Optica Applicata, Vol. XLVIII, No. 2, 2018
DOI: 10.5277/oa180205

A semi-analytic approach to calculating the Strehl ratio for a circularly symmetric system. Part 2: dynamic wavefront

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Close-form expression for the Strehl ratio calculated in the spatial frequency domain of the optical transfer function (SOTF) is considered for the case of time-varying dynamic optical system that has circular symmetry. Specifically, closed-form expressions for the temporally averaged SOTF are considered, which can be easily evaluated numerically (what we call a semi-analytic solution). As for the case of a static wavefront, described in Part 1 of this work, it is shown that the proposed methods are computationally more efficient than the commonly used approach based on the discrete Fourier transform.

Keywords: image quality assessment, optical transfer function, longitudinal vibrations, mathematical methods in physics, finite analogs of Fourier transforms.

1. Introduction

The Strehl ratio is one of the fundamental measures of image quality of an optical system but as such has mostly evaluated, as described in Part 1 of this work [1], for a static wavefront. In general, the Strehl ratio of a dynamic optical systems, whose optical transfer function (OTF) varies in time t can be expressed as

$$\mathrm{SOTF}_t = \frac{\int_{-\infty}^{\infty} \mathrm{d}f_x \int_{-\infty}^{\infty} \mathrm{d}f_y \, \mathrm{OTF}_t(f_x, f_y)}{\int_{-\infty}^{\infty} \mathrm{d}f_x \int_{-\infty}^{\infty} \mathrm{d}f_y \, \mathrm{OTF}_{\mathrm{DL}}(f_x, f_y)} \tag{1}$$

where the subscript DL refers to the case of an aberration-free time-varying diffraction limited system.

The time-dependent SOTF, although intuitively having some potential application, has not attracted much attention. On the other hand, there has been some interest in evaluating image quality of an optical system over a period of time in which the measure of quality is temporally integrated [2–4]. A specific case of such an application considers an optical system in which longitudinal vibrations occur. In 1965, LOHMANN and PARIS reported higher performance of a defocussed system in the presence of longitudinal vibrations when compared to a static system with the same amount of defocus [5]. Recently, a postulate has been made that such a scenario might be occurring also in the human eye [6] and be linked to accommodation microfluctuations [7].

In this work, we aim to arrive at close-form expressions for the temporally averaged Strehl ratio of a circularly symmetric optical system and analyze its computational performance in comparison to that derived using discrete Fourier transform.

2. Preliminaries

Recall, see [1], that the *diffraction integral* can be calculated as

$$\begin{aligned} U(r,\varphi) &= \frac{1}{\pi}\int_0^1 \mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\theta\, P(\rho,\theta)\exp\Big(2\pi i r\rho\cos(\theta-\varphi)\Big)\rho \\ &= \frac{1}{\pi}\mathcal{F}_2^{-1}[P](r,\varphi) \end{aligned} \tag{2}$$

where

$$P(\rho,\theta) = A(\rho,\theta)\exp\Big(-i\frac{2\pi n}{\lambda}W(\rho,\theta)\Big) \tag{3}$$

with $A(\rho,\theta) = \chi_{[0,1]}(\rho)$ being uniform circular pupil function and $W(\rho,\theta)$ being the wavefront error of a system. Here and in what follows, $\mathcal{F}_2$ and $\mathcal{F}_2^{-1}$ stand for the 2D direct and inverse Fourier transforms, respectively. Accordingly, the point spread function (PSF) and the OTF are given by:

$$\mathrm{PSF}(r,\varphi) = \big|U(r,\varphi)\big|^2 = \frac{1}{\pi^2}\Big|\mathcal{F}_2^{-1}[P](r,\varphi)\Big|^2 \tag{4}$$

and

$$\begin{aligned} \mathrm{OTF}(u,v) &= \frac{1}{\pi^2}\mathcal{F}_2\Big[\mathcal{F}_2^{-1}[P]\overline{\mathcal{F}_2^{-1}[P]}\Big](u,v) \\ &= \frac{1}{\pi^2}(P \star P)(u,v) \\ &= \frac{1}{\pi^2}\iint_{\mathbb{R}^2}\mathrm{d}x\mathrm{d}y\, P(x,y)\overline{P(x-u,y-v)} \end{aligned} \tag{5}$$

where $\star$ is the convolution operation and the bar denotes complex conjugation.

Finally, the Strehl ratio based on the OTF (*the SOTF metric*) in the radially symmetric case is given by (see [1]) the average in the frequency domain of the calculated OTF

$$\mathrm{SOTF} = \iint_{\mathbb{R}^2} \mathrm{d}u\,\mathrm{d}v\,\mathrm{OTF}(u, v) \tag{6}$$

3. Averaging the radially-symmetric aberrations

Assume that $W_t(\rho, \theta)$ is a radially symmetric wavefront, depending on a parameter t, and periodic in t with period $T = 1$. In consequence, the pupil function $P(\rho, \theta)$ depends on t, $P(\rho, \theta) = P_t(\rho, \theta)$ and this yields the dependence on t for the rest of the functions (U, PSF, OTF and finally the SOTF), which we denote by subindex t.

The first step is to give a meaning to the notion of "averaged SOTF". A direct examination of the expressions in Section 2 shows that there are only two alternatives:

1) Replacing U_t with its average,

$$\hat{U}(r, \varphi) = \int_0^1 \mathrm{d}t\,U_t(r, \varphi) \tag{7}$$

which, by linearity in P_t, is equivalent to replacing P_t by its average:

$$\hat{U}(r, \varphi) = \hat{U}(r) = \frac{1}{\pi}\mathcal{H}_0[\hat{P}](r) \tag{8}$$

$$\hat{P}(\rho) = \int_0^1 \mathrm{d}t\,P_t(\rho, \theta) \tag{9}$$

with $\mathcal{H}_0[P](r)$ being the Hankel transform of order 0 of P [8, 9]. The corresponding averaged metric we denote by $\widehat{\mathrm{SOTF}}$.

2) Replacing PSF_t by its average,

$$\widetilde{\mathrm{PSF}}(r, \varphi) = \int_0^1 \mathrm{d}t\,\mathrm{PSF}_t(r, \varphi) = \int_0^1 \mathrm{d}t\left|U_t(r, \varphi)\right|^2 \tag{10}$$

The corresponding averaged metric we denote by $\widetilde{\mathrm{SOTF}}$.

In this part of work, the wavefront error will consist of a linear combination of the second (defocus) and the fourth order (spherical) aberrations, with possible periodic longitudinal vibrations:

$$W(\rho) = f(t)\,\rho^2 + s(t)\,\rho^4 \tag{11}$$

where $f(t) = f_0 + f_1\sin(2\pi t)$ and $s(t) = s_0 + s_1\sin(2\pi t)$, with $t \in [0, 1]$, $f_0, s_0 \in \mathbb{R}$, and $f_1, s_1 \geq 0$. We call f_0 the defocus constant, s_0 the spherical aberration constant and f_1 and s_1 the amplitude of vibration for the defocus and spherical aberration term, respectively. The case where $f_1 = s_1 = 0$ (static wavefront) was considered in the first part of this discourse [1]. Thus, our next goal is finding a simplified closed form expression for $\widehat{\text{SOTF}}$ and $\widetilde{\text{SOTF}}$ in the case (11).

4. Calculating $\widehat{\text{SOTF}}$

Consider the wavefront (11)

$$\begin{aligned} W(\rho) &= f(t)\,\rho^2 + s(t)\,\rho^4 \\ &= \Big(f_0 + f_1 \sin(2\pi t)\Big)\,\rho^2 + \Big(s_0 + s_1 \sin(2\pi t)\Big)\,\rho^4 \\ &= f_0\,\rho^2 + s_0\,\rho^4 + \sin(2\pi t)\Big(f_1\rho^2 + s_1\rho^4\Big) \end{aligned} \tag{12}$$

Using the identity

$$\int_0^1 \mathrm{d}t \exp\left(-i\,\frac{2\pi}{\lambda}\sin(2\pi t)\Big(f_1\rho^2 + s_1\rho^4\Big)\right) = J_0\Big(\frac{2\pi}{\lambda}\Big(f_1\rho^2 + s_1\rho^4\Big)\Big) \tag{13}$$

where $J_n(\cdot)$ is the Bessel function of the first kind of order n, we obtain

$$\hat{P}(\rho) = P_{00}(\rho)\,J_0\Big(\frac{2\pi}{\lambda}\Big(f_1\,\rho^2 + s_1\,\rho^4\Big)\Big) \tag{14}$$

$$P_{00}(\rho) := \chi_{[0,\,1]}(\rho)\exp\Big(-i\,\frac{2\pi}{\lambda}\Big(f_0\,\rho^2 + s_0\,\rho^4\Big)\Big) \tag{15}$$

Hence, we are basically comparing

$$\hat{U}(r) = \frac{1}{\pi}\mathcal{H}_0\Big[P_{00}(\rho)\,J_0\Big(\frac{2\pi}{\lambda}\Big(f_1\rho^2 + s_1\rho^4\Big)\Big)\Big](r) \tag{16}$$

with $U_{00}(r) = \frac{1}{\pi}\mathcal{H}_0\Big[P_{00}(\rho)\Big](r)$.

Keeping in mind that

$$\left|J_0\Big(\frac{2\pi}{\lambda}\Big(f_1\rho^2 + s_1\rho^4\Big)\Big)\right| \le 1 \tag{17}$$

which is consistent with the idea of $\hat{U}$ being a "regularization" of U_{00} and using (see Eq. (15) in [1])

$$\text{OTF}(\rho) = \frac{1}{\pi^2}\iint_{S_\rho} \mathrm{d}x\mathrm{d}y\, P\Big(x + \frac{\rho}{2}, y\Big)\,\overline{P\Big(x - \frac{\rho}{2}, y\Big)} \tag{18}$$

with

$$S_\rho = \left\{(x, y) \in \mathbb{R}^2\colon \left(x + \frac{\rho}{2}\right)^2 + y^2 \le 1\right\} \cap \left\{(x, y) \in \mathbb{R}^2\colon \left(x - \frac{\rho}{2}\right)^2 + y^2 \le 1\right\} \quad (19)$$

we obtain

$$\begin{aligned}\widehat{\mathrm{OTF}}(\rho) &= \frac{1}{\pi^2}\iint_{S_\rho} \mathrm{d}x\mathrm{d}y\, \hat{P}\left(x + \frac{\rho}{2}, y\right) \overline{\hat{P}\left(x - \frac{\rho}{2}, y\right)} \\ &= \frac{4}{\pi^2}\mathrm{Re}\left(\int_0^{1-\rho/2} \mathrm{d}x \int_0^{\sqrt{1-(x+\rho/2)^2}} \mathrm{d}y\, V(x, y, \rho)\right)\end{aligned} \quad (20)$$

with

$$V(x, y, \rho) := \exp\left(i\frac{2\pi}{\lambda}\left(W\left(\sqrt{\left(x - \frac{\rho}{2}\right)^2 + y^2}\right) - W\left(\sqrt{\left(x + \frac{\rho}{2}\right)^2 + y^2}\right)\right)\right) \quad (21)$$

Hence,

$$\begin{aligned}\widehat{\mathrm{OTF}}(\rho) &= \frac{4}{\pi^2}\mathrm{Re}\Bigg(\int_0^{1-\rho/2} \mathrm{d}x \int_0^{\sqrt{1-(x+\rho/2)^2}} \mathrm{d}y\, \exp\left(-i\frac{2\pi}{\lambda}\rho x\left(2f_0 + s_0\left(\rho^2 + 4(x^2 + y^2)\right)\right)\right) \\ &\quad \times J_0\left(\frac{2\pi}{\lambda}\left(f_1\left(\left(x - \frac{\rho}{2}\right)^2 + y^2\right) + s_1\left(\left(x - \frac{\rho}{2}\right)^2 + y^2\right)^2\right)\right) \\ &\quad \times J_0\left(\frac{2\pi}{\lambda}\left(f_1\left(\left(x + \frac{\rho}{2}\right)^2 + y^2\right) + s_1\left(\left(x + \frac{\rho}{2}\right)^2 + y^2\right)^2\right)\right)\Bigg) \\ &= \frac{4}{\pi^2}\int_0^{1-\rho/2} \mathrm{d}x \int_0^{\sqrt{1-(x+\rho/2)^2}} \mathrm{d}y\, \cos\left(\frac{2\pi}{\lambda}\rho x\left(2f_0 + s_0\left(\rho^2 + 4(x^2 + y^2)\right)\right)\right) \\ &\quad \times J_0\left(\frac{2\pi}{\lambda}\left(f_1\left(\left(x - \frac{\rho}{2}\right)^2 + y^2\right) + s_1\left(\left(x - \frac{\rho}{2}\right)^2 + y^2\right)^2\right)\right) \\ &\quad \times J_0\left(\frac{2\pi}{\lambda}\left(f_1\left(\left(x + \frac{\rho}{2}\right)^2 + y^2\right) + s_1\left(\left(x + \frac{\rho}{2}\right)^2 + y^2\right)^2\right)\right)\end{aligned} \quad (22)$$

Replacing it in (6), we obtain

$$\begin{aligned}\widehat{\mathrm{SOTF}} &= \frac{8}{\pi}\int_0^2 \mathrm{d}\rho \int_0^{1-\rho/2} \mathrm{d}x \int_0^{\sqrt{1-(x+\rho/2)^2}} \mathrm{d}y\, Q_{f_0, f_1, s_0, s_1}(x, y, \rho)\rho \\ &= \frac{8}{\pi}\int_0^1 \mathrm{d}x \int_0^{2(1-x)} \mathrm{d}\rho \int_0^{\sqrt{1-(x+\rho/2)^2}} \mathrm{d}y\, Q_{f_0, f_1, s_0, s_1}(x, y, \rho)\rho \\ &= \frac{8}{\pi}\int_0^1 \mathrm{d}x \int_0^{\sqrt{1-x^2}} \mathrm{d}y \int_0^{2(\sqrt{1-y^2}-x)} \mathrm{d}\rho\, Q_{f_0, f_1, s_0, s_1}(x, y, \rho)\rho \end{aligned} \tag{23}$$

with

$$\begin{aligned} Q_{f_0, f_1, s_0, s_1}(x, y, \rho) &:= \cos\left(\frac{2\pi}{\lambda}\rho x\left(2f_0 + s_0\left(\rho^2 + 4(x^2+y^2)\right)\right)\right) \\ &\times J_0\left(\frac{2\pi}{\lambda}\left(f_1\left(\left(x-\frac{\rho}{2}\right)^2 + y^2\right) + s_1\left(\left(x-\frac{\rho}{2}\right)^2 + y^2\right)^2\right)\right) \\ &\times J_0\left(\frac{2\pi}{\lambda}\left(f_1\left(\left(x+\frac{\rho}{2}\right)^2 + y^2\right) + s_1\left(\left(x+\frac{\rho}{2}\right)^2 + y^2\right)^2\right)\right) \end{aligned} \tag{24}$$

Equation (23) provides a semi-analytic solution for the temporally averaged SOTF when vibrations in wavefront error have a sinusoidal character and when the averaging occurs in the pupil plane.

5. Calculating $\widetilde{\mathrm{SOTF}}$

Using (18),

$$\widetilde{\mathrm{OTF}}(\rho) = \frac{1}{\pi^2}\iint_{S_\rho} \mathrm{d}x\mathrm{d}y\, \tilde{V}(x, y, \rho) \tag{25}$$

where

$$\tilde{V}(x, y, \rho) := \int_0^1 \mathrm{d}t \exp\left(i\frac{2\pi}{\lambda}\left(W_t\left(\sqrt{\left(x-\frac{\rho}{2}\right)^2 + y^2}\right) - W_t\left(\sqrt{\left(x+\frac{\rho}{2}\right)^2 + y^2}\right)\right)\right) \tag{26}$$

is given now by the average. Observe that

$$\tilde{V}(x, y, \rho) + \tilde{V}(-x, y, \rho) = 2\mathrm{Re}\left(\tilde{V}(x, y, \rho)\right) \tag{27}$$

$$\tilde{V}(x, -y, \rho) = \tilde{V}(x, y, \rho) \tag{28}$$

Thus, by case (11),

$$\begin{aligned}
\tilde{V}(x, y, \rho) &= \int_0^1 \mathrm{d}t \exp\left(i\frac{2\pi}{\lambda}\left(W_t\left(\sqrt{\left(x-\frac{\rho}{2}\right)^2+y^2}\right) - W_t\left(\sqrt{\left(x+\frac{\rho}{2}\right)^2+y^2}\right)\right)\right) \\
&= \int_0^1 \mathrm{d}t \exp\left(-i\frac{2\pi}{\lambda}\rho x\left(2f(t)+s(t)\left(\rho^2+4(x^2+y^2)\right)\right)\right) \\
&= \int_0^1 \mathrm{d}t \exp\left(-i\frac{2\pi}{\lambda}\rho x\left(2\left(f_0+f_1\sin(2\pi t)\right)\right.\right. \\
&\qquad \left.\left. + \left(s_0+s_1\sin(2\pi t)\right)\left(\rho^2+4(x^2+y^2)\right)\right)\right) \\
&= \exp\left(-i\frac{2\pi}{\lambda}\rho x\left(2f_0+s_0\left(\rho^2+4(x^2+y^2)\right)\right)\right) \\
&\quad \times\int_0^1 \mathrm{d}t \exp\left(-i\frac{2\pi}{\lambda}\rho x\, 2f_1\sin(2\pi t)\right) \\
&\quad \times\int_0^1 \mathrm{d}t \exp\left(-i\frac{2\pi}{\lambda}\rho x s_1\sin(2\pi t)\left(\rho^2+4(x^2+y^2)\right)\right) \\
&= \exp\left(-i\frac{2\pi}{\lambda}\rho x\left(2f_0+s_0\left(\rho^2+4(x^2+y^2)\right)\right)\right) J_0\left(\frac{4\pi}{\lambda}f_1\rho x\right) \\
&\quad \times J_0\left(\frac{2\pi}{\lambda}s_1\rho x\left(\rho^2+4(x^2+y^2)\right)\right) \qquad (29)
\end{aligned}$$

Then,

$$\widetilde{\mathrm{OTF}}(\rho, \theta) = \frac{4}{\pi^2}\mathrm{Re}\left(\int_0^{1-\rho/2}\mathrm{d}x\int_0^{\sqrt{1-(x+\rho/2)^2}}\mathrm{d}y\,\tilde{V}(x, y, \rho)\right) \qquad (30)$$

Finally,

$$\begin{aligned}
\widetilde{\mathrm{SOTF}} &= \frac{8}{\pi}\int_0^2\mathrm{d}\rho\int_0^{1-\rho/2}\mathrm{d}x\int_0^{\sqrt{1-(x+\rho/2)^2}}\mathrm{d}y\,Q_{f_0, f_1, s_0, s_1}(x, y, \rho)\rho \\
&= \frac{8}{\pi}\int_0^1\mathrm{d}x\int_0^{\sqrt{1-x^2}}\mathrm{d}y\int_0^{2(\sqrt{1-y^2}-x)}\mathrm{d}\rho\,Q_{f_0, f_1, s_0, s_1}(x, y, \rho)\rho \qquad (31)
\end{aligned}$$

with

$$\begin{aligned}
Q_{f_0, f_1, s_0, s_1}(x, y, \rho) &:= \cos\left(\frac{2\pi}{\lambda}\rho x\left(2f_0+s_0\left(\rho^2+4(x^2+y^2)\right)\right)\right) \\
&\qquad \times J_0\left(\frac{4\pi}{\lambda}\rho x f_1\right) J_0\left(\frac{2\pi}{\lambda}\rho x s_1\left(\rho^2+4(x^2+y^2)\right)\right) \qquad (32)
\end{aligned}$$

Equation (31) provides a semi-analytic solution for the temporally averaged SOTF when vibrations in wavefront error have a sinusoidal character and when the averaging occurs in the image plane.

6. Special cases

6.1. Averaging pure defocus

As in [5], we will assume a wavefront of the form:

$$W_t(\rho, \theta) = f(t)\rho^2, \quad f(t) = f_0 + f_1 \sin(2\pi t), \quad f_0 \in \mathbb{R}, \quad f_1 \geq 0 \tag{33}$$

Since this is a special case of (11), we can obtain the necessary expressions by particularizing the results from Sections 4 and 5. This yields

$$\begin{aligned} \widehat{\mathrm{SOTF}} &= \frac{8}{\pi}\int_0^2 \mathrm{d}\rho \int_0^{1-\rho/2} \mathrm{d}x \int_0^{\sqrt{1-(x+\rho/2)^2}} \mathrm{d}y\, Q_{f_0,f_1}(x,y,\rho)\rho \\ &= \frac{8}{\pi}\int_0^1 \mathrm{d}x \int_0^{\sqrt{1-x^2}} \mathrm{d}y \int_0^{2(\sqrt{1-y^2}-x)} \mathrm{d}\rho\, Q_{f_0,f_1}(x,y,\rho)\rho \end{aligned} \tag{34}$$

with

$$\begin{aligned} Q_{f_0,f_1}(x,y,\rho) &:= \cos\left(\frac{4\pi}{\lambda}f_0\rho x\right) J_0\left(\frac{2\pi}{\lambda}f_1\left(\left(x-\frac{\rho}{2}\right)^2+y^2\right)\right) \\ &\quad \times J_0\left(\frac{2\pi}{\lambda}f_1\left(\left(x+\frac{\rho}{2}\right)^2+y^2\right)\right) \end{aligned} \tag{35}$$

Figure 1 shows simulation results for $\widehat{\mathrm{SOTF}}$ as a function of the vibration amplitude f_1 with W given by (33) and $f_0 = 0$ and f_0 taking values between 0 and 1.5 diopters.

Similarly, for $\widetilde{\mathrm{SOTF}}$, we see that using (18),

$$\widetilde{\mathrm{OTF}}(\rho) = \frac{1}{\pi^2}\iint_{S_\rho} \mathrm{d}x\mathrm{d}y\, \tilde{V}(x,y,\rho) \tag{36}$$

where

$$\tilde{V}(x,y,\rho) := \int_0^1 \mathrm{d}t \exp\left(i\frac{2\pi}{\lambda}\left(W_t\left(\sqrt{\left(x-\frac{\rho}{2}\right)^2+y^2}\right) - W_t\left(\sqrt{\left(x+\frac{\rho}{2}\right)^2+y^2}\right)\right)\right) \tag{37}$$

As before,

$$\tilde{V}(x,y,\rho) + \tilde{V}(-x,y,\rho) = 2\mathrm{Re}\left(\tilde{V}(x,y,\rho)\right) \tag{38}$$

$$\tilde{V}(x,-y,\rho) = \tilde{V}(x,y,\rho) \tag{39}$$

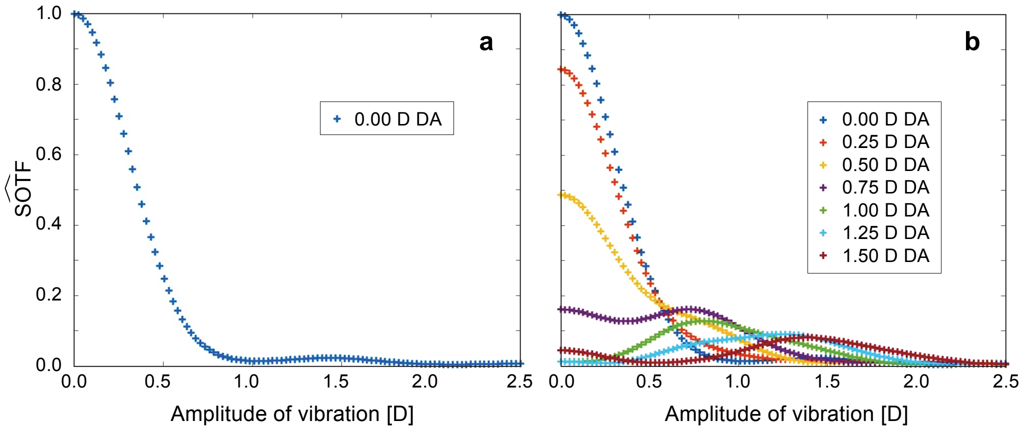

Fig. 1. $\widehat{\text{SOTF}}$ for W given by (33) and $f_0 = 0$ (**a**), and as a function of f_1, for different values of defocus aberration f_0, denoted by DA (**b**).

and we conclude that in this case

$$\widetilde{\text{OTF}}(\rho, \theta) = \frac{4}{\pi^2}\text{Re}\left(\int_0^{1-\rho/2} \mathrm{d}x \int_0^{\sqrt{1-(x+\rho/2)^2}} \mathrm{d}y\, \tilde{V}(x, y, \rho)\right) \tag{40}$$

In the case (33),

$$\begin{aligned}
\tilde{V}(x, y, \rho) &= \int_0^1 \mathrm{d}t \exp\left(i\frac{2\pi}{\lambda}\left(W_t\left(\sqrt{\left(x-\frac{\rho}{2}\right)^2+y^2}\right) - W_t\left(\sqrt{\left(x+\frac{\rho}{2}\right)^2+y^2}\right)\right)\right) \\
&= \int_0^1 \mathrm{d}t \exp\left(-i\frac{4\pi}{\lambda}f(t)x\rho\right) \\
&= \exp\left(-i\frac{4\pi}{\lambda}f_0 x\rho\right)\int_0^1 \mathrm{d}t \exp\left(-i\frac{4\pi}{\lambda}f_1 x\rho\sin(2\pi t)\right) \\
&= \exp\left(-i\frac{4\pi}{\lambda}f_0 x\rho\right) J_0\left(\frac{4\pi}{\lambda}f_1 x\rho\right)
\end{aligned} \tag{41}$$

By (40),

$$\widetilde{\text{OTF}}(\rho, \theta) = \frac{4}{\pi^2}\int_0^{1-\rho/2} \mathrm{d}x \sqrt{1-\left(x+\frac{\rho}{2}\right)^2} \cos\left(\frac{4\pi}{\lambda}f_0 x\rho\right) J_0\left(\frac{4\pi}{\lambda}f_1 x\rho\right) \tag{42}$$

This last equation has the same structure than the OTF that was calculated by LOHMANN and PARIS [5], using the same wavefront (33). Finally, noting that for a circularly symmetric system [1],

$$\mathrm{SOTF} = 2\pi\int_0^{+\infty} \mathrm{d}\rho\, \mathrm{OTF}(\rho)\rho \tag{43}$$

we get

$$\begin{aligned}\widetilde{\mathrm{SOTF}} &= \frac{8}{\pi}\int_0^2 \mathrm{d}\rho \int_0^{1-\rho/2} \mathrm{d}x \sqrt{1-\left(x+\frac{\rho}{2}\right)^2} \cos\left(\frac{4\pi}{\lambda} f_0 \rho x\right) J_0\left(\frac{4\pi}{\lambda} f_1 x \rho\right)\rho \\ &= \frac{8}{\pi}\int_0^1 \mathrm{d}x \int_0^{2(1-x)} \mathrm{d}\rho \sqrt{1-\left(x+\frac{\rho}{2}\right)^2} \cos\left(\frac{4\pi}{\lambda} f_0 \rho x\right) J_0\left(\frac{4\pi}{\lambda} f_1 x \rho\right)\rho\end{aligned} \tag{44}$$

The $\widehat{\mathrm{SOTF}}$ has also been computed using the traditional method of discrete Fourier transform achieving practically the same results. In order to appreciate the closeness between the two representations root mean square error (RMSE) was calculated. Figure 2 shows the calculated RMSE between the close-form solution for $\widehat{\mathrm{SOTF}}$ given in (23) and the discrete Fourier transform method as a function of defocus and the amplitude of vibration.

An exemplary execution times for $\widehat{\mathrm{SOTF}}$ were 628.12 and 3.37×10^3 seconds, using the semi-analytic approach and the discrete Fourier transform method, respectively. This makes the former over five times faster than the latter method. The computational efficiency of the semi-analytic solution is even more striking in the case of $\widetilde{\mathrm{SOTF}}$ where, for example, the execution times are 4.87 and 3.47×10^3 seconds, respectively.

6.2. Averaging spherical aberrations (with fixed defocus)

Here, we assume the wavefront of a very specific form

$$W_t(\rho, \theta) = f\rho^2 + s(t)\rho^4, \quad s(t) = s_0 + s_1 \sin(2\pi t), \quad s_0 \in \mathbb{R}, \quad s_1 \geq 0 \tag{45}$$

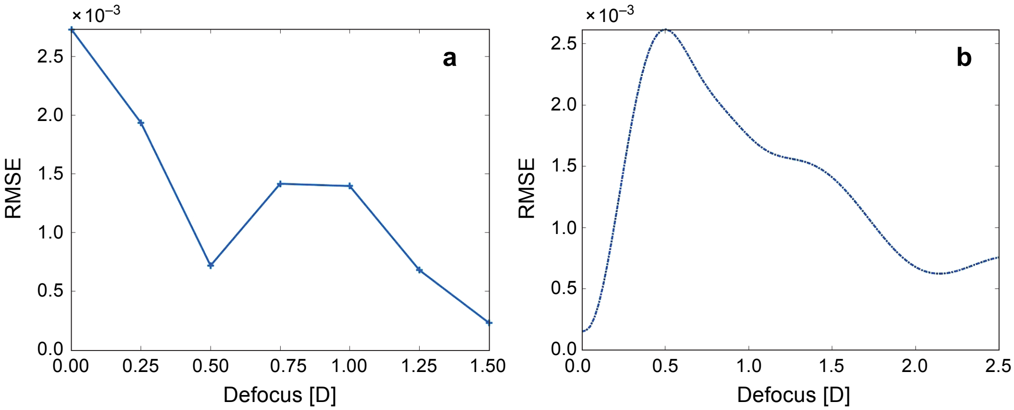

Fig. 2. RMSE of $\widehat{\mathrm{SOTF}}$ as function of the defocus f_0 (**a**) and vibration amplitude f_1 (**b**).

where $f \in \mathbb{R}$ is the defocus. Again, this is a special case of (11), and we obtain the necessary expressions by particularizing the results from Sections 4 and 5:

$$\begin{aligned}\widehat{\mathrm{SOTF}} &= \frac{8}{\pi}\int_0^2 \mathrm{d}\rho \int_0^{1-\rho/2} \mathrm{d}x \int_0^{\sqrt{1-(x+\rho/2)^2}} \mathrm{d}y\, Q_{f,s_0,s_1}(x,y,\rho)\rho \\ &= \frac{8}{\pi}\int_0^1 \mathrm{d}x \int_0^{2(1-x)} \mathrm{d}\rho \int_0^{\sqrt{1-(x+\rho/2)^2}} \mathrm{d}y\, Q_{f,s_0,s_1}(x,y,\rho)\rho \\ &= \frac{8}{\pi}\int_0^1 \mathrm{d}x \int_0^{\sqrt{1-x^2}} \mathrm{d}y \int_0^{2(\sqrt{1-y^2}-x)} \mathrm{d}\rho\, Q_{f,s_0,s_1}(x,y,\rho)\rho \end{aligned} \tag{46}$$

with

$$\begin{aligned} Q_{f,s_0,s_1}(x,y,\rho) &:= \cos\left(\frac{2\pi}{\lambda}\rho x\left(2f + 4s_0(x^2+y^2) + \rho^2 s_0\right)\right) \\ &\quad \times J_0\left(\frac{2\pi}{\lambda}s_1\left(\left(x-\frac{\rho}{2}\right)^2+y^2\right)^2\right) J_0\left(\frac{2\pi}{\lambda}s_1\left(\left(x+\frac{\rho}{2}\right)^2+y^2\right)^2\right) \end{aligned} \tag{47}$$

Similarly, following the same step as in the previous case

$$\begin{aligned} \widetilde{\mathrm{SOTF}} &= \frac{8}{\pi}\int_0^2 \mathrm{d}\rho \int_0^{1-\rho/2} \mathrm{d}x \int_0^{\sqrt{1-(x+\rho/2)^2}} \mathrm{d}y \Bigg| \cos\left(\frac{2\pi}{\lambda}\rho x\left(2f + s_0\left(\rho^2 + 4(x^2+y^2)\right)\right)\right) \\ &\quad \times J_0\left(\frac{2\pi}{\lambda}\rho x s_1\left(\rho^2+4(x^2+y^2)\right)\right)\Bigg|\rho \end{aligned} \tag{48}$$

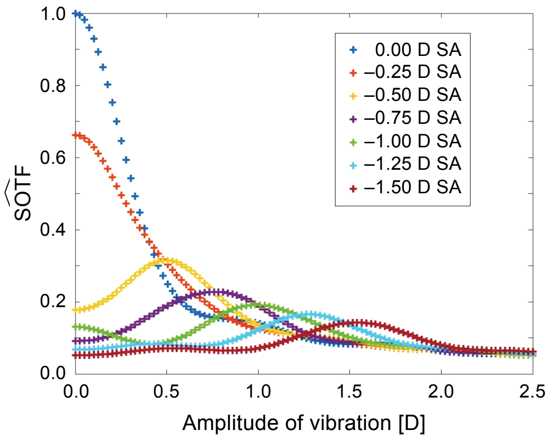

Fig. 3. $\widehat{\mathrm{SOTF}}$ for W given by (45), where defocus $f = 0$ and for different values of spherical aberration (SA) constant s_0 from 0 to –1.5 diopters, as a function amplitude of vibration s_1.

As shown in Fig. 3, it is interesting to note that for a constant defocus and given amount of static spherical aberration s_0 the temporally averaged SOTF in the presence of longitudinal vibration affecting only spherical aberration term one achieves similar amplification of SOTF as in the case of the defocus alone.

The execution times for the case of averaging the SOTF for the spherical aberration in comparison to those achieved with discrete Fourier transformation are still much shorter (*e.g.*, 529.54 *versus* 3.34×10^3 seconds or 404.08 *versus* 3.37×10^3 seconds for the $\widetilde{\text{SOTF}}$).

7. Conclusions

The Strehl ratio is used to assess image quality and this task is typically performed using discrete Fourier transform. As indicated in Part 1 of this discourse [1] such an approach has limitations. The semi-analytical solutions presented here show that finite analogs of Fourier transformations can surpass the computational efficiency of discreet Fourier transform, especially in more complicated case where longitudinal vibrations occur in an optical system and when there is a need to temporally average the Strehl ratio. The solutions presented in this work are restricted to circularly symmetric optical systems and the generalization of the provided solutions will be sought, in the future, to account for other type of systems, where, for example, the wavefront is described by a series of polynomials of order other than two and four.

Acknowledgments – The first and the second author were partially supported by the Spanish Government together with the European Regional Development Fund (ERDF) under grant MTM2014-53963-P from MINECO. Additionally, the second author (AMF) acknowledges the support of Junta de Andalucía (via the Excellence Grant P11- FQM-7276 and the research group FQM-229), and of Campus de Excelencia Internacional del Mar (CEIMAR) of the University of Almería. The third author was supported by the statutory funds from Wrocław University of Science and Technology.

References

[1] CASTAÑO-FERNANDEZ A.B., MARTÍNEZ-FINKELSHTEIN A., ISKANDER D.R., *A semi-analytic approach to calculating the Strehl ratio for a circularly symmetric system. Part 1: static wavefront*, Optica Applicata 48(2), 2017, pp. 201–210.

[2] HOFER H., CHEN L., YOON G.Y., SINGER B., YAMAUCHI Y., WILLIAMS D.R., *Improvement in retinal image quality with dynamic correction of the eye's aberrations*, Optics Express 8(11), 2001, pp. 631–643.

[3] NIGHTINGALE A.M., DUFFIN D.A., LEMMON M., GOODWINE B., JUMPER E.J., *Adaptive-optic correction of a regularized compressible shear layer*, [In] *37th AIAA Plasmadynamics and Lasers Conference, San Francisco, CA, June 5–8, 2006, article ID AIAA-2006-3072*.

[4] GORDEYEV S., CRESS J.A., SMITH A., JUMPER E.J., *Aero-optical measurements in a subsonic, turbulent boundary layer with non-adiabatic walls*, Physics of Fluids 27(4), 2015, article ID 045110.

[5] LOHMANN A.W., PARIS D.P., *Influence on longitudinal vibrations on image quality*, Applied Optics 4(4), 1965, pp. 393–397.

[6] ISKANDER D.R., *Signal processing in visual optics [Life sciences]*, IEEE Signal Processing Magazine 31(4), 2014, pp. 155–158.

[7] CHARMAN W.N., HERON G., *Microfluctuations in accommodation: an update on their characteristics and possible role*, Ophthalmic and Physiological Optics 35(5), 2015, pp. 476–499.
[8] BADDOUR N., *Operational and convolution properties of two-dimensional Fourier transforms in polar coordinates*, Journal of the Optical Society of America A 26(8), 2009, pp. 1767–1777.
[9] BADDOUR N., CHOUINARD U., *Theory and operational rules for the discrete Hankel transform*, Journal of the Optical Society of America A 32(4), 2015, pp. 611–622.

Received July 27, 2017