OMEGA BANKRUPTCY FOR DIFFERENT LÉVY MODELS

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Abstract: In this paper, we consider the so-called Omega bankruptcy model, which can be seen as an alternative to the classical approach to ruin. In contrast to the classical model, we allow the process to go below the level zero, however not further than some fixed level $-d < 0$. In addition, when the process is below zero it can be killed with some intensity function ω. Our aim is to show the relations between the Omega model and classical ruin for two important Lévy models, i.e. we consider the Crámer-Lundberg process and the Markov modulated Brownian motion. We also provide numerical experiments to confirm obtained analytical results.

Keywords: ruin probability, the Omega model, the Crámer-Lundberg process, the Markov modulated Brownian motion.

1. Introduction

In classical ruin theory, a company goes out of business when its surplus goes below the level zero. This moment is called classical ruin time. Therefore, the probability of such event is an important indicator of the financial condition in the company. Despite the fact that such a definition of bankruptcy may seem reasonable, it may not be sufficient for all economic circumstances. In particular, we do not know how long the process will stay below zero. It can be the case that this is only a short--term situation and a company will regain liquidity, unless the situation is very bad. We can also think about companies that can perform their business even when they do not have funds. This can occur, for example, for companies owned by the state. Therefore there is a need to define another definition of bankruptcy.

One can consider the idea of the Omega model, introduced in [Albrecher, Gerber, Shiu 2011] and further investigated in e.g. [Gerber, Shu, Yang 2012; Li, Palmowski 2018]. In this model, there is a distinction between technical ruin, i.e. down-crossing level zero and bankruptcy. Namely, a company can do business as usual even after technical

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ruin. However, when surplus is on the negative half-line, a company can go bankrupt with the intensity function $\omega(x)$, where x denotes the value of the surplus process. Therefore, one can define the bankruptcy moment as

$$\tau_\omega := \inf\left\{t > 0: \int_0^t \omega(X_s)ds > e_1\right\},$$

where X is a stochastic process which models surplus level and e_1 is an independent exponential random variable with the parameter 1. It turns out that such an approach is very general and covers the following particular examples:

- If we set $\omega(x) = 0$ for $x \geq 0$ and $\omega(x) = \infty$ for $x < 0$, then we have the case of classical ruin time. This is the only case when technical ruin is the same as bankruptcy.
- A non trivial example can be the following

$$\omega(x) = \begin{cases} 0 \text{ if } x > 0, \\ \gamma_0 + \gamma_1(x + d) \text{ if } x \in [-d, 0], \\ +\infty \text{ if } x < -d \end{cases}$$

where $-d < 0$,$\gamma_0 > 0$, and $\gamma_1 \leq 0$ then $\omega(x) \geq 0$ on $x \in [-d, 0]$ and ω is an a non-increasing function on this interval. In such a model we have a linear relationship between intensity and the position of the surplus process. The assumption that ω is non-increasing on $[-d, 0]$ suits the intuition that the penalty for being close to zero should not be greater than the penalty for being far from it. In the literature, such an interval is often called *the red zone*. Note that if the surplus down-crosses level $-d$ it is immediately killed. Therefore, we remove the company from a situation when the surplus is too low at the moment of bankruptcy. This model was investigated in e.g. [Gerber, Shu, Yang 2012; Li, Palmowski 2018].

A special case is when $\gamma_1 = 0$ and $\gamma_0 > 0$. In this situation we consider occupation time in the so-called *red zone*. Here we calculate the time when the surplus process stays below zero. Thus, bankruptcy means that the surplus process stays too long in the interval $[-d, 0]$ or down--crossed level $-d$. For more details see Section 6.2 in [Loeffen, Renaud, Zhou 2014].

One can also consider the so-called Parisian ruin time with the random clock. In this model, every time the surplus is below zero (or another barrier level) we run the so-called implementation clock, which is an indendent exponential random variable with the parameter $q > 0$. If the surplus stays below zero longer than the implementation clock the company is out of business. It turns out (see Section 5 in [Renaud 2014] with $\alpha = 0$) that the probability of such a bankruptcy has the same value as in the Omega model for

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$$\omega(x) = \begin{cases} 0 \text{ if } x \geq 0 \\ q \text{ if } x < 0 \end{cases}.$$

Let us note that in the Parisian ruin example we do not consider bankruptcy when the process goes below some fixed level. However, in this paper we will always assume that such level, let us call it $-d$, exists. Therefore, to avoid technical issues related with infinity one can consider an equivalent, in this case the definition of the Omega bankruptcy time is:

$$\tau_\omega^d := \inf\left\{t > 0: \int_0^t \omega(X_s)ds > e_1 \vee X_t < -d\right\},$$

then one can re-define the Omega function and set any values of $\omega(x)$ for $x < -d$.

An important assumption in our considerations is the choice of the underlying stochastic process for a company's surplus level. In the risk theory and the actuarial science, we often consider the general Cramér--Lundberg process, which is defined as follows

$$X_t = x + pt - \sum_{i=1}^{N_t} U_i + \sigma B_t,$$

where $x \in \mathbb{R}$ denotes the value of the initial capital, $p > 0$ is a constant intensity of the premium income, $\{N_t\}_{t\in[0,\infty)}$ is a homogeneous Poisson process with the intensity $\lambda > 0$, U_i are positive $i.i.d.$ random variables with the common distribution function F, $\sigma > 0$ and B_t is a standard Brownian motion which models the aggregation of small claims. The above process posseses the following probabilistic properties:

- $X_0 = const$ a.s.
- $X_t - X_s \overset{d}{=} X_{t-s}$, where $t \geq s \geq 0$ and $\overset{d}{=}$ means equality in the distribution.
- $X_t - X_s$ for $t \geq s$ is independent from $\mathcal{F}_s$, filtration generated by σ-field $\sigma(X_i: i \leq s)$.
- This process has right-continuous paths with left limits (càdlàg paths) and does not have positive jumps.

The processes that satisfy the above conditions are called spectrally negative Lévy processes. This class of processes contain for example the linear Brownian motion, the Crámer-Lundberg process with phase-type jumps and α-stable processes. In addition, we exclude the case of processes with monotone paths. More information about such processes can be found for example in [Bertoin 1996] and [Kyprianou 2014].

Moreover, we would like to state another generalization. Let us imagine that one would like to apply stochastic processes to the phenomenon which has a different structure with respect to change of season or the economic situation in the market. In such situations, Lévy processes can be an inappropriate choice because of stationarity and independents of the increments. Therefore later we will formally introduce spectrally negative Markov additive processes (see e.g. [Asmussen 2003; Breuer 2012; D'Auria et al. 2010; Dieker, Mandjes 2011; Ivanovs, Mandjes 2010]). An element of this class is a bivariate process (X, J) where X can be responsible for a surplus level and J can be seen as random environment which can have different states. When J is on state i then X is behaving as X^i which is a spectrally negative Lévy process. Therefore, distribution of increments of X depends on the current state of J. In addition, if J has only one state, then X is just a spectrally negative Lévy process.

The rest of the paper is organized as follows. First, we introduce some basic notation and definitions related to Lévy processes and Markov additive processes. In Section 2, we provide definitions associated to the Omega model. The last two sections are the main parts of this article. In Section 3, we consider the Crámer-Lundberg process with the exponential claims as a model example. This process is one of the most important processes in ruin theory, often used as a starting point of analysis and has numerous practical applications. For this model we provide the formula for Omega bankruptcy probability. In addition, we show that in this case, this probability is a linear function of the classical ruin probability. In section 4 we provide numerical results for the approximation of the probability in the Omega model for the Markov modulated Brownian motion (MMBM). This process can be seen as a Brownian motion in the random environment and is one of the most important examples of the Markov additive processes. Brownian motion appears in almost every application of stochastic processes and, as was mentioned, before can be responsible for the small claims in the non-life insurance model. Moreover, adding random environment makes this process even more flexible.

1.1. Lévy processes and scale functions

Let X be the spectrally negative Lévy process defined on filtrated probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t : t \geq 0\}, \mathbb{P})$ which satisfies the usual conditions. Every spectrally negative Lévy process can be represented by the triple (a, σ, Π) where $a \in \mathbb{R}$, $\sigma \geq 0$ and Π is the measure of $(-\infty, 0)$ which satisfy

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$$\int_{(-\infty,0)} (1 \wedge x^2)\Pi(dx) < \infty.$$

Then the exponent of the Laplace transform of process X is defined through

$$\psi(\theta) = \log\left(\mathbb{E}\left(e^{\theta X_1}\right)\right) =$$
$$a\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{(-\infty,0)} \left(e^{\theta x} - 1 - \theta x 1_{\{-1<x<0\}}\right)\Pi(dx)$$

and is well-defined for $\theta \geq 0$. In addition, we would like to consider different starting points of the process, thus we need to use the classical Markovian notation, namely we will write $\mathbb{P}_x(A)$ for the probability of event A given that $X_0 = x$ almost certainly. Similarly this will be valid for expectations. Further, we will write $\mathbb{P}(\cdot)$ and $\mathbb{E}(\cdot)$ when $x = 0$.

For $a \in \mathbb{R}$ let us define the following stopping times:

$$\tau_a^+ = \inf\{t > 0: X_t \geq a\}, \tau_a^- = \inf\{t > 0: X_t < a\}.$$

We will be interested in the representation of the following expression

$$\mathbb{E}_x\left(e^{-q\tau_b^+}, \tau_b^+ < \tau_a^-\right), \tag{1}$$

for $a \leq x \leq b$. This expression can be seen in two ways. The first one is connected with the important idea of the killing of the process. Let us define e_q as an exponential independent random variable with parameter $q \geq 0$ (with the convention that for $q = 0$, e_q is $+\infty$ a.s.). Then, if $t > e_q$ we are killing X_t, namely we put X into an absorbing state. Therefore, the above expectation is just the probability that we cross level b before we go below a and before we get killed by e_q. From another point of view, we can think that we are paying one unit when the process reaches level b before we down-cross level a and discount this unit with factor q. Thus above expectation is the expected present value of such payment. These two ways of thinking will be very useful in the spirit of this article and can be convenient for numerical calculations.

Another important quantity of the interest is the probability of classical ruin

$$\mathbb{P}_x(\tau_0^- < \infty), \text{for } x > 0.$$

This probability is very interesting from our point of view. One can see that the above expressions can be treated as functions of some parameters, therefore it will be convenient to have some analytical representation of them. To do this we need to introduce the key tools of this paper.

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For every $q \geq 0$ there exists function $W^{(q)}: \mathbb{R} \rightarrow [0, \infty)$, called (q)-scale function, which satisfies $W^{(q)}(x) = 0$ for $x < 0$ and for $[0, \infty)$ is defined as a strictly increasing and continuous function through the following Laplace trasform

$$\int_0^\infty e^{-\theta x} W^{(q)}(x) dx = \frac{1}{\psi(\theta) - q} \text{ for } \theta > \Phi(q),$$

where $\Phi(q) = \sup\{\theta \geq 0: \psi(\theta) = q\}$. We will write $W(x)$ for $q = 0$. The second scale function is defined by

$$Z^{(q)}(x) := 1 + q \int_0^x W^{(q)}(y) dy, x \in \mathbb{R}.$$

One can show that

$$\mathbb{E}_x\left[e^{-q\tau_c^+}, \tau_c^+ < \tau_0^-\right] = \frac{W^{(q)}(x)}{W^{(q)}(c)} \tag{2}$$

and

$$\mathbb{E}_x[e^{-q\tau_0^-}, \tau_0^- < \infty] = Z^{(q)}(x) - \frac{q}{\Phi(q)} W^{(q)}(x),$$

putting in above $q = 0$ (and making a limit argument for $\frac{q}{\Phi(q)}$) one can get that

$$\mathbb{P}_x(\tau_0^- < \infty) = \begin{cases} 1 - \psi'(0+) W(x) \text{ if } \psi'(0+) > 0 \\ 1 \text{ if } \psi'(0+) \leq 0 \end{cases}. \tag{3}$$

Thus, for example, one can use scale functions for a deeper analysis of classical ruin time. We refer the reader to [Bertoin1996; Hubalek, Kyprianou 2011; Kuznetsov, Kyprianou, Rivero 2012; Kyprianou 2014], for more detail about scale functions.

1.2. Markov additive processes

In this section we would like to introduce some elements of the theory of Markov additive processes. Later on, we will be interested in one particular example of this class of processes, namely, the Markov modulated Brownian motion.

Let us consider bivariate stochastic process (X, J), where X is a real--valued càdlàg process and J is a right-continuous stochastic process taking values in the finite set $E = \{1, 2, \ldots, N\}$. We say that such a process is a Markov additive process if for fixed $\{J_t = i\}$ vector $(X_{t+s} - X_t, J_{t+s})$ is independent from $\mathcal{F}_t$ and is equal in distribution to $(X_s - X_0, J_s)$ for fixed

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$\{J_0 = i\}$. Despite the formal definition there exists a very intuitive representation of every MAP, which we will present below.

At the beginning, we can see that J_t is a continuous time Markov chain. When J is on state i then process X is behaving like some Lévy process X^i. When J is changing state from i to j $(i \neq j)$ then process X can make the jump distributed like some random variable U_{ij} and then behaves like Lévy process X^j. All the above random elements are independent. This representation can be summarised in Figure 1.

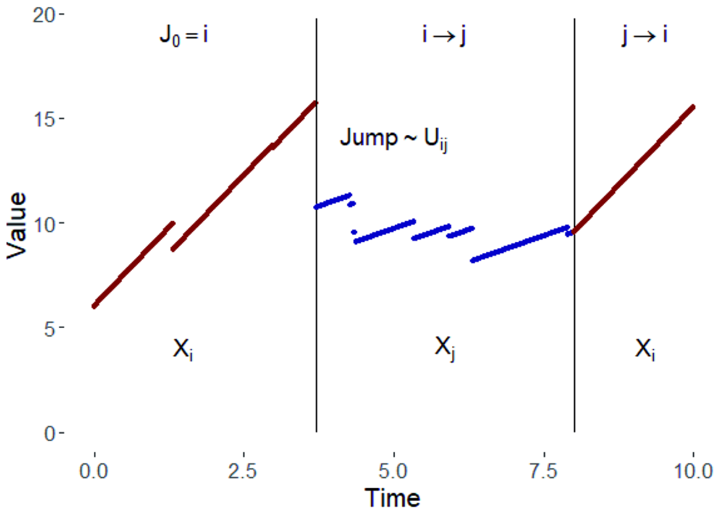

Fig. 1. An example of an approximated sample path of the MAP

Source: own elaboration.

One can see that we can divide time into intervals of occupation times of J and then treat process X as a Lévy process on each interval. These explain another well-known name for MAP, namely the „Markov--modulated Lévy process". Thus a MAP can be seen as a Lévy process in a random environment. In addition, note that if $N = 1$ (thus J has only one state) then X is a Lévy process. Because of that, we would like to consider the generalization of spectrally negative Lévy processes namely, we assume that for every $i \in E$ we have that X^i is a spectrally negative Lévy process and $U_{i,j} \leq 0$ a.s. for every $i, j \in E$. Therefore, X can have only negative jumps. We exclude the trivial case of monotonic paths of X. In addition we assume that J is an irreducible Markov chain with $\boldsymbol{Q}$ being its intensity matrix and $\boldsymbol{\pi}$ a uniquely determined stationary vector.

In the case of spectrally negative Lévy processes, we saw that the so-called scale functions turn out to be a very important tool in the

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investigation of the exit problems. In the MAP framework we can also define such functions, but now they need to be matrix-valued.

Let us define the matrix exponent of the Laplace transform for the MAP as matrix $\boldsymbol{F}(\alpha)$ which will satisfy the following

$$\mathbb{E}_x(e^{\alpha X_t}, J_t = j | J_0 = i) = (e^{\boldsymbol{F}(\boldsymbol{\alpha})t})_{i,j}, \text{for } \alpha \geq 0,$$

which has an explicit representation

$$\boldsymbol{F}(\alpha) = \text{diag}(\psi_1(\alpha), \dots, \psi_N(\alpha)) + \boldsymbol{Q} \circ \mathbb{E}(e^{\alpha U_{ij}}),$$

where $(A \circ B)_{ij} = (a_{ij} b_{ij})$ is an entry-wise (Hadamard) matrix multiplication.

From Kryprianou and Palmowski [Kryprianou, Palmowski 2008] we know that for $q \geq 0$ there exist the invertible matrix-valued function $\boldsymbol{W}^{(q)}: [0, \infty) \to \mathbb{R}^{N \times N}$ such that for $0 \leq x \leq a$,

$$\boldsymbol{E}_x\left[e^{-q\tau_a^+}, \tau_a^+ < \tau_0^-, J_{\tau_a^+} | J_0\right] = \boldsymbol{W}^{(q)}(x) \boldsymbol{W}^{(q)}(a)^{-1}. \quad (4)$$

Note that that above expected value is a $N \times N$ matrix, such that entry (i, j) means that $J_0 = i$ and $J_{\tau_a^+} = j$. At this point it can be confusing that we consider the "ending state" of the process J. However, if for $i \in E$ one is just interested in the following

$$\mathbb{E}_x[e^{-q\tau_a^+}, \tau_a^+ < \tau_0^- | J_0 = i],$$

then it is sufficient to sum up i-th row of the expression $\boldsymbol{W}^{(q)}(x) \boldsymbol{W}^{(q)}(a)^{-1}$. Ivanovs [Ivanovs 2011] and Ivanvos and Palmowski [Ivanovs, Palmowski 2012] showed that $\boldsymbol{W}^{(q)}$ can be defined through

$$\int_0^\infty e^{-\alpha x} \boldsymbol{W}^{(q)}(x) dx = (\boldsymbol{F}(\alpha) - qI)^{-1}, \text{for large enough } \alpha.$$

They showed even more, namely that first we need to consider process (X, J) killed with rate $q \geq 0$. This means, like before, that we would like to put our MAP into absorbing time after some random exponential time. Then one can express (q)-scale matrix $\boldsymbol{W}^{(q)}$ in the following way

$$\boldsymbol{W}^{(q)}(x) = e^{-\boldsymbol{\Lambda}_q^+ x} \boldsymbol{L}^q(x), \quad (5)$$

where $\boldsymbol{\Lambda}_q^+$ is the transition rate matrix of Markov chain $\{J_{\tau_x^+}\}_{x \geq 0}$, i.e. for $i, j \in E$

$$\mathbb{P}(\tau_x^+ < e_q, J_{\tau_x^+} = j | J_0 = i) = (e^{\boldsymbol{\Lambda}_q^+ x})_{ij}, \quad (6)$$

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with e_q being an independent exponential random variable of rate $q \geq 0$. We use, again, the convention that $e_0 = \infty$. Moreover, $\boldsymbol{L}^q(x)$ is a matrix of expected occupation times at 0 up to the first passage time over x. In addition, $\boldsymbol{L}^q := \boldsymbol{L}^q(\infty)$ is the expected occupation density at 0.

The second scale matrix $\boldsymbol{Z}^{(q)}$ is defined through the $\boldsymbol{W}^{(q)}$ matrix function:

$$\boldsymbol{Z}^{(q)}(x) := I - \int_0^x \boldsymbol{W}^{(q)}(y)dy(\boldsymbol{F}(0) - qI).$$

Now we are ready to formally introduce the titled Omega model.

2. Omega killing and the Omega model

In this section we assume that the background process is a spectrally negative Markov additive process. In the case of citing results connected with spectrally negative Lévy processes, one can think about taking $N = 1$ as a cardinality of the state space of J. We start with a technical remark.

Remark 2.1. *If the function is matrix-valued it will always be bold in this paper. It will be also the case for the constants.*

At the beginning let us recall some definitions and results from [Czarna et al. 2018] and [Li, Palmowski 2018].

Definition 2.1. *Let* $\omega: E \times \mathbb{R} \to \mathbb{R}^+$ *be a function defined as* $\omega(i, x) = \omega_i(x)$, *where for a fixed* $i \in E$, $\omega_i: \mathbb{R} \to \mathbb{R}^+$ *is a bounded, non-negative measurable function and its value formulates the matrix* $\boldsymbol{\omega}(x) := diag(\omega_1(x), \dots, \omega_N(x))$. *Let* $\lambda > 0$ *be the upper bound of* $|\omega_i(x)|$ *on* $[0, \infty)$ *for all* $i \in E$.

Fix ω function which satisfies the above definition and consider the definition of (ω)-scale matrices for $x \geq y$

$$\mathcal{W}^{(\omega)}(x, y) := \mathbf{W}(x - y) + \int_y^x \mathbf{W}(x - z)\boldsymbol{\omega}(z)\mathcal{W}^{(\omega)}(z, y)dz, \quad (7)$$

$$\mathcal{Z}^{(\omega)}(x, y) := I + \int_y^x \boldsymbol{W}(x - z)\boldsymbol{\omega}(z)\mathcal{Z}^{(\omega)}(z, y)dz. \quad (8)$$

Then we have that for $a \leq x \leq b$

$$\boldsymbol{E}_x\left[e^{-\int_0^{\tau_b^+} \omega_{J_s}(X_s)ds}, \tau_b^+ < \tau_a^-, J_{\tau_b^+} | J_0\right] = \mathcal{W}^{(\omega)}(x, a)\mathcal{W}^{(\omega)}(b, a)^{-1}$$

and

$$\boldsymbol{E}_x\left[e^{-\int_0^{\tau_a^-} \omega_{J_s}(X_s)ds}, \tau_a^- < \tau_b^+, J_{\tau_a^-} | J_0\right] =$$
$$\mathcal{Z}^{(\omega)}(x, a) - \mathcal{W}^{(\omega)}(x, a)\mathcal{W}^{(\omega)}(b, a)^{-1}\mathcal{Z}^{(\omega)}(b, a).$$

Let us mention some but not all of the possibilities to apply the above results. Firstly, one can see that we can use these expressions for the Omega model. As before, one can think about paying one unit when the process crosses level b before it down-crosses level a and before it is *killed* by Omega bankruptcy time. Then the first expected value is the expected present value of such payment. On the other hand, we can also apply the above results for a complex discounting structure, independent of the Omega model framework. These expressions can be used when one would like to have different factors for states of J, for example, related to the change of season or change of economic situation on the market. Therefore, yet again one can think about paying one unit when the process crosses level b before it down-crosses level a but know that every payment is discounted with intensity ω. Therefore, one can see how these expressions generalize (1) and (4).

Finally, we would like to formally introduce Omega bankruptcy time as

$$\tau_\omega^d := \inf\left\{t \geq 0: \int_0^t \omega_{J_s}(X_s)ds > e_1 \vee X_t < -d\right\}, \tag{9}$$

where $d > 0$, ω is a function which satisfies Definition 2.1 and e_1 is an independent exponential random variable with parameter 1. We will be interested in the following probability for $i \in E$ and $x \geq 0$

$$\left(\varphi^{(\omega)}(x)\right)_i = \mathbb{P}_x\left(\tau_\omega^d < \infty | J_0 = i\right).$$

Our aim is to obtain a close expression for this probability or to obtain the numerical approximation of it. However, we will begin with obvious inequalities, which will be the starting point for our analysis. Namely, for $i \in E$ and $x \geq 0$ we have that

$$\mathbb{P}_x(\tau_{-d}^- < \infty | J_0 = i) \leq \mathbb{P}_x\left(\tau_\omega^d < \infty | J_0 = i\right) \leq \mathbb{P}_x(\tau_0^- < \infty | J_0 = i). \tag{10}$$

We will need also some knowledge about one-sided problems, thus again we cite [Czarna et al. 2018] to get that for $x \geq 0$

$$\boldsymbol{E}_x\left[e^{-\int_0^{\tau_0^-} \omega_{J_s}(X_s)ds}, \tau_0^- < \infty, J_{\tau_0^-} \middle| J_0\right] = \boldsymbol{\mathcal{Z}}^{(\omega)}(x) - \boldsymbol{\mathcal{W}}^{(\omega)}(x)\boldsymbol{C}_{\mathcal{W}(\infty)^{-1}\mathcal{Z}(\infty)}, \tag{11}$$

where matrix $\boldsymbol{C}_{\mathcal{W}(\infty)^{-1}\mathcal{Z}(\infty)} := \lim_{c\to\infty} \boldsymbol{\mathcal{W}}^{(\omega)}(c)^{-1}\boldsymbol{\mathcal{Z}}^{(\omega)}(c)$ exists and has finite entries.

In the case when $N = 1$ we have from [Li, Palmowski 2018] that for fixed level $-d < 0$ and $x \geq -d$

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$$\mathbb{E}_x[e^{\int_0^\infty \omega(s)ds}; \tau_{-d}^- = \infty] = c_{\mathcal{W}^{-1}(\infty,-d)}\mathcal{W}^{(\omega)}(x,-d), \qquad (12)$$

where $c_{\mathcal{W}^{-1}(\infty,-d)} = [\lim_{c\to\infty}\mathcal{W}^{(\omega)}(c,-d)]^{-1}$.

3. Probability of Omega bankruptcy for the Crámer-Lundberg process

The first process which will be treated as the model example is the Crámer--Lundberg process. Remember that we define this process as follows

$$X_t = x + p_t - \sum_{i=1}^{N_t} U_i,$$

with the same meaning of the parameters as in the introductory section of this article. Before we proceed to the results, we would like to mention that the calculations below are done in the same manner as in [Li, Palmowski 2018] where the linear Brownian motion was the underlying process. They achieved the formula for the probability of bankruptcy in the Omega model and showed that this probability is in fact a function of classical ruin probability. As is shown below we get a similar result. Let us proceed to the calculations.

We know that the Laplace exponent for the Cramér-Lundberg process is equal to

$$\psi(\alpha) = p\alpha - \frac{\lambda\alpha}{\mu+\alpha}.$$

Thus one can obtain the formula for $W^{(q)}$, namely

$$W^{(q)}(x) = \frac{1}{p}(A^+(q)e^{q^+(q)x} - A^-(q)e^{q^-(q)x}),$$

where

$$A^\pm(q) = \frac{\mu + q^\pm(q)}{q^+(q) - q^-(q)}, q^\pm(q) = \frac{q+\lambda-\mu p \pm \sqrt{(q+\lambda-\mu p)^2 + 4pq\mu}}{2p}.$$

From (3) we know that

$$\varphi(x) = \mathbb{P}_x(\tau_0^- < \infty) = 1 - \psi'(0+)W(x),$$

if $0 < \psi'(0+) = p - \frac{\lambda}{\mu}$. Note that this assumption is a well-known net profit condition. In the language of stochastic processes, this is equivalent to the fact that the drift of the process is strictly positive. This assumption will be also considered here. From the above one can see that we need to calculate the formula for the (0)-scale function. Therefore, note that

$$q^{+}(0)=\frac{\lambda-\mu p}{p}, q^{-}(0)=0,$$

and

$$W(x)=\frac{1}{p}\left(A^{+}(0)e^{q^{+}(0)x}-A^{-}(0)e^{q^{-}(0)x}\right)=\frac{1}{\lambda-\mu p}\left(\frac{\lambda}{p}e^{\left(\frac{\lambda-\mu p}{p}\right)x}-\mu\right).$$

Thus, we can go back to the representation of the probability of classical ruin time

$$\varphi(x)=1-\left(p-\frac{\lambda}{\mu}\right)W(x)=\frac{\lambda}{\mu p}e^{\left(\frac{\lambda-\mu p}{p}\right)x}. \tag{13}$$

This result is well-known in the literature, however one can see that the scale functions are very convenient tools for this problem. This will also be the case for the probability of Omega bankruptcy.

Now we will proceed to some general calculations with some restrictions. Let us assume that function ω satisfies the following conditions

- $\omega(x)\geq 0$ for $x\in[-d,0]$ and zero otherwise,
- $\omega(x)$ is differentiable continuously function on $[-d,0]$, where at the ending points we use the left and right derivative/limit respectively.

Remember from (7) that when we put $N=1$ then the ω-scale function satisfies the following

$$\mathcal{W}^{(\omega)}(x,-d)= W(x+d)+\int_0^{x+d} W(x+d-y)\omega(y-d)\mathcal{W}^{(\omega)}(y-d,-d)dy, \text{for } x\geq -d. \tag{14}$$

The next proposition will give us the possibility for numerical calculations of the above scale function.

Proposition 3.1. *Function $\mathcal{W}^{(\omega)}$ satisfies the following differential equation for $x\in[-d,0]$*

$$p\mathcal{W}^{(\omega)''}(x,-d)-[\omega(x)+(\lambda-\mu p)]\mathcal{W}^{(\omega)'}(x,-d)-[\mu\omega(x)+\omega'(x)]\mathcal{W}^{(\omega)}(x,-d)=0,$$

with $\mathcal{W}^{(\omega)}(-d,-d)=\frac{1}{p}, \mathcal{W}^{(\omega)\prime}(-d,-d)=\frac{\lambda+\omega(-d)}{p^2}$.

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Proof. Let us take $z = x + d \geq 0$ and denote $g(z) := \mathcal{W}^{(\omega)}(z - d, -d) = \mathcal{W}^{(\omega)}(x, -d)$. Then from (14) we have that

$$g(z) = W(z) + \int_0^z W(z - y)\omega(y - d)g(y)dy. \qquad (15)$$

Let us observe, since $q^-(0) = 0$, that

$$\left(\frac{d}{dz} - q^+(0)\right)\frac{d}{dz}W(x) = 0$$

and

$$\left(\frac{d}{dz} - q^+(0)\right)\frac{d}{dz}g(z) = \frac{1}{p}[\mu\omega(z - d)g(z) + \omega'(z - d)g(z) + \omega(z - d)g'(z)].$$

Therefore

$$pg''(z) - [\omega(z - d) + (\lambda - \mu p)]g'(z) - [\mu\omega(z - d) + \omega'(z - d)]g(z) = 0,$$

with the initial values $g(0) = \frac{1}{p}$ and $g'(0) = \frac{\lambda+\omega(-d)}{p^2}$. To end this proof one needs to go back to x-domain.

The second proposition will be related to the probability of Omega bankruptcy time. Remember that

$$\tau_\omega^d = \inf\left\{t \geq 0: \int_0^t \omega(X_s)ds > e_1 \vee X_t < -d\right\}.$$

Therefore the probability of Omega bankruptcy is equal to

$$\varphi^{(\omega)}(x) = \mathbb{P}_x(\tau_\omega^d < \infty) = 1 - \mathbb{E}_x\left[e^{-\int_0^\infty \omega(s)ds}; \tau_{-d}^- = \infty\right]$$

and from (12) we know that for $x \geq -d$

$$\mathbb{E}_x\left[e^{-\int_0^\infty \omega(s)ds}; \tau_{-d}^- = \infty\right] = c_{\mathcal{W}^{-1}(\infty,-d)}\mathcal{W}^{(\omega)}(x, -d),$$

with $c_{\mathcal{W}^{-1}(\infty,-d)} = [\lim_{c\to\infty}\mathcal{W}^{(\omega)}(c, -d)]^{-1}$.

Proposition 3.2. *Function $\varphi^{(\omega)}(x)$ is given by*

$$\varphi^{(\omega)}(x) = \varphi(x)\frac{\mu p^2}{\lambda(\mu p - \lambda)}c_{\mathcal{W}^{-1}(\infty,-d)}\mathcal{W}^{(\omega)\prime}(0, -d), \text{for } x \geq 0,$$

where as before $\varphi(x) = \mathbb{P}_x(\tau_0^- < \infty) = \frac{\lambda}{\mu p}e^{\frac{\lambda-\mu p}{p}x}$ is a classical ruin probability.

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Proof. First, we again make the substitution $z = x + d$ and from (15) and the fact that $\omega(z) = 0$ for $z > d$ one can get the following

$$g'(z) = \frac{\lambda}{p^2} e^{(\frac{\lambda-\mu p}{p})z} \left[1 + \int_0^d e^{-(\frac{\lambda-\mu p}{p})y} \omega(y-d) g(y) dy\right]$$

and

$$g(z) = g(d) + \frac{p}{\mu p - \lambda}[1 - e^{\frac{\lambda-\mu p}{p}(z-d)}]g'(d).$$

Thus when we get back to the x-domain using $g(z) = \mathcal{W}^{(\omega)}(x, -d)$, then the above equation gives

$$\mathcal{W}^{(\omega)}(x, -d) = \mathcal{W}^{(\omega)}(0, -d) + \frac{p}{\mu p - \lambda}[1 - e^{\frac{\lambda-\mu p}{p}x}]\mathcal{W}^{(\omega)\prime}(0, -d).$$

From the above equation it is clear that

$$c_{\mathcal{W}^{-1}(\infty,-d)} = \frac{1}{\mathcal{W}^{(\omega)}(0, -d) + \frac{p}{\mu p - \lambda}\mathcal{W}^{(\omega)\prime}(0, -d)}.$$

Therefore

$$\varphi^{(\omega)}(x) =$$

$$1 - c_{\mathcal{W}^{-1}(\infty,-d)}\mathcal{W}^{(\omega)}(x, -d) =$$

$$\frac{p}{\mu p - \lambda}\left[e^{\frac{\lambda-\mu p}{p}x}\right]\mathcal{W}^{(\omega)\prime}(0, -d)c_{\mathcal{W}^{-1}(\infty,-d)} = \qquad (16)$$

$$\varphi(x)\frac{\mu p^2}{\lambda(\mu p-\lambda)}\mathcal{W}^{(\omega)\prime}(0, -d)c_{\mathcal{W}^{-1}(\infty,-d)}.$$

Note that we still need to calculate $\mathcal{W}^{(\omega)}(0, -d)$ and $\mathcal{W}^{(\omega)\prime}(0, -d)$. We will use numerical methods to approximate them.

The main question here is how $\varphi^{(\omega)}(x)$ is related with the probability of the classical ruin time. In particular if $\omega(x) \equiv 0$ then

$$\varphi^{(\omega)}(x - d) = \varphi(x).$$

This is due to the translation of process X by constant d (more precisely due to the spatial homogeneity of the process). Let us recall the inequalities noted at (10)

$$\varphi(x + d) \leq \varphi^{(\omega)}(x) \leq \varphi(x). \qquad (17)$$

The first inequality is not of the same form as in (2.4), but this is the same expression due to, again, the spatial homogeneously of process X.

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We will show these inequalities in Figure 2, but to give numerical examples one needs to fix the Omega function. We will consider the shape of the Omega function which was mentioned in the introductory section. Namely, let us take

$$\omega(x) = [\gamma_0 + \gamma_1(x+d)]1_{\{x\in[-d,0]\}}.$$

As we mentioned before, we need to calculate $\mathcal{W}^{(\omega)}(0,-d)$ and $\mathcal{W}^{(\omega)\prime}(0,-d)$ with the use of numerical methods. Note that if we set

$$\gamma_0 = -\gamma_1 d$$

then ω will be continuous at zero. This gives us a more reasonable interpretation because the penalty will be decreasing continuously to zero. Thus,

$$\omega(x) = \gamma_1 x 1_{\{x\in[-d,0]\}}.$$

For such a model we have that from Proposition 3.1 the following differential equation for $x \in [-d,0]$, holds

$$p\mathcal{W}^{(\omega)''}(x,-d) - [\gamma_1 x + (\lambda - \mu p)]\mathcal{W}^{(\omega)'}(x,-d) - \gamma_1$$

$$[\mu x + 1]\mathcal{W}^{(\omega)}(x,-d) = 0, \qquad (18)$$

with the initial values $\mathcal{W}^{(\omega)}(-d;-d) = \frac{1}{p}$ and $\mathcal{W}^{(\omega)\prime}(-d;-d) = \frac{\lambda-\gamma_1 d}{p^2}$.

It is straightforward that $\gamma_1 \le 0$ because we have the assumption that $\omega(x) \ge 0$ for all x. Before we proceed to numerical examples let us recall the basic procedure for dealing with such differential equations with the use of numerical methods.

To start one can set $f(x) := \mathcal{W}^{(\omega)}(x;-d)$ and $h(x) := \mathcal{W}^{(\omega)\prime}(x;-d)$ for $x \in [-d,0]$. Then (18) became

$$\begin{cases} \dfrac{df}{dx} = h(x) \\ \dfrac{dh}{dx} = \dfrac{(\gamma_1 x + (\lambda - \mu p))}{p} h(x) + \dfrac{\gamma_1(\mu x + 1)}{p} f(x) \end{cases}$$

and $f(-d) = \frac{1}{p}$ and $h(-d) = \frac{\lambda-\gamma_1 d}{p^2}$. On the interval of interest, this system has a unique solution h and f due to the Lipschitz condition with respect to the dependent variables and continuity. Therefore one can obtain an approximation for $\mathcal{W}^{(\omega)}(0,-d)$ and $\mathcal{W}^{(\omega)\prime}(0,-d)$ using one of the iterative methods, e.g. the classical Runge-Kutta method for the system of ODE.

Hence we are ready to show relations between the probabilities in the inequalities (17). Thus, let us fix the following

$$\lambda = 1, \mu = 1, \gamma_1 = -0.2, d = 3, p = 1.25$$

and consider Figure 2.

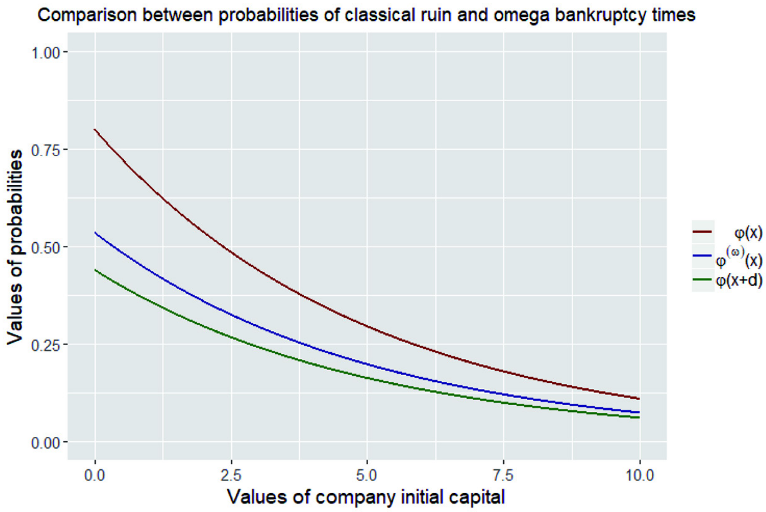

Fig. 2. Comparison between $\varphi(x)$, $\varphi^{(\omega)}(x)$ and $\varphi(x+d)$

Source: own elaboration.

From Figure 2 one can see the relations between these three probabilities and the trivial observation that if one increases capital then probabilities becomes smalle exponentially fast. Note, that if we increase the values of the penalty function ω then the probability of Omega bankruptcy time will become closer to the classical ruin time. However, if we choose to behave conversely then we will be close to the $\varphi(x+d)$.

4. Numerical approach for the Omega model for the Markov modulated Brownian motion

In this section, we will consider the Markov modulated Brownian motion (MMBM in short) as the underlying process. Namely, we will consider (X, J), where X will be an additive component and J is a continuous time Markov chain. We assume that the state space $E = \{1,2,\dots,N\}$ of process J can be of any finite size. However, when we proceed to numerical examples we will always fix the state space to be $E = \{1,2\}$. We state this assumption only for the clarity of the numerical examples presented in this section.

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When process J is on the state $i \in E$ then X is behaving like a linear Brownian Motion with the parameters $\sigma_i > 0$ and μ_i. To achieve almost certainly continuity of the paths we exclude all jumps from the reasoning, also these connected with the changing of the state. Moreover, we denote $\boldsymbol{\sigma}$ and $\boldsymbol{\mu}$ as the (column) vectors of σ_i and μ_i, and $\Delta_{\boldsymbol{v}}$ as the diagonal matrix (of the proper size) with vector $\boldsymbol{v}$ on the diagonal. Therefore, the matrix Laplace exponent $\boldsymbol{F}(s)$ is given by

$$\boldsymbol{F}(s) = \frac{1}{2}\Delta_{\boldsymbol{\sigma}}^2 s^2 + \Delta_{\boldsymbol{\mu}} s + \boldsymbol{Q}.$$

For this process we know from Ivanovs [Ivanovs 2011] that despite the case when $\kappa := \boldsymbol{\pi}^T \boldsymbol{\mu} = 0$ and $q = 0$ we have that

$$\boldsymbol{W}^{(q)}(x) = (e^{-\boldsymbol{\Lambda}_q^+ x} - e^{\boldsymbol{\Lambda}_q^- x})\Xi_q, \tag{19}$$

where $\Xi_q^{-1} = -\frac{1}{2}\Delta_{\boldsymbol{\sigma}}(\boldsymbol{\Lambda}_q^+ + \boldsymbol{\Lambda}_q^-)$ and $\boldsymbol{\Lambda}_q^\pm$ are the (unique) right solutions to the matrix equation $\boldsymbol{F}(\mp\boldsymbol{\Lambda}_q^\pm) = qI$, namely

$$\frac{1}{2}\Delta_{\boldsymbol{\sigma}}^2(\boldsymbol{\Lambda}_q^\pm)^2 \mp \Delta_{\boldsymbol{\mu}}\boldsymbol{\Lambda}_q^\pm + (\boldsymbol{Q} - qI) = \boldsymbol{0}.$$

We assume that $\kappa > 0$ to obtain positive asymptotic drift. From our point of view the general definition of $\boldsymbol{\Lambda}_q^\pm$ will be important. Thus, let us recall from (6) that for $i, j \in E$

$$\mathbb{P}(\tau_x^+ < e_q, J_{\tau_x^+} = j | J_0 = i) = (e^{\boldsymbol{\Lambda}_q^+ x})_{ij}.$$

Matrix $\boldsymbol{\Lambda}_q^-$ plays the same role for process $(-X, J)$ (which is MMBM but with the drift vector $-\boldsymbol{\mu}$). The next proposition can be used for the identification of the classical ruin time for the MMBM. Note that the same can be proven just using spatial homogeneity, however, we would like to show that (11) involves quantities that can be computed explicitly.

Proposition 4.1. *For $x \geq 0$ and $q \geq 0$ we have that*

$$\boldsymbol{P}_x(\tau_0^- < e_q, J_{\tau_0^-}) = \boldsymbol{Z}^{(q)}(x) - \boldsymbol{W}^{(q)}(x)\boldsymbol{C}_{W(\infty)^{-1}Z(\infty)} = e^{\boldsymbol{\Lambda}_q^- x},$$

where e_q is an independent exponential random variable with parameter q (if $q = 0$ then we set $e_q = \infty$) and $\boldsymbol{C}_{W(\infty)^{-1}Z(\infty)} := \lim_{c\to\infty}\boldsymbol{W}^{(q)}(c)^{-1}\boldsymbol{Z}^{(q)}(c)$.

We leave the proof of this proposition for the Appendix due to its long calculations.

Therefore, we have the formula for the classical ruin probability for MMBM, namely after we set $q = 0$ in the above proposition, one can get that for $x \geq 0$

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$$\boldsymbol{P}_x(\tau_0^- < \infty | J_0) = e^{\boldsymbol{\Lambda}^- x} 1,$$

where 1 is a column vector of ones of the size $N \times 1$.

Before we state our numerical method for the approximation of the probability of Omega bankruptcy, we need to consider the numerical method to obtain an approximation of the $\boldsymbol{\Lambda}_q^{\pm}$ matrices. Formally our approximation will be valid for $q \geq 0$, but in the examples we will be interested in the case when $q = 0$. Note that for this particular case there exist explicit formulas for these matrices (and even scale matrices) as was noted in [Czarna et al. 2018]. However, we would like to have a general formula that can work for any choice of the parameters in the MMBM model, as well as a different choice of parameter N. Here we will cite the results from [Breuer 2008] where an iterative method was derived. One can use other methods, for example, involving spectral analysis of the matrices $\boldsymbol{\Lambda}_q^{\pm}$, see [D'Auria et al. 2010].

Let us recall that $\Phi(q) = \sup\{\theta \geq 0 : \psi(\theta) = q\}$ is an inverse of function $\psi(\theta)$. In the case of the linear Brownian motion, one can obtain an explicit formula for this function, namely

$$\Phi(q) = \frac{-\mu + \sqrt{\mu^2 + 2q\sigma^2}}{\sigma^2}. \quad (20)$$

Let us denote $\Phi_i(q)$ as a function related to the linear Brownian motion with the parameters μ_i and σ_i for $i \in E$. Then we denote $\Delta_\Phi := diag(\Phi(q_i + q))_{i \in E}$, where $q_i = -q_{ii}$ and q_{ii} is (i,i)-entry of the matrix $\boldsymbol{Q}$.

Let $\boldsymbol{U}_0 := -\Delta_\Phi$ and $\boldsymbol{U}_{n+1} := g(\boldsymbol{U}_n)$ for $n > 0$ where row i of the matrix $g(\boldsymbol{U}_n)$ is defined as follows

$$(e_i)^T g(\boldsymbol{U}_n) \coloneqq$$
$$-\Phi_i(q_i + q)(e_i)^T + q_i\left(\textstyle\sum_{k \in E} p_{ik}(e_k)^T\right)[\Phi_i(q_i + q)I + \boldsymbol{U}(n)] \cdot \quad (21)$$
$$[-\frac{\sigma_i^2}{2}\boldsymbol{U}(n)^2 + \mu_i \boldsymbol{U}(n) + (q_i + q)I]^{-1},$$

where e_i is vector of zeros despite i'th position (canonical vector) and p_{ik} is the probability that if process J exits from state i then it will go to state k. It was proven that $\boldsymbol{U}_n$ converges to matrix $\boldsymbol{\Lambda}_q^+$. As we mentioned before, to obtain a numerical method for matrix $\boldsymbol{\Lambda}_q^-$ one needs to consider process $(-X, J)$ as the background for the above algorithm.

As a first example, we will consider the following parameters

$$\Delta_{\boldsymbol{\mu}} = \begin{pmatrix} 0.25 & 0 \\ 0 & 0.1 \end{pmatrix}, \Delta_{\boldsymbol{\sigma}} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.3 \end{pmatrix}, \boldsymbol{Q} = \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}, q = 0, x_0 = 1.$$

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Then we can apply above numerical method to derive an approximation of matrix $\mathbf{\Lambda}^-$ and therefore the probability of classical ruin time. Hence we get the following approximation

$$\mathbf{\Lambda}^- \approx \begin{pmatrix} -4.819 & 0.618 \\ 3.915 & -23.08 \end{pmatrix}, e^{\mathbf{\Lambda}^-} \approx \begin{pmatrix} 0.00915 & 0.000307 \\ 0.00195 & 0.0000654 \end{pmatrix}.$$

Note that the (i,j) cell of matrix $e^{\mathbf{\Lambda}^-}$ is the probability that

$$\mathbb{P}_1(\tau_0^- < \infty, J(\tau_0^-) = j | J_0 = i).$$

Thus to obtain the desired probability we need to add up the cells in row i. One can be interested in how the probability of classical ruin time is different for MMBM and Brownian Motion X^i with parameters μ_i and σ_i for $i \in E$. Remember that for Brownian Motion probability of classical ruin time takes the following form

$$\mathbb{P}_x(\tau_0^- < \infty) = e^{-\frac{2\mu}{\sigma^2}x}. \tag{22}$$

Consider Figure 3:

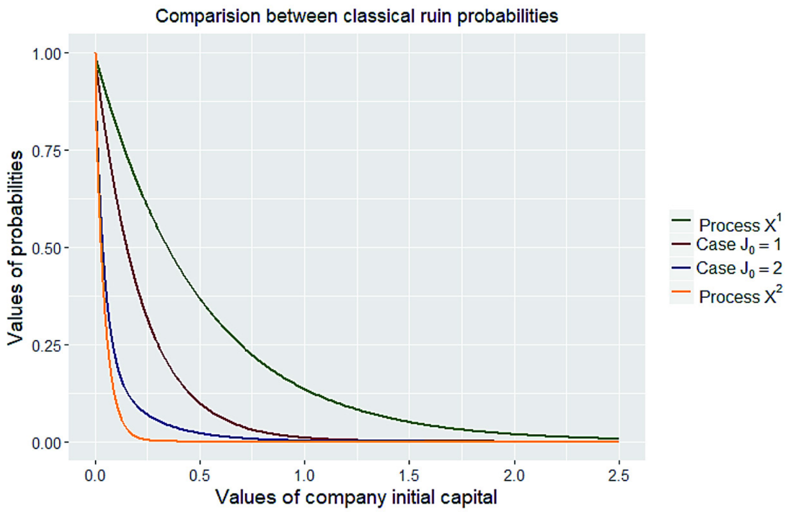

Fig. 3. Comparison between the classical probabilities for two cases of MMBM when $J_0 = 1$ and $J_0 = 2$ and also between linear Brownian Motions X^1 and X^2

Source: own elaboration.

One can see that in Figure 3 the probabilities for the Markov modulated Brownian motion stay between the probabilities for X^1 and X^2. This is a somewhat trivial observation but will become important later on.

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Let us proceed to the numerical example for the probability of Omega bankruptcy for the MMBM. Our aim is to approximate

$$(\varphi^{(\omega)}(x))_i = 1 - \mathbb{E}_x\left[e^{-\int_0^\infty \omega_{J_s}(X_s)ds}; \tau_{-d}^- = \infty | J_0 = i\right],$$

for all $i \in E$. Denote $\varphi_T^{(\omega)}(x)$ as the following vector of expected values for all $i \in E$

$$(\varphi_T^{(\omega)}(x))_i := 1 - \mathbb{E}_x\left[e^{-\int_0^T \omega_{J_s}(X_s)ds}; \tau_{-d}^- = \infty | J_0 = i\right].$$

Therefore this is a modification of our bankruptcy time in such a way that we allow to be killed by penalty function only before time T. If we let $T \to \infty$ then $\varphi_T^{(\omega)}$ converge to $\varphi^{(\omega)}(x)$ entry-wise, by dominated convergence theorem. From now we will hold the assumption that $\kappa > 0$, thus roughly speaking, after a certain long time the process should be saved from the penalty. Therefore we can use that to set a big enough T to approximate our ruin time. Therefore, because we will approximate $\varphi^{(\omega)}$ using an approximation of $\varphi_T^{(\omega)}$ we will face a so-called cut-off error.

Thus, we turn our problem into an approximation of $\varphi_T^{(\omega)}(x)$ and for that we will use the Monte Carlo methods. However we need to consider a few problems related to our method of approximation

- How to simulate a sample path of the Markov modulated Brownian motion?
- How to deal with the different starting points of process X?
- How big should the parameter T be?
- How many simulations are sufficient to get a trustworthy approximation?

Simulation of the sample path of the *MMBM*

At the start let us recall the method of simulation of process J. Let us assume that $J_0 = i$. Then we know that the time until J change the state from i to j is distributed like an exponential distributed random variable with parameter $q_i = -q_{i,i}$. Then, when J is leaving state i it can go to state j with probability $p_{i,j}$. Note that these probabilities can be determinant from matrix $\boldsymbol{Q}$. For more details, we refer to [Norris 1997].

Therefore, one can see that if we would like to simulate J until some time T then we need to simulate random numbers from the exponential distributions until their sum crosses level T.

Let us assume that we simulate the sample path of process J and $(X_0, J_0) = (0, i)$ for some $i \in E$. Then, let $0 = T_0, T_1, \ldots$ be a sequence of

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the successive jumps epoch of J (namely, the times when J changes the state). In the interval $[T_n, T_{n+1})$ we now that J is constant and equal to some $i \in E$. Thus in this interval, we can simulate increments of X the same as for the linear Brownian motion with the parameters μ_i and σ_i. In brief, we divide the time interval with the use of the occupation times of J and then use well-known methods for the simulation of the linear Brownian motion.

Different starting points of process *X*

As we mentioned before, we would like to simulate (X, J) efficiently with a different choice of X_0. Note that if we sample the random path of process (X, J) with $X_0 = 0$ then we can translate this sample path of X by the constant x to obtain the sample path of process (X, J) with $X_0 = x$. Therefore, after one simulation of process (X, J) with one starting point, we will have one simulation per every starting point.

Choice of parameter *T*

We need to choose such T that X_T will be "safe" with high probability. Note that if we choose T for $X_0 = 0$ then for a larger starting point this T will be also sufficient (because of the probability of ruin decrease when X_0 increase$s$). Thus, we will only consider $X_0 = 0$ and we will consider the following criteria. Let us take (if such $max\ arg$ is unique)

$$i = \max \arg_{k \in E} -\frac{2\mu_k}{\sigma_k}.$$

This means that we will take such X^i for which probability of classical ruin time is the highest from all possible $i \in E$. If such maximum is not unique then take these i's which satisfy this maximum and take the one with the smallest drift. Note, that such X^i will have a higher probability of classical ruin than process X itself, therefore this will be our worst-case scenario. Note that X_T^i is distributed as $N(\mu_i T, \sigma_i^2 T)$, hence we know that X_T^i will be greater than $\mu_i T - 3\sigma\sqrt{T}$ with high probability. Thus, our aim is to set T big enough that the probability that the linear Brownian motion, which starts with the value $\mu_i T - 3\sigma\sqrt{T}$, ever crosses level zero is less than some fixed ϵ. Then we must take the lowest value of T which satisfy

$$e^{-\frac{2\mu_i\sqrt{T}}{\sigma_i^2}(\mu_i T - 3\sigma_i\sqrt{T})} \leq \epsilon,$$

due to (22). Let us assume that $\epsilon = 10^{-4}$.

Note that this method is somehow trivial and restrictive. One can find another bound for T which is better for numerical approximation. However, in the example below parameter T will have reasonable value.

Number of simulations

From the theory of the Monte Carlo simulations we know that the rate of convergence is $n^{-\frac{1}{2}}$, or to be more precise, on the confidence level $\alpha = 0.05$ the relation between error (call it b), sample variance and the number of simulations is

$$b = \frac{1.96\hat{S}_n}{\sqrt{n}}.$$

Thus, to obtain a trustworthy approximation of parameter n one needs to make some pilot simulations to get $\hat{S}_n$ and then we also need to choose an acceptable error on the chosen confidence level.

Example of simulations

Finally, we are ready to make an approximation of the probability of Omega bankruptcy ruin. Let us take the same parameters as was chosen for the classical probability of ruin. Namely,

$$\Delta_{\boldsymbol{\mu}} = \begin{pmatrix} 0.25 & 0 \\ 0 & 0.1 \end{pmatrix}, \Delta_{\boldsymbol{\sigma}} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.3 \end{pmatrix}, \boldsymbol{Q} = \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}.$$

In addition for all $i \in E$ we take

$$\omega_i(x) = -0.02x1_{\{x\in[-5,0]\}},$$

thus $d = 5$. For such parameters we take that $T = 42$ and $\hat{S}_n \approx 0.066$. Therefore, we have that

$$b \approx \frac{0.129}{\sqrt{n}},$$

on the confidence level $\alpha = 0.05$. We will show the result for the error of the size 10^{-3}, then it is sufficient to take $N = 10^4$. Let us consider Figure 4.

Note that we used the formula for the probability of Omega bankruptcy for the linear Brownian motion from [Li, Palmowski 2018]. One can see that, las before, the probabilities for the MMBM are between these for X^1 and X^2, however here one can see that there is a little difference between the cases of $J_0 = 1$ and $J_0 = 2$.

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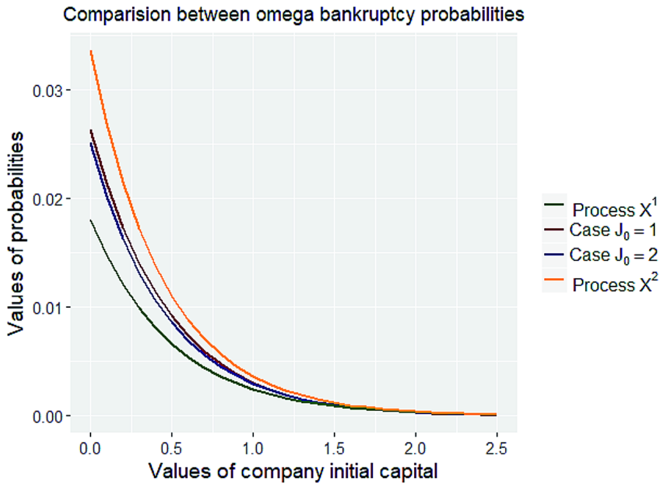

Fig. 4. Comparison between Omega bankruptcy probabilities for MMBM with different values of J_0, X^1 and X^2

Source: own elaboration.

5. Concluding remarks

In this paper we compared the Omega and classical ruin models for the two stochastic processes. An interesting result of this work seems to be Proposition 3.2. Namely, under the general assumptions about the ω function, we obtain that the probability of Omega bankruptcy is a linear function of the probability of classical ruin time. It is also important to say that the same type of relation was obtained for the linear Brownian motion in [Li, Palmowski 2018]. They achieve this result for function ω being a linear function. However, it seems that it should be possible to obtain the same type of relation for more general ω, as was done in this paper. Hence, the following question arises. Is this relationship, between the probability of Omega bankruptcy and the probability of classical ruin, true for the general spectrally negative Lévy processes? Unfortunately, the methods used in this paper and in [Li, Palmowski 2018] depend on the shape of specific scale functions, therefore it cannot be done straightforwardly.

In the second part of this article, we focused on the numerical analysis of the probability of classical ruin and the probability of Omega bankruptcy for the Markov modulated Brownian motion. For this case, we also noticed some relation between these probabilities, however it was only a numerical observation, and an important step would be to find such a relation with the use of analytic tools.

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Appendix. Proof of Proposition 4.1

Recall from (11) that

$$\boldsymbol{E}_x[e^{-\int_0^{\tau_0^-}\omega_{J_s}(X_s)ds}, \tau_0^- < \infty, J_{\tau_0^-}|J_0] = \boldsymbol{\mathcal{Z}}^{(\omega)}(x) - \boldsymbol{\mathcal{W}}^{(\omega)}(x)\boldsymbol{C}_{\mathcal{W}(\infty)^{-1}\mathcal{Z}(\infty)}.$$

If we take $\omega(i,x) = 0$ for all $i \in E$ and $x \geq 0$, then the above turns into

$$\boldsymbol{P}_x[\tau_0^- < \infty, J_{\tau_0^-}|J_0] = \boldsymbol{Z}^{(q)}(x) - \boldsymbol{W}^{(q)}(x)\boldsymbol{C}_{W(\infty)^{-1}Z(\infty)},$$

where $\boldsymbol{C}_{\mathcal{W}(\infty)^{-1}\mathcal{Z}(\infty)} = \lim_{c\to\infty}\boldsymbol{W}^{(q)}(c)^{-1}\boldsymbol{Z}^{(q)}(c)$. Recall that

$$\boldsymbol{Z}^{(q)}(x) = I - \int_0^x \boldsymbol{W}^{(q)}(z)dz(\boldsymbol{Q} - qI).$$

Before we state the proof, let us recall from [Czarna et al. 2018] the relation between $\boldsymbol{\Lambda}_q^+$, $\boldsymbol{\Lambda}_q^-$ and the model parameters

$$\boldsymbol{C}_q\boldsymbol{\Lambda}_q^+ = \Delta_{\frac{2}{\sigma^2}}[-\boldsymbol{Q} + qI], \tag{A1}$$

where $\boldsymbol{C}_q = (\boldsymbol{\Lambda}_q^+ + \boldsymbol{\Lambda}_q^-)\boldsymbol{\Lambda}_q^-(\boldsymbol{\Lambda}_q^+ + \boldsymbol{\Lambda}_q^-)^{-1}$.

Recall also (see [Ivanovs 2011]) that in the case of the Markov modulated Brownian motion we have that

$$\lim_{x\to\infty}\boldsymbol{L}^{(q)}(x) = \boldsymbol{\Xi}_{\boldsymbol{q}}, \tag{A2}$$

where $\boldsymbol{L}^{(q)}$ was defined in (5).

One can see that our proof can be divided into a few parts. Thus we will need the two lemmas.

Lemma A1. *For* $x \geq 0$

$$\int_0^x \boldsymbol{W}^{(q)}(z)dz(\boldsymbol{Q} - qI) = I - e^{-\boldsymbol{\Lambda}_q^+ x} - \boldsymbol{W}^{(q)}(x)\Delta_{\frac{\boldsymbol{\sigma}^2}{2}}\boldsymbol{\Lambda}_q^+.$$

Proof. After simple calculations, one can obtain the following

$$\begin{gathered}\int_0^x \boldsymbol{W}^{(q)}(z)dz(\boldsymbol{Q} - qI) = \\ -[(\boldsymbol{\Lambda}_q^+)^{-1}e^{-\boldsymbol{\Lambda}_q^+ x} + (\boldsymbol{\Lambda}_q^-)^{-1}e^{\boldsymbol{\Lambda}_q^- x}]\Xi_q(\boldsymbol{Q} - qI) + \\ [(\boldsymbol{\Lambda}_q^+)^{-1} + (\boldsymbol{\Lambda}_q^-)^{-1}]\Xi_q(\boldsymbol{Q} - qI).\end{gathered} \tag{A3}$$

We will divide our calculations into two parts, namely

$$-[(\boldsymbol{\Lambda}_q^+)^{-1}e^{-\boldsymbol{\Lambda}_q^+ x} + (\boldsymbol{\Lambda}_q^-)^{-1}e^{\boldsymbol{\Lambda}_q^- x}]\Xi_q(\boldsymbol{Q} - qI) =$$

$$[(\boldsymbol{\Lambda}_q^+)^{-1}e^{-\boldsymbol{\Lambda}_q^+ x} + (\boldsymbol{\Lambda}_q^-)^{-1}e^{\boldsymbol{\Lambda}_q^- x}]\left(\boldsymbol{\Lambda}_q^+ + \boldsymbol{\Lambda}_q^-\right)^{-1}\Delta_{\frac{2}{\boldsymbol{\sigma}^2}}(\boldsymbol{Q} - qI) \overset{(A.1)}{=}$$

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$$-[(\boldsymbol{\Lambda}_q^+)^{-1}e^{-\boldsymbol{\Lambda}_q^+ x} + (\boldsymbol{\Lambda}_q^-)^{-1}e^{\boldsymbol{\Lambda}_q^- x}]\boldsymbol{\Lambda}_q^-(\boldsymbol{\Lambda}_q^+ + \boldsymbol{\Lambda}_q^-)^{-1}\boldsymbol{\Lambda}_q^+ =$$
$$-[e^{-\boldsymbol{\Lambda}_q^+ x}(\boldsymbol{\Lambda}_q^+)^{-1}\boldsymbol{\Lambda}_q^- + e^{\boldsymbol{\Lambda}_q^- x}](\boldsymbol{\Lambda}_q^+ + \boldsymbol{\Lambda}_q^-)^{-1}\boldsymbol{\Lambda}_q^+ =$$
$$-[e^{-\boldsymbol{\Lambda}_q^+ x}((\boldsymbol{\Lambda}_q^+)^{-1}(\boldsymbol{\Lambda}_q^+ + \boldsymbol{\Lambda}_q^-) - I) + e^{\boldsymbol{\Lambda}_q^- x}](\boldsymbol{\Lambda}_q^+ + \boldsymbol{\Lambda}_q^-)^{-1}\boldsymbol{\Lambda}_q^+ =$$
$$-e^{-\boldsymbol{\Lambda}_q^+ x} + [e^{-\boldsymbol{\Lambda}_q^+ x} - e^{\boldsymbol{\Lambda}_q^- x}](\boldsymbol{\Lambda}_q^+ + \boldsymbol{\Lambda}_q^-)^{-1}\boldsymbol{\Lambda}_q^+ = -e^{-\boldsymbol{\Lambda}_q^+ x} - \boldsymbol{W}^{(q)}(x)\Delta_{\frac{\boldsymbol{\sigma}^2}{2}}\boldsymbol{\Lambda}_q^+$$

and

$$[(\boldsymbol{\Lambda}_q^+)^{-1} + (\boldsymbol{\Lambda}_q^-)^{-1}]\Xi_q(\boldsymbol{Q} - qI) =$$
$$-[(\boldsymbol{\Lambda}_q^+)^{-1} + (\boldsymbol{\Lambda}_q^-)^{-1}][\boldsymbol{\Lambda}_q^+ + \boldsymbol{\Lambda}_q^-]^{-1}\Delta_{\frac{2}{\boldsymbol{\sigma}^2}}(\boldsymbol{Q} - qI) \overset{(A.1)}{=}$$
$$[(\boldsymbol{\Lambda}_q^+)^{-1} + (\boldsymbol{\Lambda}_q^-)^{-1}]\boldsymbol{\Lambda}_q^-(\boldsymbol{\Lambda}_q^+ + \boldsymbol{\Lambda}_q^-)^{-1}\boldsymbol{\Lambda}_q^+ =$$
$$(\boldsymbol{\Lambda}_q^+)^{-1}[\boldsymbol{\Lambda}_q^- + \boldsymbol{\Lambda}_q^+][\boldsymbol{\Lambda}_q^+ + \boldsymbol{\Lambda}_q^-]^{-1}\boldsymbol{\Lambda}_q^+ = I.$$

The above calculations end the proof.

Lemma A2. *We have that*

$$\boldsymbol{C}_{\mathcal{W}(\infty)^{-1}\mathcal{Z}(\infty)} = -\Delta_{\frac{\boldsymbol{\sigma}^2}{2}}\boldsymbol{\Lambda}_q^-.$$

Proof. We have from the previous proposition, the fact that $\boldsymbol{\Xi}$ is invertible, the definition of scale matrix (5) and (A2) that

$$\begin{aligned}\boldsymbol{C}_{\mathcal{W}(\infty)^{-1}\mathcal{Z}(\infty)} &= \lim_{a\to\infty}\boldsymbol{W}^{-1}(a)\boldsymbol{Z}^{(q)}(a) = \lim_{a\to\infty}\boldsymbol{W}^{-1}(a)[e^{-\boldsymbol{\Lambda}_q^+ a} + \boldsymbol{W}^{(q)}(a)\Delta_{\frac{\boldsymbol{\sigma}^2}{2}}\boldsymbol{\Lambda}_q^+] \\ &= \lim_{a\to\infty}[\boldsymbol{L}^{(q)}(a)]^{-1} + \Delta_{\frac{\boldsymbol{\sigma}^2}{2}}\boldsymbol{\Lambda}_q^+ = (\boldsymbol{\Xi_q})^{-1} + \Delta_{\frac{\boldsymbol{\sigma}^2}{2}}\boldsymbol{\Lambda}_q^+ \\ &= -\Delta_{\frac{\boldsymbol{\sigma}^2}{2}}(\boldsymbol{\Lambda}_q^+ + \boldsymbol{\Lambda}_q^-) + \Delta_{\frac{\boldsymbol{\sigma}^2}{2}}\boldsymbol{\Lambda}_q^+ = -\Delta_{\frac{\boldsymbol{\sigma}^2}{2}}\boldsymbol{\Lambda}_q^-.\end{aligned}$$

Now we are ready to prove Proposition 4.1

Proof of Proposition 4.1

$$\boldsymbol{Z}^{(q)}(x) - \boldsymbol{W}^{(q)}(x)\boldsymbol{C}_{\mathcal{W}(\infty)^{-1}\mathcal{Z}(\infty)} =$$
$$e^{-\boldsymbol{\Lambda}_q^+ x} + \boldsymbol{W}^{(q)}(x)\Delta_{\frac{\boldsymbol{\sigma}^2}{2}}\boldsymbol{\Lambda}_q^+ + \boldsymbol{W}^{(q)}(x)\Delta_{\frac{\boldsymbol{\sigma}^2}{2}}\boldsymbol{\Lambda}_q^- = +$$
$$\boldsymbol{W}^{(q)}(\boldsymbol{x})\boldsymbol{\Delta}_{\frac{\boldsymbol{\sigma}^2}{2}}(\boldsymbol{\Lambda}_q^+ + \boldsymbol{\Lambda}_q^-) = \boldsymbol{e}^{-\boldsymbol{\Lambda}_q^+ x} - (\boldsymbol{e}^{-\boldsymbol{\Lambda}_q^+ x} - \boldsymbol{e}^{\boldsymbol{\Lambda}_q^- x}) = \boldsymbol{e}^{\boldsymbol{\Lambda}_q^- x}$$

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BANKRUCTWO TYPU OMEGA DLA RÓŻNYCH MODELI LÉVY'EGO

Streszczenie: W niniejszym artykule rozważamy model bankructwa typu Omega, który może być traktowany jako alternatywa wobec klasycznego pojęcia ruiny. W odróżnieniu od klasycznego modelu pozwalamy, aby proces znalazł się poniżej zera, jednakże nie poniżej ustalonego poziomu $-d < 0$. Gdy proces znajduje się poniżej zera, jest on „zabijany" z funkcją intensywności ω. Naszym celem jest ukazanie relacji pomiędzy modelem Omega a klasyczną ruiną dla dwóch istotnych modeli typu Lévy'ego, a więc rozważać będziemy proces Crámera-Lundberga oraz markowsko modulowany ruch Browna. W pracy podamy również wyniki numeryczne, które będą ilustrować wyniki z analiz.

Słowa kluczowe: prawdopodobieństwo ruiny, model Omega, proces Crámera-Lundberga, markowsko modulowany ruch Browna.