.8	1.65179
26.02	1410.9	1.64853	19.22	1057	1.65162
			19.54	1055.2	1.65145
			26.1	1027.8	1.64882

depending on the date of production. The difference is usually of the order of 3×10^{-4} (according to the Central Office of Measurements in Warsaw, Poland).

The temperature has been measured with the accuracy of 0.1 °C. More accurate values are estimated in the Table (with ±0.02 °C). For counting the number of fringes on a scale there were photographed 20 pictures of the scale of the cuvette. The temperature of the immersion liquid was recorded before and after counting the fringes. It has been shown that, in general, the difference is below 0.1 °C. The mean value has been used in further calculations. The results are presented in the Table.

Using a standard procedure of the Microsoft Excel one can give an experimental dependence of the refractive index of α-bromonaphtalene against the temperature t (in °C): $n_i = -0.000435\,t + 1.65999$.

The estimate error of a factor of variable t is 10^{-5}, but the estimate error of the constant is less than 2×10^{-4}. The calculation has been done for points given in the Table. For instance, for 20 °C calculated value of n_i is equal to 1.6513 and for 25 °C it is equal to 1.6491. The estimate error of the measurement is less than 0.0002.

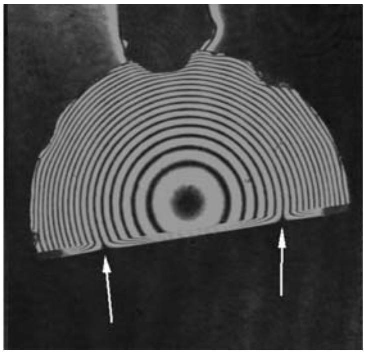
5. Measurement of the distribution of refractive index for gradient-index elements

In order to measure the distribution of the refractive index in a gradient-index element we cut out (from a cylinder or a plate) a flat section plate perpendicularly to the surface of the GRIN element [3–5]. This measurement can be performed for any section of the glass plate or cylinder. Especially, it is necessary for elements made of glass of low homogeneity. For these glasses – cylinders, a non-destructive method cannot be used. In such a method, the cylinder is submerged in an immersion oil and the beam of light falls perpendicularly onto the surface of the cylinder. The beam deflects according to the index profile which is the subject of investigation.

The thickness of the polished glass plate was about 0.3 mm. The plate was cut along the diameter in such a way as to make a wedge of an angle close to 60°. The glass plate prepared in this way was inserted in the cuvette, filled with an immersion liquid and put in one arm of the Mach–Zehnder interferometer (Fig. 5). A special holder was designed enabling rotation and moving the sample inside the cuvette. The plate measured in the homogenous interference field and an interferogram of the plate was recorded by means of the computerized image acquisition system.

In Figure 10, the arrow points to the place where the refractive index of the surrounding immersion oil is the same as that of the GRIN plate. In this case, the fringe in the plate connects with the surroundings.

When in the interference pattern of the section plate there is no such fringe for which the refractive index is equal to the index of the immersion liquid, the value of the refractive index in any place in the plate can be calculated by adding (or subtracting) the value λ/d multiplied by the number of fringes between the background and the place in question to the refractive index of the immersion liquid. It is relatively easy



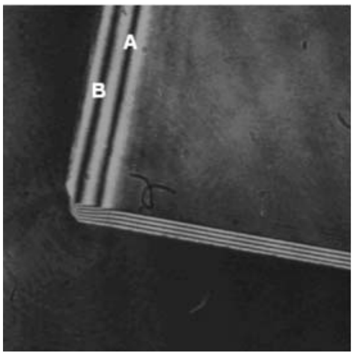


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Fig. 10. A cross-section plate cut from a gradient-index cylinder and immersed in α-bromonaphtalene, in the Mach–Zehnder interferometer in the dark homogenous interferometric field. Arrows show the places and the ring in the plate for which the refractive index is the same as that for the immersion oil.

Fig. 11. Refractive index for the second from the edge of the plate black “vertical” interferometric fringe (fringe A) differs from the refractive index of surroundings (immersion liquid) by $5(\lambda/d)$, but for the first of “vertical” fringes from the edge (fringe B) – by $6(\lambda/d)$. The plate is plane-parallel with an edge of a wedge (the wedge is along lower edge of the plate). The element disperses the light and its range of indices of refraction is higher than the index of immersion liquid.

to check by visual inspection whether the element (the section plate) is positive or negative.

For instance, if we know that the element is negative and after putting in immersion liquid we see in the center of the plate (with wedge of the edge) a smaller number of fringes on the wedge than near the edge of the plate, then the range of indices of the plate is higher than index of the immersion liquid (Fig. 11). If we see, for the same element, in the center of the plate a larger number of fringes on the wedge than near the edge of the plate, then the range of indices of the element is lower than index of the immersion liquid.

It is worth mentioning that slices of cylinders can also be measured when their surfaces and wedges of edges are matt – not polished at all. In immersion liquid one can observe interference fringes through such matt slice on its surface and on the wedge of the edge.

6. Conclusions

The method measuring the parameters of gradient-index elements is specially useful for glasses with some (but not too high) inhomogeneity, obtained from small laboratory melts. Such melts of the weight of the order of 0.5–1 kg have been made for the Institute of Applied Optics (INOS) in the Institute for Electronic Materials (ITME) in Warsaw. Gradients of refractive index have been produced in ion diffusion processes to glass. All the work associated with the production of plates or cylinders of glass, together with carrying on the diffusion processes and measurements of plates cut from glass elements were done in the Institute of Applied Optics in Warsaw.

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