Alena Tartaľová
Technical University of Košice

THE IMPACT OF LARGE CLAIMS ON ACTUARIAL DECISIONS*

Summary: The article discusses the application of extreme value theory in the analysis of extreme values what makes their statistical analysis easier. The application of this method in the analysis of reinsurance contracts, reinsurance of loss excess and excess of loss ratio is described. Furthermore, in the case of agreements of the reinsurance of loss excess two ways of establishing an optimal level of retention are analyzed.

Key words: claim, extra event, excess of loss reinsurance.

1. Introduction

The increasing number of catastrophic events and the total claim amount connected with them have the strong influence on the insurance market. It forces insurance companies to use effective risk management tools. It means to have better reinsurance to transfer a high cost. In this paper its author concerns two methods used in extreme value theory for the registration of extreme events, which simplify the statistical analysis. We describes the application of these methods in reinsurance treaties, Largest Claim Reinsurance (LCR(p)) and Excess of Loss Reinsurance (XL(M)). For the XL reinsurance she analyzes two ways of setting optimal retention M level.

2. Methods for registration of extreme events

Let $\left(X_1, X_2, \dots, X_{N(t)}\right)$ be a sequence of iid (independent and identically distributed) positive random variables representing claim sizes. Assume that claims occur according to a counting process $\{N(t), t \geq 0\}$, i.e. the random variable $N(t)$ counts the number of claims up to time t. We further assume that the claim

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number process $\{N(t), t \geq 0\}$ is independent of claim size process $\{X_i, i \geq 1\}$. Let $\left(X^*_1, X^*_2, \ldots, X^*_{N(t)}\right)$ be the sequence of the order statistics, arranged in the increasing order from the random vector $\left(X_1, X_2, \ldots, X_{N(t)}\right)$.

For the registration of extreme values we use two methods – method of block-maxima and peaks over threshold method.

2.1. Method of block-maxima

By this method as extreme values only the maxima in the block – e.g. annual, monthly or daily are recorded. We can see the principle of this method in Fig. 1.

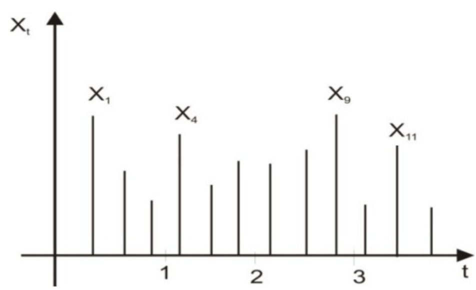

Fig. 1. Method of block-maxima

Let $M_n = \max\{X_1, X_2, \ldots, X_n\}$ be a maximum of iid random variables. The limit distribution of M_n is given by Fisher-Tippet theorem.

Fisher-Tippet theorem: Let $\{X_n\}_n$ be a sequence of iid random variables. If there exist norming constants $c_n \in R,\ d_n > 0$ and some non-degenerate distribution function H such as

$$d_n^{-1}\left(M_n - c_n\right) \xrightarrow{d} X, \text{ for } n \to \infty.$$

then H belongs to the type of one of the following three ***standard extreme value distributions***:

1. Fréchet: $\Phi_\alpha(x) = \exp\left\{-x^{-\alpha}\right\},\ x > 0,\ \alpha > 0$ (else $\Phi_\alpha(x) = 0$).
2. Weibull: $\Psi_\alpha(x) = \exp\left\{-(-x)^\alpha\right\},\ x \leq 0,\ \alpha > 0$ (else $\Psi_\alpha(x) = 1$).
3. Gumbel: $\Lambda(x) = \exp\left\{-e^{-x}\right\},\ x \in R\cdot$

Proof. For the sketch of the proof see [Embrechts et al. 1997].

There exists a one-parameter representation of the three standard cases in one family of dfs. The general extreme value distribution $H_{\xi,\mu,\sigma}$ is called ***the generalized***

extreme value distribution (GEV). The parameter ξ is called ***extreme value index (EVI)***.

$$H_{\xi,\mu,\sigma}(x)=\begin{cases}\exp\left\{-\left(1+\xi\frac{x-\mu}{\sigma}\right)_+^{-1/\xi}\right\}, \xi\neq 0,\\ \exp\left[\exp\left\{\frac{x-\mu}{\sigma}\right\}\right], \xi=0.\end{cases}$$

H_ξ corresponds to

- Fréchet distribution for $\xi=\alpha^{-1}>0$, $x>-\xi^{-1}$,
- Weibull distribution for $\xi=-\alpha^{-1}<0$, $x<-\xi^{-1}$,
- Gumbel distribution for $\xi=0$, $x\in R$.

The three standard extreme value distributions have different tails and serve as limit distribution for different types of distributions:

- Fréchet – long tail (for Pareto, Cauchy, Student and loggamma distributions),
- Gumbel – moderately long tail (for exponential, normal, lognormal and gamma distributions),
- Weibull – short tail (for uniform and beta distributions).

This method could be used in the Largest Claim reinsurance (LCR(p)). The reinsured amount in LCR treaty equals:

$$L_p(t)=\sum_{i=1}^{p}X^*_{N(t)=i+1},\ r\geq 1,\ t\geq 0.$$

A lot of theory was published on the LCR (see e.g. [Kremer 2000]) but there is a few information about the choice of p – the number of largest claims taken over by the reinsurer. More about this problem in [Kremer 2000].

2.2. Peaks over threshold method

The method of block-maxima has a disadvantage: it uses only one value per block, but in the block can be another big value, which we do not consider. Another way how to record extreme values is to use peaks over threshold (POT) method.

In POT method we record all exceedances over given threshold u. In Fig. 2. we can see the principle of this method.

The possible limit distribution for exceedances is given by Pickands-Balkema-de Haan theorem.To model the tail of the underlying distribution F, we follow the excesses above sufficiently high threshold u. Let X be a random variable with df F and with right endpoint xF. We define the ***excess distribution function*** $F_u(x)$ for $u<x_F$ by

$$F_u(x)=P\left(X-u\leq x\mid X>u\right)=\frac{F(x+u)-F(u)}{\overline{F}(u)},\ x\geq 0.$$

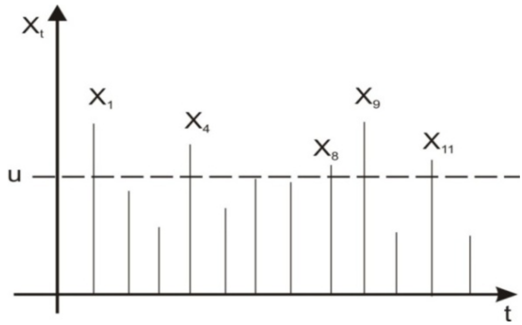

Fig. 2. Peaks over threshold method

The following theorem says that the only possible limit distribution for the excesses over high threshold is the generalized Pareto distribution. First we define this function.

Definition. The distribution function $G_\xi(x)$ defined for $1+\xi x>0$ by

$$G_\xi(x) = \begin{cases} 1-(1+\xi x)^{-1/\xi}, \text{ if } \xi\neq 0, \\ 1-e^{-x}, \text{ if } \xi=0. \end{cases}$$

is called ***the generalized Pareto distribution (GPD)***.

For $\xi > 0$ is $x \geq 0$ and for $\xi < 0$ is $0 \leq x \leq \xi^{-1}$. For $\xi = 0$ we get the exponential distribution.

We can extend the family of Pareto distributions adding scaling parameter β and location parameter γ as

$$G_{\xi,\beta}(x) := G_\xi(x/\beta) \text{ or } G_{\xi,\beta,\gamma}(x) := G_\xi\left(\frac{x-\gamma}{\beta}\right).$$

Pickands, Balkema and de Haan theorem: $F_u(x)$ is an excess distribution function if and only if we can find a positive measurable function $\beta=\beta(u)$ for every $\xi > 0$ such as

$$\lim_{x\uparrow x_F} \sup_{0\leq x\leq x-x_F} \left|F_u(x)-G_{\xi,\beta(u)}(x)\right| = 0.$$

This method is used in Excess of Loss reinsurance.

3. Excess of loss reinsurance

In an excess of loss agreement (or treaty) a reinsurer handled the excess of each claim X over an agreed excess level M. Individual claim X is split into two components $X = X_I + X_R$, which are respectively handled by the insurance (X_I) and reinsurance companies (X_R).

$$X_I = \begin{cases} X & \text{if } X \le M \\ M & \text{if } X > M \end{cases} \text{ and } X_R = \begin{cases} 0 & \text{if } X \le M \\ X - M & \text{if } X > M \end{cases}.$$

The expected payment per claim for the insurer is reduced from *E(X)* to

$$E\left(X_I\right) = \int_0^M x.f(x)dx + M.\bar{F}(M).$$

If we take $E(X_I)$ as a function of excess level *M*, we get a well-known function called ***limited expected value function (LEV)*** $L_X\left(M\right) = E\left(X_I\right)$. The value of this function at point *M* is equal to the expected value of random variable *X* truncated at point *M*. The limited expected value function for some most often used loss distribution is given below:

- If *X* has lognormal distribution, $X \sim LN(\mu, \sigma^2)$:

$$L_X(M) = \exp\left(\mu + \frac{\sigma^2}{2}\right).\Phi\left(\frac{\log M - \mu - \sigma^2}{\sigma}\right) + \\ + M.\left[1 - \Phi\left(\frac{\log M - \mu}{\sigma}\right)\right]$$

- If X has exponential distribution, $X \sim Exp(\lambda)$:

$$L_X(M) = \frac{1}{\lambda}\left(1 - e^{-\lambda M}\right)$$

- If X has Pareto distribution, $X \sim Pareto(\alpha, \lambda)$:

$$L_X(M) = \frac{\lambda - \lambda^{\alpha}\left(\lambda + M\right)^{1-\alpha}}{\alpha - 1}.$$

The LEV function for other types of distribution is in [Cizek et.al. 2005]. The LEV function is a suitable tool for choosing the excess level M and has some important properties such as:

- L is concave, continuous and increasing,
- $L(M) \to E(X)$, if $M \to \infty$,
- $F(x) = 1 - L'(x)$, where *L'* is the derivative of function *L*.

Our main interest is to set optimal retention level M.

A reinsurance company charges for sharing the risk of an insurance company and this affects the retention level and the type of agreement. If we denote θ as a loading (security or safety) factor, ξ is a corresponding loading factor used by a reinsurer. Normally, the reinsurer uses the loading factor ξ greater than the loading factor of insurance company (ξ ≥ θ). We denote P as an insurer's profit or net premium minus claims, and so the expected profit for the insurer is:

$$E(P)=(1+\theta).E(S)-(1+\xi).E(S_R)-E(S_I),$$

where $S=S_I+S_R$ is the total claim amount, $S=\sum_{i=1}^{N(t)}X_i$,

S_I is the amount paid by the insurer,
S_R is the amount paid by the reinsurer.

The insurer wants to have a nonnegative profit, $E(P)\geq 0,$ so i.e. the minimum excess level M* satisfying:

$$\frac{E(S_I)}{E(S_R)}=\frac{E(X_I)}{E(X_R)}\geq\frac{\xi-\theta}{\theta}=\frac{\xi}{\theta}-1\equiv M^*.$$

For example, when *X~Pareto*(*α*, *λ*), the M must satisfy:

$$\frac{E(X_I)}{E(X_R)}=\frac{\int_0^M x.f(x)dx+M.\bar{F}(M)}{\int_M^\infty (x-M)f(x)dx}=$$

$$=\left[\frac{\lambda}{\alpha-1}-\left(\frac{\lambda}{\lambda+M}\right)^\alpha.\left(\frac{\lambda+M}{\alpha-1}\right)\right]\Big/\left[\left(\frac{\lambda}{\lambda+M}\right)^\alpha.\left(\frac{\lambda+M}{\alpha-1}\right)\right]=\left(1+\frac{M}{\lambda}\right)^{\alpha-1}-1\geq\frac{\xi}{\theta}-1,$$

or equivalently $M\geq\lambda\left[\left(\frac{\xi}{\theta}\right)^{\frac{1}{\alpha-1}}-1\right]\equiv M^*$

When *X~Exp(λ)*, the M must satisfy:

$$\frac{E(X_I)}{E(X_R)}=\frac{\int_0^M x.f(x)dx+M.\bar{F}(M)}{\int_M^\infty (x-M)f(x)dx}=\frac{\frac{1}{\lambda}\left(1-e^{-\lambda M}\right)}{\frac{1}{\lambda}.e^{-\lambda M}}=e^{-\lambda M}-1\geq\frac{\xi}{\theta}-1,$$

or equivalently $M\geq\log\left(\frac{\xi}{\theta}\right).\frac{1}{\lambda}.$

4. Illustrative example

We introduce an illustrative example of selecting the retention level M for the excess of loss reinsurance. We analysed the sample of values of 91 individual claims from the accident (motor) insurance from some Slovak insurance company (the sample is also analysed in [Pacáková, Šoltés 2004] and [Pacáková, Šoltés 2005]). Using maximum likelihood estimation method we find the parameter of Pareto distribution $\alpha = 1{,}7393999$ and $\lambda = 37\,277{,}8135$, to fit the data (more about distribution fitting is in [Pacáková 2007], [Pacáková, Linda 2006] or in [Skřivánková, Tartaľová 2008]). For the case of large claim we want to use the excess of loss reinsurance, so we need to set the optimal retention level.

Limited expected value function for Pareto distribution with parameters α = 1, 7393999 and λ = 37 277, 8135 is in Fig. 3.

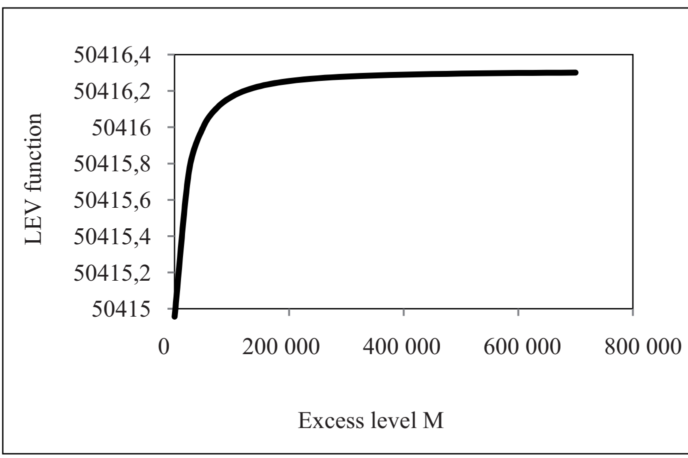

Fig. 3. LEV function for Pareto distribution

Source: own calculations.

Table 1. Minimum excess level M* for Pareto distribution

θ/ξ	0,1	0,2	0,3	0,4	0,5
0,1	0	57909,24	127436,8	205777,6	291400,5
0,2	0	0	27228,87	57909,24	91441,48
0,3	0	0	0	17729,9	37107,87
0,4	0	0	0	0	13132,13

Source: own calculations.

The LEV function satisfies all properties described before, it converges to expected value $E(X)$=50416,30885 for very large M. The minimum excess level M* for Pareto distribution for various values of θ and ξ is in Tab. 1.

Another way, how to find a suitable excess level, is to use POT method described in section 2.2. By the Pickands-Balkema-de Haan theorem the best model for exceedances over threshold is the generalized Pareto distribution, for our data with three parameters: ξ = 0,629417, β = 6738,52 and γ = 19009,6. The graphical agreement with two-parameter Pareto distribution $G_{\xi,\beta,\gamma}$ is in Fig. 4.

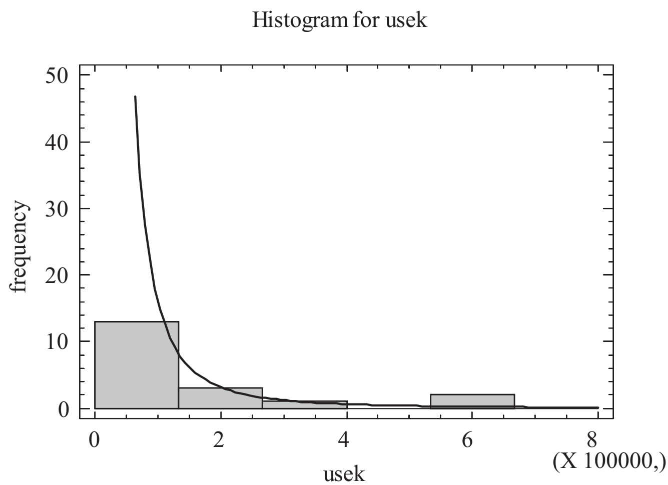

Fig. 4. The tail histogram and Pareto distribution

In Fig. 9 we see the noticeable agreement between the quantiles of Pareto distribution and theoretical distribution of 19 exceedances over threshold M = 57487,0. The best fit is for this excess level when p-value of goodness of fit test (Kolmogorov-Smirnov test) is the largest.

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WPŁYW WYSTĘPOWANIA WYSOKICH SZKÓD NA DECYZJE AKTUARIALNE

Streszczenie: W artykule uwaga została poświęcona zastosowaniu teorii wartości ekstremalnych w analizie zdarzeń ekstremalnych, co upraszcza ich analizę statystyczną. Opisane zostało zastosowanie tej metody w analizie kontraktów reasekuracyjnych, reasekuracji nadwyżki szkody oraz nadwyżki szkodowości. Ponadto w przypadku umów reasekuracji nadwyżki szkodowości przeanalizowano dwa sposoby ustalania optymalnego poziomu retencji.