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Analytical relations for thin-film external-reflection phase retarders

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Thin-film coated external reflection phase retarders at given angle of incidence and specific phase retardation are analytically approached for a two-reflection system with two, one coated and the other uncoated, mirrors. Single-layer and specific symmetrical triple-layer dielectric coatings on absorbing substrates are considered. Useful diagrams of solution zones for refractive indices are obtained. Simple relations for one-reflection systems in the case of non-absorbing substrates are also infereed.

1. Introduction

Thin-film phase retarders are obtained by coating transparent thin films on appropriate substrates to provide equal p and s polarization reflectances and specified differential phase shift (retardation) Δ. These retarders are commonly used in laser systems as intracavity laser optics and also in optical trains (delivery systems). The common values of phase retardation are $\Delta = k\pi$ and $(2k+1)\pi/2$, $k = 0, 1, 2\ldots$, corresponding to half-wave and quarter-wave retarders, respectively.

One- and two-reflection systems can be used, as is shown in Fig. 1. The first device (A in Fig. 1) has one thin-film coated mirror. In the case of the two-reflection device (B in Fig. 1), one mirror is uncoated and produces its proper phase shift $\bar{\Delta}_r$.

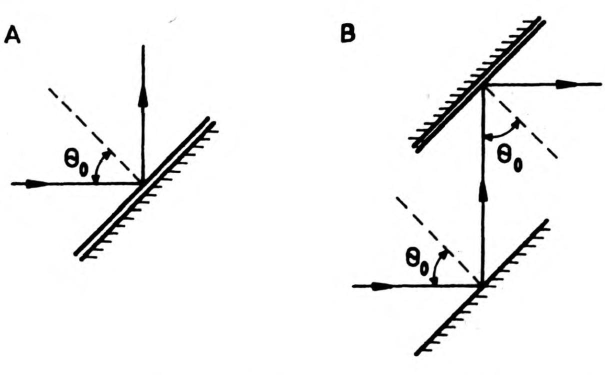

Fig. 1. One- and two-reflection thin-film phase retarding systems; the system A contains one thin-film coated mirror, and the two-reflection system B contains two mirrors, one coated and the other uncoated

Previous works [1]–[5] have mostly been pursued numerically for single-layer coatings in one- and two-reflection systems and for double-layer coatings in one-reflection systems at given angle of incidence and specified values of phase retardation.

Analytical relations for single-layer and specific symmetrical triple-layer coatings on substrates of complex refractive indices are presented in this paper for two-reflection systems. Then, simple relations are deduced for one-reflection systems in the case of transparent substrates at given angle of incidence and specified values of phase retardation. The symmetrical triple-layer coatings which are considered contain one non-quarterwave layer sandwiched between two identical quarterwave layers. These specific triple-layer coatings which are of practical interest are similar to the single-layer coatings but they have greater solution zones of layer refractive indices available to achieve a specified phase retardation.

2. General considerations

The change of the state of polarization of light by external reflection on an uncoated substrate of complex refractive index $\hat{n} = n - jk$ is determined by the ratio

$$\hat{\rho} \equiv \bar{r}_p/\bar{r}_s = (|\bar{r}_p|/|\bar{r}_s|)\exp[j(\bar{\varphi}_p - \bar{\varphi}_s)] \tag{1}$$

where $\bar{r}_p$ and $\bar{r}_s$ are complex Fresnel reflection coefficients for the linear p and s polarizations [6]. A phase shift (retardation) $\bar{\Delta} = \bar{\varphi}_p - \bar{\varphi}_s$ is produced upon reflection.

If the substrate is coated with a transparent thin film of refractive index n_1 and geometrical thickness d_1, the change of the state of polarization of light is determined by the ratio

$$\rho \equiv r_p/r_s = (|r_p|/|r_s|)\exp[j(\varphi_p - \varphi_s)] \tag{2}$$

and a phase shift $\Delta_1 = \varphi_p - \varphi_s$ is produced. The complex reflection coefficients r_p and r_s are given by

$$r_v = (r_{01v} + r'_v X_1)/(1 + r_{01v} r'_v X_1), \quad v = p,\ s \tag{3}$$

where r_{01v} and r'_v are Fresnel reflection coefficients for v polarization on ambient/film and film/substrate interfaces, respectively. Generally, the Fresnel reflection coefficient r_{ijv} for the ij interface and v polarization is determined from

$$r_{ijv} = (\mu_i - \mu_j)/(\mu_i + \mu_j) \tag{4}$$

where $\mu_{i,j}$ are effective refractive indices defined by

$$\mu_i = \begin{cases} p_i = (n_i^2 - n_0^2 \sin^2\theta_0)^{1/2} & \text{for } s \text{ polarization} \\ q_i = n_i^2/p_i & \text{for } p \text{ polarization} \end{cases} \tag{5}$$

where n_0 and n_i are refractive indices of ambient and i-th media and θ_0 is the angle of incidence. This paper uses the Abele's convention $r_p = r_s$ at normal incidence. In Equation (3), X_1 is a complex periodic function of the layer phase thickness

$$X_1 = \exp(-j2\gamma_1) \tag{6}$$

where the layer phase thickness is defined by

$$\gamma_1 = 2\pi d_1 p_1/\lambda. \tag{7}$$

A normalized film thickness is defined by [5]

$$\zeta_1 = 2d_1 p_1/\lambda = \gamma_1/\pi. \tag{8}$$

In the case of the two-reflection system (B in Fig. 1), the change of the state of polarization of light is determined from

$$\rho_t = \rho\bar{\rho} = |r_p\bar{r}_p/(r_s\bar{r}_s)|\exp[j(\Delta_1+\bar{\Delta}). \tag{9}$$

By a suitable choice of the film refractive index n_1 and phase thickness γ_1 it is possible to achieve a specific value of the differential phase shift $\Delta = \Delta_1 + \bar{\Delta}$. Then

$$\rho = r_p/r_s = (\bar{r}_s/\bar{r}_p)\exp(j\Delta). \tag{10}$$

Inserting Equation (3) into (10), one obtains [5]

$$\hat{a}X_1^2 + \hat{b}X_1 + \hat{c} = 0 \tag{11}$$

where the complex coefficients $\hat{a}$, $\hat{b}$ and $\hat{c}$ are given by

$$\hat{a} = r'_p r'_s(r_{01s} - \rho r_{01p}), \tag{12a}$$

$$\hat{b} = r_{01p} r_{01s}(r'_s - \rho r'_p) + (r'_p - r'_s), \tag{12b}$$

$$\hat{c} = r_{01p} - \rho r_{01s}. \tag{12c}$$

By using the constraint $|X_1| = 1$ in Eq. (11), a quadratic equation results

$$At^2 + Bt + C = 0, \tag{13}$$

for the unknown layer phase thickness γ_1

$$t = \tan\gamma_1 \tag{14}$$

where real coefficients A, B and C are the functions of layer refractive index at given angle of incidence and specific value of the phase shift Δ. Solution of Eq. (13) gives

$$t = [-B \pm (B^2 - 4AC)^{1/2}]/(2A). \tag{15}$$

Because t must be real, from the condition $B^2 - 4AC \geqslant 0$ refractive index sets are obtained as solution zones. It should be noted that false solutions can be obtained [5]. When a forward calculation is made with refractive index pair taken from the solution zones and t given by Eq. (15) in many cases the p and s polarization reflectances are not equal and the differential phase shift is not the value specified. Appropriate solution zones of refractive indices are obtained by imposing constraints on the reflectance and phase errors:

$$\varepsilon_r \equiv \text{reflectance error} = |r_p\bar{r}_p/(r_s\bar{r}_s)| - 1, \tag{16a}$$

$$\varepsilon_{ph} \equiv \text{phase error} = (\Delta_1 + \bar{\Delta}) - \Delta. \tag{16b}$$

The case of symmetrical triple-layer coatings containing one non-quarterwave layer of refractive index n_2 and phase thickness γ_2 sandwiched between two identical quarterwave layers of refractive index n_1 and $\gamma_1 = \pi/2$, can be treated similarly to the previous single-layer case [6]. Greater solution zones in the plane (n_1, n_2) are obtained for given refractive indices of the substrates at specified angle of incidence and phase retardation compared to the single-layer coatings.

The thin films are assumed nonabsorbing, isotropic and homogeneous throughout this paper.

3. General relations for two-reflection systems with substrates of complex refractive indices

Let us consider the general case of a two-reflection system B with substrates of complex refractive indices. The substrates for the coated and uncoated mirrors can be different. The effective refractive indices for a substrate of complex refractive index $\hat{n} = n - jk$ can be written in the form

$$\mu = \begin{cases} p = u - jw & \text{for } s \text{ polarization} \\ q = \alpha - j\beta & \text{for } p \text{ polarization} \end{cases} \tag{17}$$

where u, w, α and β are given by:

$$2u^2 = \eta - S + [(\eta - S)^2 + 4n^2k^2]^{1/2}, \tag{18a}$$
$$2w^2 = -\eta + S + [(\eta - S)^2 + 4n^2k^2]^{1/2} \tag{18b}$$

where: $\eta = n^2 - k^2$ and $S = n_0^2 \sin^2\theta_0$, and

$$\alpha = (u\eta + 2nkw)/(u^2 + w^2), \tag{19a}$$
$$\beta = (2nku - w\eta)/(u^2 + w^2). \tag{19b}$$

Inserting the Fresnel reflection coefficients $\bar{r}_s$ and $\bar{r}_p$ of the uncoated substrate of complex refractive index $\hat{\bar{n}} = \bar{n} - j\bar{k}$ into Eq. (10) one obtains

$$\rho = \rho_r + j\rho_i = \exp(j\Delta)[p_0^2(\bar{u}^2 + \bar{w}^2) - S^2 + 2jp_0\bar{w}S]/[(p_0\bar{u} - S)^2 + p_0^2\bar{w}^2]. \tag{20}$$

Then one obtains after laborious and unsignificant algebra the following relations for the real coefficients A, B and C of Eq. (13):

$$A = (a_1 + a_3)^2 + (a_2 + a_4)^2 - a_5^2 - a_6^2, \tag{21a}$$
$$B = 4(a_2a_3 - a_1a_4), \tag{21b}$$
$$C = A - 4(a_1a_3 + a_2a_4). \tag{21c}$$

For single-layer coatings the coefficients $a_1 - a_6$ are given by the following relations:

$$a_1 = 2(n_0^2 + n_1^2)[(n_1^2 - \eta)(\rho_r - 1) - 2nk\rho_i] + 2c_+[a_-(\rho_r + 1) + b_-\rho_i], \tag{22a}$$
$$a_2 = 2(n_0^2 + n_1^2)[\rho_i(n_1^2 - \eta) + 2nk(\rho_r - 1)] + 2c_+[a_-\rho_i - b_-(\rho_r + 1)], \tag{22b}$$
$$a_3 = d_+e_- - \rho_i(b_+ - 2nk)(n_1^2n_0^2 + c_-), \tag{22c}$$
$$a_4 = d_+(b_+ - 2nk) + e_-\rho_i(n_1^2 - n_0^2 - c_-), \tag{22d}$$
$$a_5 = d_-e_+ + \rho_i(b_+ + 2nk)(n_1^2 - n_0^2 - c_-), \tag{22e}$$

$$a_6 = d_-(b_+ + 2nk) - e_+\rho_i(n_1^2 - n_0^2 - c_-) \tag{22f}$$

where:

$$a_\pm = \alpha p_1 \pm q_1 u, \tag{23a}$$
$$b_\pm = \beta p_1 \pm q_1 w, \tag{23b}$$
$$c_\pm = p_0 q_1 \pm p_1 q_0, \tag{23c}$$
$$d_\pm = (n_1^2 - n_0^2)(\rho_r - 1) \pm c_-(\rho_r + 1), \tag{23d}$$
$$e_\pm = n_1^2 + \eta \pm a_+. \tag{23e}$$

For specific triple-layer coatings, one obtains:

$$a_1 = 2(n_1^4 + n_0^2 n_2^2)[(n_1^4 - \eta n_2^2)(\rho_r - 1) - 2nkn_2^2\rho_i] + 2c_+[a_-(\rho_r + 1) + b_-\rho_i], \tag{24a}$$
$$a_2 = 2(n_1^4 + n_0^2 n_2^2)[\rho_i(n_1^4 - \eta n_2^2) + 2nkn_2^2(\rho_r - 1)] + 2c_+[a_-\rho_i - b_-(\rho_r + 1)], \tag{24b}$$
$$a_3 = d_+e_- - \rho_i(b_+ - 2nkn_2^2)(n_1^4 - n_0^2 n_2^2 + c_-), \tag{24c}$$
$$a_4 = d_+(b_+ - 2nkn_2^2) + e_-\rho_i(n_1^4 - n_0^2 n_2^2 + c_-), \tag{24d}$$
$$a_5 = d_-e_+ + \rho_i(b_+ + 2nkn_2^2)(n_1^4 - n_0^2 n_2^2 - c_-), \tag{24e}$$
$$a_6 = d_-(b_+ + 2nkn_2^2) - e_+\rho_i(n_1^4 - n_0^2 n_2^2 - c_-) \tag{24f}$$

where:

$$a_\pm = \alpha p_1^2 q_2 \pm p_2 q_1^2 u, \tag{25a}$$
$$b_\pm = \beta p_1^2 q_2 \pm p_2 q_1^2 w, \tag{25b}$$
$$c_\pm = p_0 p_2 q_1^2 \pm p_1^2 q_0 q_2, \tag{25c}$$
$$d_\pm = (n_1^4 - n_0^2 n_2^2)(\rho_r - 1) + c_-(\rho_r + 1), \tag{25d}$$
$$e_\pm = n_1^4 + \eta n_2^2 \pm a_+. \tag{25e}$$

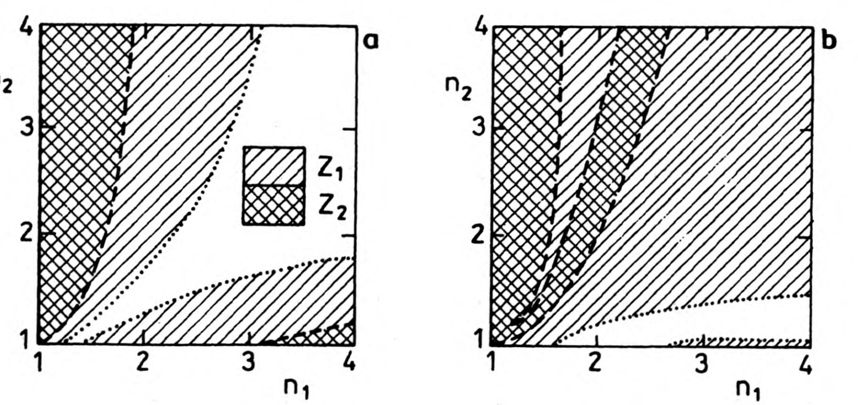

Fig. 2. Diagram showing solution zones (shaded areas Z_1) of refractive index sets satisfying the condition of real solutions of Eq. (13) and appropriate solution zones (shaded areas Z_2) with error constraints (reflectance error $|\varepsilon_R| \leqslant 0.1$, and phase error $|\varepsilon_{ph}| \leqslant 0.02$) in the case of the one-reflectance system (A, Fig. 1) — **a**, and in the case of the two-reflection system (B, Fig. 1) — **b**. In both cases a specific symmetrical triple-layer coating is considered on a silver substrate of complex refractive index $\hat{n} = 9.5 - j73$ (for $\lambda = 10.6$ μm) at an angle of incidence $\theta_0 = 45°$ and phase shift $\Delta = 0$. The uncoated mirror in the case **b** has the same complex refractive index and angle of incidence θ_0 as the coated mirror

Relations for one-reflection systems (A, Fig. 1) at given angle of incidence and specified phase retardation result from the above relations as a particular case for $\rho_r = \cos\Delta$ and $\rho_i = -\sin\Delta$.

From the constraint of real solutions of Eq. (13), solution zones of refractive indices result for a specific phase retardation at given angle of incidence. An example is shown in Fig. 2 for specific triple-layer coatings on silver substrates, $\hat{n} = 9.5 - j73$, at zero phase shift for one- and two-reflection systems at an angle of incidence 45°. One can see that greater solution zones result for the two-reflection system (B, Fig. 1) compared to the one-reflection system (A, Fig. 1). By imposing constraints on the reflectance and phase errors, $|\varepsilon_R| \leqslant 0.1$ and $|\varepsilon_{ph}| \leqslant 0.02$, significant smaller appropriate solution zones result.

Diagrams of refractive indices available to get a specified phase retardation are obtained simply for any substrates of given complex refractive index by using the analytical relations given above.

4. Simple relations for one-reflection systems on non-absorbing substrates at specified phase retardation

Let us consider a transparent substrate of real refractive index n and effective refractive indices p and q for s and p polarizations, respectively, at given angle of incidence θ_0.

At $\Delta = 0$ one obtains for the real coefficients A, B and C of Eq. (13), the following simple relations in the case of single-layer coatings:

$$A_{01} = (p_1^2 q_0 q - p_0 p q_1^2)[(p_0 q_1 + p_1 q_0)(p_1 q - p q_1) + (n_1^2 + n^2)(p_0 q_1 - p_1 q_0)], \quad (26a)$$
$$B_{01} = 0, \quad (26b)$$
$$C_{01} = n_1^2 (p_0 q - p q_0)[(p_0 q_1 + p_1 q_0)(p_1 q - p q_1) - (n_1^2 + n^2)(p_0 q_1 - p_1 q_0)]. \quad (26c)$$

In the case of specific symmetrical triple-coatings one obtains at $\Delta = 0$:

$$A_{03} = (p_1^4 q_0 q_2^2 q - p_0 p_2^2 p q_1^4)[(p_0 p_2 q_1^2 + p_1^2 q_0 q_2)(p_1^2 q_2 q - p_2 p q_1^2) + (n_1^4 + n_2^2 n^2)(p_0 p_2 q_1^2 - p_1^2 q_0 q_2), \quad (27a)$$
$$B_{03} = 0, \quad (27b)$$
$$C_{03} = (n_1^4 n_2^2 (p_0 q - p q_0)[(p_0 p_2 q_1^2 + p_1^2 q_0 q_2)(p_1^2 q_2 q - p_2 p q_1^2) - (n_1^4 + n_2^2 n^2)(p_0 p_2 q_1^2 - p_1^2 q_0 q_2). \quad (27c)$$

At $\Delta = \pi$ (for half-wave retarders) one obtains for single-layer coatings:

$$A_{H1} = (n_1^4 - n_0^2 n^2)[(n_0^2 + n_1^2)(n_1^2 - n^2) + (n_0^2 - n_1^2)(p_1 q + p q_1), \quad (28a)$$
$$B_{H1} = 0, \quad (28b)$$
$$C_{H1} = n_1^2 (n_0^2 - n^2)[(n_0^2 + n_1^2)(n_1^2 - n^2) - (n_0^2 - n_1^2)(p_1 q + p q_1), \quad (28c)$$

and for triple-layer coatings:

$$A_{H3} = (n_1^8 - n_0^2 n_2^4 n^2)[(n_1^4 + n_0^2 n_2^2)(n_1^4 - n_2^2 n^2) + (n_1^2 n_2^2 - n_1^4)(p_1^2 q_2 q + p_2 p q_1^2)], \quad (29a)$$

$$B_{H3} = 0, \tag{29b}$$
$$C_{H3} = n_1^4 n_2^2 (n_0^2 - n^2)\left[(n_1^4 + n_0^2 n_2^2)(n_1^4 - n_2^2 n^2) - (n_0^2 n_2^2 - n_1^4)(p_1^2 q_2 q + p_2 p q_1^2)\right]. \tag{29c}$$

Because $B = 0$ at $\Delta = 0$ and π for single-layer and specific triple-layer coatings, Eq. (13) has the following solutions:

$$t_{\pm} = \pm (C/A)^{1/2} \tag{30}$$

and then

$$\gamma_{k+} + \gamma_{k-} = \pi \text{ or} \tag{31a}$$

$$\zeta_{k+} + \zeta_{k-} = 1 \tag{31b}$$

where $k = 1$ and 2 for single- and triple-layer coatings, respectively. Relation (31b) was found also numerically [5] for a double-layer coating.

At $\Delta = \pm\pi/2$ (for quarterwave retarders) one obtains for coefficients A and C of Eq. (13):

$$A_{Qi} = (A_{0i} + A_{Hi})/2, \tag{32a}$$
$$C_{Qi} = (C_{0i} + C_{Hi})/2, \quad i = 1,\ 3 \tag{32b}$$

where A_{0i}, C_{0i}, A_{Hi} and C_{Hi}, $i = 1, 3$, are given by Eqs. (26) and (28) for single-layer coatings ($i = 1$) and by Eqs. (27) and (29) for specific triple-layer coatings ($i = 3$). For the coefficient B of Eq. (13), one obtains in the case of single-layer coatings:

$$B_{Q1} = \pm(1/2)(p_1 - p)(q_1 - q)\left[(n_0^2 + n_1^2)(n_1^2 - n^2)(p_0 q_1 - p_1 q_0) + (n_0^2 - n_1^2)(p_1 q - p q_1)(p_0 q_1 + p_1 q_0)\right] \tag{33}$$

and of specific triple-layer coatings

$$B_{Q3} = \pm(1/2)(p_1^2 - p_2 p)(q_1^2 - q_2 q)\left[(n_1^4 + n_0^2 n_2^2)(n_1^4 - n_2^2 n^2)(p_0 p_2 q_1^2 - p_1^2 q_0 q_2) + (n_0^2 n_2^2 - n_1^4)(p_1^2 q_0 q - p_2 p q_1^2)(p_0 p_2 q_1^2 + p_1^2 q_0 q_2)\right]. \tag{34}$$

The real coefficients A, B and C can also be deduced directly from the general relations (21)–(25) up to a constant factor of 1/32.

It should be noted that very narrow solution zones of refractive indices are obtained in the case of non-absorbing one-reflection phase retarding systems. The simple relations presented are useful to reduce the time of computation and to understand the comportment of layer-phase thickness solutions at $\Delta = 0$ and π.

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