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Ryszard Gonczarek

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Recenzenci
Maria STĘŚLICKA
Zygmunt GALASIEWICZ

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Jerzy CIEKOT

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Ruta CZAPLIŃSKA

Opracowanie
Ewa SOBESTO

Korekta
Halina CISŁAK
Stanisław GANCARZ
Ewa SOBESTO

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Ryszard GONCZAREK *

QUALITATIVE EFFECTS GENERATED BY FERMI LIQUID INTERACTION IN SUPERCONDUCTING AND SUPERFLUID SYSTEMS

The monograph is devoted to theoretical investigations on properties of superconducting and superfluid systems. The qualitative effects generated by the Fermi liquid interaction are subject of particular interest. The Green function theory, including the presence of a strong magnetic field, has been elaborated. The monograph contains the most essential results defining properties of superconductors, superfluid ^3He and ^3He-^4He mixtures in strong magnetic fields at T = 0 and T close to T_c. Moreover, BCS and BW type systems in the linear response approach have been examined. The developed theory permits to take into account additional pairing harmonics, dipole-dipole interaction, influence of temperature, high frequencies and strong inhomogeneity of a system. The applied approaches allowed us to obtain several qualitative results. In the last part some mathematical methods extended by the author are presented.

LIST OF UNIVERSAL SYMBOLS

a_l, A_l, b_l, F_l^s, F_l^a	- Landau parameters,
F	- Maki and Ebisawa function**,
f_l, g_l	- pairing parameters,
g	- dimensionless dipole contribution parameter,
H	- external magnetic field**,
H_p	- paramagnetic field,
H_T	- total magnetic field,
h	- total magnetic field in energy scale**,
H_c, H_{c1}, H_{c2}	- critical magnetic field,
J, M	- quantum numbers of two-particle states**,
$\underline{k}$	- wave vector,
M, M_p	- paramagnetic magnetization**,
M_d, M	- diamagnetic magnetization,
N	- total number of quasiparticles**,

* Institute of Physics, Technical University of Wrocław, Wybrzeże Wyspiańskiego 27, 50-370 Wrocław.

** most often appearing.

N_n - number of normal quasiparticles,
N_s - number of superfluid quasiparticles,
p_0 - Fermi momentum,
v - Fermi velocity,
v_s - superfluid velocity,
Δ - energy gap,
ε_F - Fermi energy,
$\xi(s)$ - Riemann's ξ-function,
$\theta(x)$ - Heaviside step function,
$\varkappa$ - Ginzburg-Landau parameter,
μ_B - Bohr's magneton,
μ^*_B - Bohr's effective magneton,
$\nu(\xi)$ - density of states with definite spin,
$\nu_0 \equiv 2\nu(0)$
ξ_p - cut-off parameter,
ξ_c - coherence distans,
σ^0 - unit matrix (2×2),
σ^i - Pauli matrices,
χ_{ij} - spin susceptibility tensor,
$\chi_p^0 = \nu_0\,\mu_B^2$ - paramagnetic susceptibility of a free electron gas,
$\chi_d^0 = -1/4\pi$ - diamagnetic susceptibility of a Meissner superconductor,
ω - frequency
ω_0 - Debye characteristic frequency,
$\langle\ldots\rangle$ - averaging over spherical angles,
$\langle\ldots\rangle_{\underline{R}}$ - averaging over $\underline{R}$ - vector space.

INTRODUCTION

Theoretical investigations of the present problems concerning many -body systems are dominated by two research methods, i.e., numerical and analytical ones which can be characterized as follows:

Numerical method comprises standard physical theories, highly complicated numerical computations and numerical results, whereas the analytical one covers non-trivial and still improved physical theories, far advanced mathematical methods and analytical solutions which can be examined in the numerical way. The advantage of the latter method becomes particularly visible while researching small qualitalive effects, since such effects could be misleadingly interpreted on the ground of the numerical results only. Moreover, the analytical method makes it possible to in-spect microscopic processes and to eliminate the insignificant effects taking place in the system under consideration. Hence, the analytical research should precede the numerical elaborations, particularly in fundamental investigations. Besides, extensive application of modern theories to theoretical and experimental investigations of real physical systems requires a fresh approach to the course of problems.

The present monograph is devoted the search for qualitative effects connected with the quasiparticle interactions in the superconducting and superfluid Fermi systems. Hence, the Fermi liquids are the principal objects of our interest. The formalism is developed within Landau's concept of quasiparticles [74-76, 87, 106, 122] which states that the macroscopic properties of the strongly interacting (normal) Fermi liquid may be mapped onto a gas of elementary excitations, the Landau quasiparticles. The interactions of these quasiparticles are described in terms of the scattering amplitude for binary collisions. After permission of the pairing interaction, the quasiparticles are coupled in Cooper's pairs [22, 56, 78, 79].

All the presented considerations are performed in the Green function formalism which is developed in two main directions: the linear response of the system in the most comprehensive form, and the non--linear inclusion of the strong magnetic field.

In the former case we based our considerations on the formalism developed by Larkin, Migdal [79] and Czerwonko [22]. Although their original works concerned the zero-temperature case, Leggett [81] has shown that the whole Larkin and Migdal formalism can be applied in non-zero temperatures if some characteristic functions appearing in the basic equations of the theory are replaced by the appropriate temperature-dependent functions. The possibility of application of the modified Czerwonko formalism to the non-zero temperature case was demonstrated in [39, 40, 43].

The equations formulated by Maki and Ebisawa [93] were also obtained by the microscopic approach and can be resolved into the form of Czerwonko's equations after some transformations connected with the used symbols. All microscopic approaches are then coherent. Since in the presented formalism we do not impose any restrictions on the frequency and the wave vector, the temperature dependence of the formalism is given by means of four characteristic functions, which in turn can be always expressed by the so-called Maki-Ebisawa function which was computed correctly for the first time in [43].

The main advantage of the Green function formalism is manifested in the fact that the particle-hole interaction, i.e., the Fermi liquid interaction and the total particle-particle interaction, which can be composed of the pairing and dipole-dipole interactions, can be included in the straight manner.

The latter developed direction is connected with the reconstruction of the Green function theory in the presence of a constant strong magnetic field. The magnetic field is included by means of the nuclear paramagnetic term in case of the neutral systems and by the Pauli paramagnetic term and the vector potential in case of the charged BCS-systems. In the present approach the static case of the theory is discussed with particular care. Basing on standard properties of the normal and anomalous Green functions and on the fundamental equations of the LMC theory [22, 79] we define principal parameters of the system and find their interrelations. While considering the quasiparticle interactions we state that the inclusion of the dipole-dipole interactions is realized in composition with the pairing interaction by the gap equation, whereas taking account of the Fermi liquid interaction requires a special treatment.

The developed formalism is based on the LMC theory which is valid within the pale of the weak coupling model with the spherical Fermi surface, where all the considerations are performed in the collisionless regime. Thus, these assumptions shift automatically. On the other hand, we simultaneously determine the possibilities of regarding some ordinarily neglected effects such as the particle-hole asymmetry [101, 102], strong coupling corrections, feedback effects [85, 112, 136] and the effects connected with the dependence of the quasiparticle scattering amplitude on the length of the quasiparticle momentum i.e. in the mass operator or the pairing interaction [112-115, 124], which are ordinarily neglected as small effects limitrophe to accuracy of the theory. As we shall show these four kinds of effects can give new qualitative results probably of the same order, thus they should be considered jointly. The developed formalism is kept in the spirit of the Landau quasiparticles. It constitutes consistent reconstruction of the existing microscopic theories [22, 79, 87, 106] in the presence of strong and constant external magnetic field. Therefore we concentrate only on some essential problems which lead to new results and we omit the whole formal discussion of the applied theory which was precisely described in [3, 22, 35, 79, 82, 86, 106]. In order to achieve the intended purpose we apply the self-consistent renormalizing procedure in which superfluid properties of the system are considered also in the presence of the strong magnetic field. The reconstructed quasiparticles conserve general properties of initial particles, though some characteristic parameters can be renormalized. The quasiparticles are still the fermions gifted an effective mass, which conserve the charge. The interaction of the magnetic moment of a charged quasiparticle with the magnetic field differs from the corresponding quantity of a free quasiparticle since the former interaction is realized through the magnetic moment and the motion of the electric charges. Hence in this case, Bohr's magneton must be replaced by its effective value [100]. The individual problem is connected with microscopic estimation of modified parameters. However, complexity of processes occurring in real many-body systems does not allow us to compute them precisely.

The developed theory is employed twofold. We investigate dynamic properties of the superconducting BCS and superfluid BW systems applying the linear response approach when the external additional magnetic field is excluded. The theory after strong magnetic field is included becomes the non-linear theory and is applied to investigate statis systems only. With small modifications it is adapted to the superfluid ^{3}He and almost isotropic superconductors. It can also succeed in ^{3}He-^{4}He mixtures with s-wave-pairing [112].

The principal purpose of this monograph is to show the significance of the Fermi liquid interaction in the superconducting and superfluid systems which are considered with in the frame of the theory based on Landau quasiparticle concept. We study the qualitative effects due to the Fermi liquid interaction with a special solicitude and prove that they are determined by certain values of the Landau parameters. It is indicated moreover, that fundamental properties of the system must be independent of the applied formalism and must be assigned univocally before being revealed in some suitable conditions. We do not introduce any additional parameters and confine ourselves to the ones existing in the initial theories. We consistently develop the Green function formalism making it coherent within all known limits. We also demonstrate that the reconstructed formalism creates great chances to reveal the effects which can occur in the superconducting or superfluid systems if some additional conditions are fulfilled. The suitable remarks are given in comments C. The applied Green function formalism required the development of special mathematical methods which are inserted in the last part of this monograph. We consider only volume properties of the systems and all the presented calculations refer to the unit volume.

GREEN FUNCTION FORMALISM

I. Forms and properties of Green functions

1. Remarks on applicability of the theory

The developed formalism is founded on the concept of the Landau quasiparticles which express the low-lying excitations of the system with a small energy if compared with the Fermi energy. Therefore all the parameters appearing in the theory and expressed in the energy scale must be small also with respect to ε_F. However, the inequalities $T \ll \varepsilon_F$ and $\Delta \ll \varepsilon_F$ are always fulfilled in the superconducting and superfluid systems and we only have to demand the total magnetic field expressed in the energy scale to fulfil the relation $h \ll \varepsilon_F$. As to the linear response of the system it can be considered correctly if the slow-varying field fulfils the conditions $k \ll p_F$ and $\omega \ll \varepsilon_F$ where $\underline{k}$ is the wave vector expressing the inhomogeneity of the external perturbation and ω is the frequency of this perturbation. Within the frame of the applied formalism no extra relations among the parameters Δ , T, h, kv and ω are required. However, if h or ω is of order of Δ then the Cooper pairs are destroyed [44, 45, 50-52], whereas in the Pippard limit the Cooper phenomenon is not observed [22, 39].

2. General principles of the magnetic field inclusion [51]

In order to make the reconstructed formalism reliable, we begin our consideration ab ovo, i.e., from the Hamiltonian. It will not be quoted here in an explicit form. We assume, however, that it is a typical Hamiltonian adequate to the appropriate system and write down only the term which in such a Hamiltonian appears in the presence of the external magnetic field. That additional term has the form

$$\mathcal{H}_E = -\mu_B \sum_{p,\alpha,\beta} \underline{H}\underline{\sigma}_{\alpha\beta}\, a^+_{\underline{p},\alpha} a_{\underline{p},\beta} \tag{1}$$

where μ_B denotes Bohr's magneton, $\underline{H}$ is the external magnetic field and $\underline{\sigma}^i$ are the Pauli matrices [66, 85]. The form (1) of the Hamiltonian allows us to derive directly the Fourier-transform of the one-particle Green function and to obtain the free-particle Green function in the form

$$\underline{G}_0(p,\underline{H}_E) = \left[(\varepsilon - \varepsilon_{\underline{p}})\sigma^0 + \mu_B \underline{H}\,\underline{\sigma} + i0^+ \mathrm{sgn}\,\varepsilon\sigma^0\right]^{-1} \tag{2}$$

where p is four-vector energy ε and momentum $\underline{p}$, ε_p is the particle energy ($\varepsilon_p = (|\underline{p}| - p_0)v$), σ^0 is the unit matrix (2×2). Using the Dyson equation we find the Green function (only its singular part) of the interaction system in the form

$$\underline{G}(p,\underline{h}) = Z\left[(\varepsilon - \xi)\sigma^0 + \underline{h\sigma} + i0^+ \mathrm{sgn}\,\varepsilon\,\sigma^0\right]^{-1} \tag{3}$$

where

$$\xi = Z\left(1 + \frac{m}{p_0}\,\frac{\partial \Sigma^0}{\partial |\underline{p}|}\right)\varepsilon_{\underline{p}}$$

is the free quasi-particle energy measured from the Fermi level,

$$Z = \left(1 - \frac{\partial \Sigma^0}{\partial \varepsilon}\right)^{-1}$$

is the discontinuity of the particle density on the Fermi surface

$$\underline{H}_T = Z\left(1 - \frac{1}{3\mu_B}\,\frac{\partial \Sigma^i}{\partial H_E^i}\right)\underline{H}$$

is the total magnetic field obtained in the linear approximation (the normal state). All the derivatives are taken on the Fermi surface and for $\underline{H} = 0$. The mass operator is of the form

$$\Sigma(p,\underline{H}) = \Sigma^0(p,H^2)\sigma^0 + \Sigma^i(p^2,\underline{H})\,\sigma^i.$$

We also assume that in the superconducting or superfluid system the total magnetic field can be of the energy-gap order and that it fulfils then the relation $0 \leqslant h \leqslant h_0$ where $\Delta \lesssim h_0 \ll \varepsilon_F$. Moreover, we can suspect that the external and total magnetic fields are of the same order, according to which $\underline{H}$ is the strong magnetic field. The function $\underline{G}(p, \underline{h})$ can be transformed into a more convenient form

$$\underline{G}(p,\underline{h}) = Z\,\frac{(\varepsilon - \xi)\sigma^0 - \underline{h\sigma}}{(\varepsilon - \xi + i0^+ \mathrm{sgn}\,\varepsilon)^2 - h^2} \tag{4}$$

from which some retarded and advanced Green functions or the appropriate Matsubara Green functions can be easily obtained. Moreover, from now on the parameter Z defining the discontinuity of the particle density on the Fermi surface will be ignored in all expressions, since it can always be eliminated from the final results in the systematic manner (cf. [51, 53, 86]). Hence, we can put it equal to one in the above expressions.

3. General properties of superconducting and superfluid systems [51]

Although general properties of the normal and anomalous one-particle Green functions of superconducting and superfluid systems can be considered jointly if the vector potential is omitted, the inclusion of strong magnetic field requires a particularly careful treatment. To this end we distinguish the following types of the Green functions: $\underline{G}_n$ is a normal Green function, i.e., the Green function which is renormalized by the Fermi liquid effects in a normal state, $\underline{G}_q$ is a quasi-normal Green function, i.e., the like normal Green function which, however, is renormalized by the Fermi liquid effects appearing in the superconducting (superfluid) state, and $\underline{G}_s$ is the superconducting (superfluid) normal Green function. (Such differentiation of the Green functions $\underline{G}_n$ and $\underline{G}_q$ is not obligatory if a magnetic field is excluded). Moreover $\underline{F}_1$ and $\underline{F}_2$ are the anomalous Green functions connected with creation and anihilation of the quasi-particle pairs, respectively, and Δ_1, Δ_2 are the related matrices being the irreducible parts of the appropriate Dyson equations (cf. [3, 22, 79, 86]). They have the following symmetry properties referred to time inversion

$$\underline{F}_1 = -\underline{F}_2^+ \quad \text{and} \quad \underline{\Delta}_1 = -\underline{\Delta}_2^+ . \tag{1}$$

Using the above symbols the superconducting (superfluid) Dyson equations can be written in the following two quite equivalent forms

$$\begin{aligned} \underline{G}_s &= \underline{G} + \underline{G}\underline{\Delta}_2\underline{F}_1, \quad \underline{G}_s = \underline{F}_2\underline{\Delta}_1\underline{G} + \underline{G}, \\ \underline{F}_1 &= \underline{\bar{G}}\underline{\Delta}_1\underline{G}_s, \quad \underline{F}_1 = \underline{\bar{G}}_s\underline{\Delta}_1\underline{G} , \\ \underline{F}_2 &= \underline{G}\underline{\Delta}_2\underline{\bar{G}}_s, \quad \underline{F}_2 = \underline{G}_s\underline{\Delta}_2\underline{\bar{G}} \end{aligned} \tag{2}$$

where symbol "-" over the normal (quasi-normal) Green function $\underline{G}$ denotes time inversion. All Green functions are complex matrices which in the case of the function $\underline{G}$ is due to the spin dependence. That is why the time inverse operation induces the synchronous conversion of signs of the following quantities: $\underline{\xi}$, $\underline{p}$, $\underline{h}$, $\underline{\sigma}$ and the function $\underline{\bar{G}}$ receives the form

$$\underline{\bar{G}}(\tilde{p}) = -\frac{(\varepsilon+\xi)\sigma^0+\underline{h\sigma}}{(\varepsilon+\xi+i0^+\operatorname{sgn}\varepsilon)^2-h^2} . \tag{3}$$

Appylying the typical procedure to Eqs. (2) we can express the function $\underline{G}_s$ in two equivalent forms

$$\underline{G}_s = [(\underline{G\overline{G}})^{-1} - \underline{\overline{G}}^{-1}\,\underline{\Delta}_2\underline{\overline{G}}\underline{\Delta}_1]^{-1}\underline{\overline{G}}^{-1} \tag{4}$$

and

$$\underline{G}_s = \underline{\overline{G}}^{-1}\left[(\underline{\overline{G}G})^{-1} - \underline{\Delta}_2\,\underline{\overline{G}\,\Delta}_1\underline{\overline{G}}^{-1}\right]^{-1}, \tag{5}$$

which depend only on the well-known functions $\underline{G}$ and two complex matrices $\underline{\Delta}_1$ and $\underline{\Delta}_2$. Using the obtained results, the functions $\underline{F}_1$ and $\underline{F}_2$ can be performed in the analogous way. In order to make our general considerations complete we introduce the gap equation in two quite equivalent forms

$$\underline{\Delta}_2 = \underline{VF}_2, \quad \text{or} \quad \underline{\Delta}_1 = \underline{F}_1\underline{V}, \tag{6}$$

where $\underline{V}$ is the pairing interaction which can be supplemented by the dipole-dipole interaction. We define some extra symbols, namely

$$\underline{\Delta}_2 \equiv \hat{\Delta} \text{ then } \quad \underline{\Delta}_2\,\underline{\Delta}_1 = -\hat{\Delta}\hat{\Delta}^+, \tag{7}$$

and

$$\underline{F}_2 \equiv \hat{F}\,\frac{\underline{\Delta}_2}{\underline{\Delta}} \quad \text{then} \quad \underline{F}_1 \equiv \frac{\underline{\Delta}_1}{\underline{\Delta}}\,\underline{\hat{F}}^+. \tag{8}$$

Although the above formalism is common for superconducting and superfluid systems, the particular considerations can be continued only in the individual way. Nevertheless, the obtained results are coherent and relatively simple. Moreover, the tangible difference between the normal and quasi-normal Green functions becomes subsequently obvious.

4. The normal and anomalous Green functions [51]

a) The neutral BCS system

Hereafter in order to make our calculations simpler we fix the coordinate system so that $\underline{h} = h\hat{z}$, and if possible we return to the general vectorial notation. Such a procedure is fully justified because the final results cannot depend on the coordinate system choice.

We consider the standard BCS system (pure S-pairing) in presence of strong external magnetic field ($\Delta \sim \mu_B H$). The matrix $\hat{\Delta}$ can be taken in the form

$$\hat{\Delta} = \Delta i \sigma^y \tag{1}$$

Using Eqs. (3.1) and the desired commutation rules for Pauli matrices we get

$$\underline{\Delta\overline{G}} = \underline{\overline{G}\Delta} - \underline{\overline{G}}\,[(\varepsilon + \xi)\sigma^0 + h\sigma^z]^{-1} \cdot 2h\sigma^z\underline{\Delta}. \tag{2}$$

Inserting Eq. (2) into Eq. (3.4) and applying Eq. (3.3) after some evaluations we obtain

$$\underline{G}_s(\overset{\approx}{p}) = (\varepsilon_0^2 - E_+^2)^{-1}(\varepsilon_0^2 - E_-^2)^{-1}$$

$$\times\left\{\left[(\varepsilon+\xi)(\varepsilon^2-E^2-h^2) + 2\xi h^2\right]\sigma^0 - \left[(\varepsilon+\xi)^2-h^2 + \Delta^2\right]\right\}\underline{h}\underline{\sigma}\ , \tag{3}$$

where

$$E_\pm = |E\pm h|, \quad E^2 = \xi^2 + \Delta^2, \quad \varepsilon_0 = \varepsilon + i0^+ \mathrm{sgn}\,\varepsilon \tag{4}$$

and according to the introduced convention we replaced $h\sigma^z$ by $\underline{h}\underline{\sigma}$. In similar way, using Eqs. (3.2), (3.8) and (3), we find

$$\hat{F}(\overset{\approx}{p}) = -\Delta(\varepsilon_0^2 - E_+^2)^{-1}(\varepsilon_0^2 - E_-^2)^{-1}$$

$$\times\left[(\varepsilon^2 - E^2 + h^2)\sigma^0 - 2\varepsilon\underline{h}\underline{\sigma}\right]. \tag{5}$$

b) The superfluid system

In order to consider in the possibly most general way the superfluid system with the pure P-pairing in the presence of a strong magnetic field, we must choose the equilibrium state as the non-unitary state [10, 11, 85, 98, 99]. Then the order parameter can be written in the form

$$\hat{\Delta} = \Delta\,\underline{d}\,\underline{\sigma}\,i\sigma^y \tag{6}$$

where the complex vector $\underline{d}$ is a linear function of the unit vector $\hat{p}$, i.e., $d_i = d_{i\alpha}\hat{p}_\alpha$, and the real vector

$$\underline{l} = i\underline{d} \times \underline{d}^* \tag{7}$$

expresses the non-unitarity of the equilibrium state.

We begin our consideration with computing the superfluid Green functions. Using Eqs. (3.3), (3.4) and (6) and the commutation rules for Pauli matrices we can find that

$$\underline{\hat{\Delta}}\,\underline{\bar{G}} = \underline{\bar{G}}\,\underline{\hat{\Delta}} - \underline{\bar{G}}\left[(\varepsilon + \xi)\sigma^0 + h\sigma^z\right]^{-1} 2hd_z\,\Delta\, i\sigma^y. \tag{8}$$

Proceeding in the similar way as before, i.e., inserting Eq. (8) into Eq. (3.4) and taking into account Eq. (3.3), after the arduous and rather complicated evaluations we get the normal Green function in the form

$$G_s(\tilde{\tilde{p}}) = (\varepsilon_0^2-E_+^2)^{-1}(\varepsilon_0^2-E_-^2)^{-1}$$
$$\times \left\{[(\varepsilon+\xi)(\varepsilon^2-E^2-h^2) + 2\xi h^2 - \Delta^2\underline{hl}]\sigma^0\right.$$
$$- [(\varepsilon^2 - E^2 - h^2) + 2\xi(\varepsilon + \xi)]\,\underline{h\sigma}$$
$$\left. + \Delta^2[(\varepsilon+\xi)\underline{l\sigma} + (\underline{hd}^*)(\underline{d\sigma}) + (\underline{hd})\,(\underline{d}^*\underline{\sigma})]\right\} \tag{9}$$

where

$$E_{\pm} = \left[E^2 + h^2 \pm \sqrt{(\Delta^2\underline{l} - 2\xi\underline{h})^2 + 4\Delta^2|\underline{hd}|^2}\right]^{1/2}, \tag{10}$$
$$E = \sqrt{\xi^2 + \Delta^2|\underline{d}|^2},$$

and

$$\varepsilon_0 = \varepsilon + i0^+ \operatorname{sgn}\varepsilon .$$

Again, using Eqs. (3.2) and (9) we derive the anomalous Green function in the form

$$\hat{F}(\tilde{\tilde{p}}) = -\Delta(\varepsilon_0^2 - E_+^2)^{-1}(\varepsilon_0^2 - E_-^2)^{-1}$$
$$\times \left\{-2\varepsilon\,\underline{hd}\sigma^0 + (\varepsilon^2 - E^2 - h^2)\underline{d\sigma} + 2(\underline{dh})(\underline{h\sigma})\right.$$
$$\left. + i\,[(\Delta^2\underline{l} - 2\xi\underline{h}) \times \underline{d}]\cdot\underline{\sigma}\right\}(\underline{d\sigma})^{-1}, \tag{11}$$

where the term $(\underline{d\sigma})^{-1}$ is connected with the chosen form of notation.

c) The charged superconducting system

The problem of the charged superconductor in a strong magnetic field should be considered in connection with space-inhomogeneity of the system. The problem formulated in this way proves to be very complicated and it finds solution only in some specific limit, i.e., in the local limit or in the Pippard limit [35, 124]. Now we consider slightly inhomogeneous systems only (the local limit), for which we formulate two related approaches.

1° The generalized Gorkov approach [54]

In order to derive the explicit forms of the normal and anomalous Green function we have to solve the proper Gorkov equations [35, 57, 124]. While passing to the centre-of-mass coordinate system we first replace the coordinate $\underline{r}_1$ and $\underline{r}_2$ by

$$\underline{r} = \underline{r}_1 - \underline{r}_2 \quad \text{and} \quad \underline{R} = \frac{1}{2}(\underline{r}_1 + \underline{r}_2) \tag{12}$$

and next we perform the Fourier transformation by $\underline{r}$. The new representation is diagonal in momentum with in the local approximation limit, i.e., if we assume that the gradients of the quantities Δ , $\underline{v}_s$ and $\underline{h}$ disappear and the functional dependence on $\underline{R}$ can be made implicit. In general case we must, however, include the space inhomogeneity of the system. This can be done e.g. by means of the perturbation method. The state constructed in this way is inhomogeneous and the fermion pairs possess the non-vanishing total momentum (cf. [37, 80, 89, 124, 138]).

In order to derive the normal and anomalous Green function we repeat the formalism developed in Section 3. Restricting ourselves to the superconducting systems only (pure S-pairing state) the basic equations for the normal and anomalous Green functions reduce to the forms

$$\begin{aligned} \underline{G}_s &= \underline{G} - \underline{G}\,\underline{\Delta}\,\underline{F} , \\ \underline{F} &= \underline{\bar{G}}\,\underline{\Delta}\,\underline{G}_s \end{aligned} \tag{13}$$

where we assume that the order parameter is of the form

$$\underline{\Delta} = -i\Delta\sigma^y . \tag{14}$$

Such an assumption allows us to eliminate the additional differentation, connected with creation and annihilation of the Cooper pairs, by putting

$$\underline{\Delta} \equiv \underline{\Delta}_1 = -\underline{\Delta}_2 = -\hat{\Delta} \quad \text{and} \quad \underline{F} \equiv \underline{F}_1 . \tag{15}$$

Let us note that Eqs. (13) are equivalent to Gorkov's equations [57] and that they can be rewritten in the form

$$\begin{aligned} \underline{G}^{-1}\underline{G}_s + \underline{\Delta}\underline{F} &= 1 , \\ -\underline{\bar{G}}^{-1}\underline{F} + \underline{\Delta}\underline{G}_s &= 0 \end{aligned} \tag{16}$$

where the quasi-normal Green function should be taken in the following form

$$\underline{G} = [(\varepsilon_0 - \xi - \underline{p}\underline{v}_s)\sigma^0 + \underline{h\sigma}]^{-1} \tag{17}$$

and

$$\begin{aligned} &\underline{v}_s = \nabla\chi - \frac{e}{m}\underline{A} , \qquad \varepsilon_0 = \varepsilon + i0^+ \mathrm{sqn}\,\varepsilon , \\ &\xi = \frac{p^2}{2m} - \mu' , \quad \mu' = \mu - \frac{mv_s^2}{2} , \quad \mathrm{rot}\ \underline{v}_s = -\frac{e}{m}\underline{H} \end{aligned} \tag{18}$$

where $\underline{v}_s$ is superfluid velocity and χ is the phase of the order parameter. We emphasize that the superconducting velocity is the gauge invariant due to its physical meaning in opposition to χ and $\underline{\Delta}$. Let us note that now the function $\underline{G}$ should be identified with the quasi-normal Green function $\underline{G}_q$ since the parameters $\underline{v}_s$, χ , $\underline{h}$ are connected with the superconducting state i.e. by the Fermi liquid interaction and the electron-phonon interaction which is consistently omitted (cf. [37, 80, 89, 124, 138]). Moreover, we assume that now the symbol "-" over the quasi-normal Green function denotes the time inversion of the microscopic variables only, then we have (cf. [22, 79])

$$\bar{\underline{G}} = \left[(-\varepsilon_0 - \xi + \underline{p}\underline{v}_s)\sigma^0 + h\right]^{-1}. \tag{19}$$

The assumed form of the Green function $\underline{G}$ allows us to eliminate the phase dependence from all the other functions $\underline{G}_s$, $\underline{F}$ and $\underline{\Delta}$ in Eqs. (13) and (16) which become the real matrix functions. Hereafter we again assume that the external magnetic field is chosen along z axis, whereas the direction of the superfluid velocity is quite arbitrary. Now applying the previous methods we can derive the normal and anomalous Green functions. Their forms can be obtained from Eqs. (3) and (5) by the following repleacement

$$\varepsilon \longrightarrow \varepsilon - \underline{p}\underline{v}_s \tag{20}$$

which also results from the forms of Eqs. (13) and (16). Putting $\underline{v}_s = 0$ we will restrict ourselves to the paramagnetic effects.

Ⓒ The anomalous Green function obtained in the generalized Gorkov approach combined with the gap equation allow us to notice that the presence of the external magnetic field can lead to the appearance of the triplet state Cooper's pairs if the pairing interaction contains also the first harmonic responsible for the creation of triplet state pair. However, in such a situation the equilibrium state should be a mixed state composed of the pure S- and P-pairing states.

2^0 The paramagnetic approach

In competition to the Gorkov-type approaches [31, 37, 54, 57, 62, 80, 89, 132, 138] we formulate also the accessory independent approach based on the assumptions of the LMC theory [50, 55]. This approach is also in opposition to the consideration of the paramagnetic field in superconductors presented in [18, 19] (cf. [138]).

Let us specify main assumptions of the paramagnetic approach. It is assumed namely that the total influence of the magnetic field on the system can be expressed by means of paramagnetic terms only and the omitted interaction of the magnetic field (vector potential) with the quasiparticle charge can be compensated by the renormalisation of Bohr's magneton. Hence, the conformable equations of the paramagnetic theory keep their forms and only Bohr's magneton is replaced by the effective Bohr's magneton.

The constructed formalism is set in the centre-of-mass coordinate system. It also can be applied to quite homogeneous systems in extremely broad range [50]. Moreover, similary as before, if considerations are carried on in the local limit, the suitable quantities become the functions of the centre-of-mass co-ordinate $\underline{R}$ creating the inhomogeneous state (cf. [37]). From now on all the calculations will be carried on in the centre-of-mass coordinate system. Hence, some quantities must be replaced by their reduced forms.

5. Fundamental quantities of the formalism [51, 54]

Let us define now the fundamental quantities characterizing the superconducting or superfluid system in the strong magnetic field. Trying to stick to the Landau quasiparticle concept we should consider only the quasiparticles localized near the Fermi surface. Due to such an assumption the number of quasiparticles taken into account is of the order of $\nu(0)\ \Delta(0)$ which is in opposition to the approaches based on the Gorkov equations [54]. However, according to Feynman's view [36] the numbers of quasiparticles are solely the auxiliary quantities which cannot be treated too literally. Therefore we permit of another approach.

a) The paramagnetic approach

The total number of quasiparticles and average paramagnetic field can be derived from the following equations

$$N = T \lim_{\delta\to 0^+}\Bigg[\sum_{\varepsilon_n}\ \sum_{\underline{p}\,|\underline{p}|>p_0} e^{i\varepsilon_n\delta}\, \mathrm{tr}\,\sigma^0\, \mathcal{G}_s(\tilde{p}) + \sum_{\varepsilon_n}\ \sum_{\underline{p}\,|\underline{p}|<p_0} e^{-i\varepsilon_n\delta}\, \mathrm{tr}\,\sigma^0\, \mathcal{G}_s(\tilde{p})\Bigg], \tag{1}$$

$$H_p = \frac{T}{2\nu(0)\ \mu_B} \lim_{\delta\to 0^+} \sum_{\varepsilon_n} \sum_{\underline{p}} e^{i\varepsilon_n\delta} \operatorname{tr} \underline{\sigma}\, \mathcal{G}_s(\tilde{p}). \tag{2}$$

Hence, the paramagnetic magnetization is expressed as follows

$$M_p = \chi_p^0 H_p. \tag{3}$$

The gap equation, according to (3.6), can be written in the form

$$\hat{\Delta} = -\frac{T}{2} \lim_{\delta\to 0^+} \sum_{\varepsilon_n} \sum_{\underline{p}} e^{i\varepsilon_n\delta}\, \theta\left[\xi_p - \xi(\underline{p})\right] \operatorname{tr} \hat{\underline{V}}\, F\,(\tilde{p}) \tag{4}$$

where ξ_p $(\Delta \ll \xi_p \ll \varepsilon_F)$ is a so-called cut-off parameter and it is identified with the Debye frequency ω_D for the superconducting systems. Equation (4) is common for both discussed approaches.

b) The generalized Gorkov approach

In the Gorkov-type approach the number of quasiparticles is defined in the form

$$N = T \lim_{\delta\to 0^+} \sum_{\varepsilon_n} \sum_{\underline{p}} e^{i\varepsilon_n\delta} \operatorname{tr} \sigma^0\, \mathcal{G}_s(\tilde{p}) \tag{5}$$

where the summation is extended over the deep region within the Fermi sphere, hence the density of states cannot be put constant. However, we need not derive the value of the expression (5) since we assume that the total number of quasi-particles N is constant and equal to $\frac{4}{3}\ \nu(0)\ \varepsilon_F$. Moreover, the average current can be derived from the equation

$$j = \frac{e}{m} T \lim_{\delta\to 0^+} \sum_{\varepsilon_n} \sum_{\underline{p}} (\underline{p}+mv_s) e^{i\varepsilon_n\delta} \operatorname{tr} \sigma^0\, \mathcal{G}_s(\tilde{p}) \tag{6}$$

and the paramagnetic magnetization keeps its previous form. Let us complete the above equations with the expression allowing us to evaluate the thermodynamic potential difference of the superconducting (superfluid) state relative to the normal state. According to the formula given i.a. in [35] it is of the form

$$\Delta\Omega = \int_0^{\Delta} d\Delta' \, (\Delta')^2 \frac{\partial}{\partial\Delta'} \left(\frac{1}{g_i}\right) \tag{7}$$

where g_i is the parameter of the pairing interaction and employing the gap equation; it can be expressed as a function of the energy gap Δ and a few fixed parameters: H, $\varkappa$, T, ξ_p.

6. The quasiparticle interactions

In the developed formalism we consider three types of the quasiparticle interactions, i.e., the Fermi liquid interaction, the pairing interaction and the dipole-dipole interaction. However, the latter one is a weak spin interaction, so it can modify only the triplet part of pairing interaction. Hence the full interaction in the particle-particle channel $\hat{\underline{V}}$ is formed from the pairing and dipole-dipole interactions and is of the form [43-45]

$$\hat{\underline{V}} = \frac{1}{\nu(0)} \hat{\underline{\Phi}} + \frac{4}{3} g_D \hat{\underline{D}} \tag{1}$$

where

$$\underline{D}_{ij}(\hat{p},\hat{p}') = \delta_{ij} - 3(\hat{p}_i - \hat{p}'_i)(\hat{p}_j - \hat{p}'_j)/|\hat{p} - \hat{p}'|^2 .$$

Other properties of the dipole-dipole interaction are discussed in detail in Section 32. The dimensionless pairing interaction has the form [22]

$$\hat{\underline{\Phi}} = \Phi_1 (i\sigma^y)(i\,\sigma'^y) + \Phi_{-1}\,(\sigma^y i\underline{\sigma})(\underline{\sigma}' i\,\sigma'^y) \tag{2}$$

where the spin antisymmetric (singlet) and spin symmetric (triplet) pairing interactions are of the forms [22, 49]

$$\Phi_1 = \sum_{l=0}^{\infty} (2l+1) f_l^a \, P_l(\hat{p}\hat{p}') , \tag{3}$$

and

$$\Phi_{-1} = \sum_{l=0}^{\infty} (2l+1) f_l^s \, P_l(\hat{p}\hat{p}') \tag{4}$$

where

$$f_l^{a(s)} = \left(\ln \frac{2\xi_p}{r_l^{a(s)}} \right)^{-1} \tag{5}$$

and all $r_l^{a(s)} \equiv 0$ for odd (even) l, respectively, then the suitable $f_l^{a(s)} \equiv 0$ according to the Pauli exclusion principle. Therefore, below we omit the superscripts. Moreover, the dimensionless parameters f_l are often represented in the form

$$f_l = \nu(0) g_l . \tag{6}$$

The interaction in the particle-hole channel expresses the Fermi liquid interaction and can be performed in the form [22]

$$\underline{\hat{C}} = \underline{A}\,\sigma^0\sigma'^0 + \underline{B}\,\underline{\sigma}\,\underline{\sigma}' \tag{7}$$

where its spin direct and spin exchange parts are of the forms

$$A = \sum_{l=0}^{\infty} (2l+1)\, a_l\, P_l(\hat{p}\hat{p}'), \tag{8}$$

$$B = \sum_{l=0}^{\infty} (2l+1)\, b_l\, P_l(\hat{p}\hat{p}'), \tag{9}$$

and a_l and b_l are the Landau parameters denoted also by F_l^s and F_l^a or by A_l and B_l then we have

$$F_l^s = A_l = (2l+1)\, a_l \quad \text{and} \quad F_l^a = B_l = (2l+1)\, b_l. \tag{10}$$

Such specified quasiparticle interactions can be easily introduced to the linear response theory and the pairing and dipole-dipole interactions to the gap equation. However, the problem of the Fermi liquid interaction inclusion to the static non-linear theory requires an individual approach. We consider it below.

7. The Fermi liquid interaction in the non-linear theory [51]

In order to consider general properties of the Fermi liquid interaction inclusion to the theory containing the strong and constant magnetic field we construct the self-consistent approach.

Using the Dyson equation we can write

$$\underline{G}_n^{-1} = \underline{G}_0^{-1} - \underline{IG}_n, \tag{1}$$

$$\underline{G}_q^{-1} = \underline{G}_0^{-1} - \underline{IG}_s \tag{2}$$

and eliminating $\underline{G}_0$ we obtain

$$\underline{G}_q^{-1} = \underline{G}_n^{-1} - \underline{I}(\underline{G}_s - \underline{G}_n) \tag{3}$$

where we assume that within the frame of the applied formalism the mass operator $\underline{\Sigma}$ can be performed in the form [35]

$$\underline{\Sigma}_{(\ )} = \underline{IG}_{(\ )} \tag{4}$$

and $\underline{I}$ is the irreducible part of the effective two-particle interaction. According to the Bethe-Salpeter equation we have

$$\underline{\Gamma}^{\omega} = \underline{I} + \underline{\Gamma}^{\omega} (\underline{G}^2)^{\omega} \underline{I} \ , \tag{5}$$

then multiplying both sides of Eq. (3) by

$$1 + \underline{\Gamma}^{\omega} (\underline{G}^2)^{\omega} \ , \tag{6}$$

after some transformations we obtain

$$\underline{G}_q^{-1} = \underline{G}_n^{-1} - \underline{\Gamma}^{\omega} (\underline{G}_s - \underline{G}_q) + \underline{\Gamma}^{\omega} \delta_p (\underline{G}_n^{-1} - \underline{G}_q^{-1}) \tag{7}$$

where we also include that

$$\underline{\Gamma}^{\omega} (\underline{G}^2)^k \underline{G}_{n(q)}^{-1} = \underline{\Gamma}^{\omega} \underline{G}_{n(q)} \tag{8}$$

and

$$(\underline{G}^2)^{\omega} - (\underline{G}^2)^k = \delta_p \tag{9}$$

and that δ_p is equivalent to the four-dimension Dirac delta. The symbols ω and k denote the appropriate limits. The dimensionless amplitude of the Fermi liquid interactions is of the form

$$\underline{C} = 2\nu(0) \Gamma^{\omega}. \tag{10}$$

The above consideration is carried on in relation to the full forms of the Green functions when the matrix notation is applied.

From now on we can restrict ourselves to the singular parts of Green functions and assume the functions $\underline{G}_n$ and $\underline{G}_q$ in the forms:

$$\begin{aligned} \underline{G}_n^{-1} &= [(\varepsilon - \xi)\sigma^0 + \underline{h\sigma}], \\ \underline{G}_q^{-1} &= [(\mathcal{E} - \xi)\sigma^0 + \vec{\mathcal{H}}\underline{\sigma}]. \end{aligned} \tag{11}$$

Let us note that (usual notation)

$$T \sum_{\varepsilon_n} \sum_{\underline{p}} \underline{\Gamma}^{\omega} \delta_p (\underline{G}_n^{-1} - \underline{G}_q^{-1}) = \langle B(\underline{h} - \vec{\mathcal{H}})\sigma \rangle \ , \tag{12}$$

hence Eq. (7) reduces to the form

$$(\mathcal{E} - \xi)\sigma^0 + \vec{\mathcal{H}}\underline{\sigma} = (\varepsilon - \xi)\sigma^0 + \underline{h\sigma} +$$

$$\frac{2\Pi T\sigma^0}{\nu(0)} \langle A \ \Sigma_2 \ \mathrm{tr}[\sigma^0 (\underline{G}_s - \underline{G}_q)] \rangle - \frac{2\Pi T\underline{\sigma}}{\nu(0)} \langle B \ \Sigma_2 \ \mathrm{tr}[\underline{\sigma}(\underline{G}_s - \underline{G}_q)] \ \langle Bh\underline{\sigma} \rangle - \langle B \vec{\mathcal{H}} \underline{\sigma} \rangle \tag{13}$$

where we separate the integration over spherical angles and after exploiting relations

$$\frac{1}{2\nu(0)} \langle B \ \Sigma_2 \ \mathrm{tr}(\underline{\sigma G}_q) \rangle = \langle B \vec{\mathcal{H}} \rangle, \tag{14}$$

and

$$\underline{h} = \mu_B \underline{H} - \langle B\underline{h} \rangle , \tag{15}$$

for a normal system, we obtain:

$$\mathcal{E} = \varepsilon - \frac{1}{\nu_0} \langle A\delta n_s \rangle , \tag{16}$$

$$\mathcal{H} = \mu_B \underline{H}_E - \frac{1}{\mu_B \nu_o} \langle B\underline{m}_s \rangle \tag{17}$$

where

$$4\Pi T \langle A \, \Sigma_2 \, \mathrm{tr}[\sigma^0 (\underline{G}_s - \underline{G}_q)] \rangle \equiv \langle A\delta n_s \rangle ,$$

$$4\Pi T \langle B \, \Sigma_2 \, \mathrm{tr}(\underline{\sigma}\underline{G}_s) \rangle \equiv \frac{1}{\mu_B} \langle B\underline{m}_s \rangle ,$$

and $N = \langle n \rangle$ is a number of quasi-particles; $M = \langle m \rangle$ - a paramagnetic magnetization. It is easy to note that the obtained Eqs. (16) and (17) are also valid for the normal systems where

$$\delta n = 4\Pi T \, \Sigma_2 \, \mathrm{tr}\left[\sigma^0 (\underline{G}_n - \underline{G}_n(0))\right] \tag{18}$$

and

$$\underline{M} = \mu_B \nu_0 \underline{h} . \tag{19}$$

Moreover, Eq. (17) represents the development of the molecular field approximation and reduces to it if we put $B = F_0^a$, whereas Eq. (16) after being rewritten in the form

$$\delta \mathcal{E} = \delta\varepsilon - \frac{1}{\nu_0} \langle A\delta n \rangle \tag{20}$$

can be used to obtain the following Word's identity

$$\left.\frac{\partial N}{\partial \varepsilon}\right|_{\varepsilon = 0} = \frac{\nu_0}{1+F_0^a} \tag{21}$$

where we assume that

$$\nu_0 \delta \mathcal{E} = \delta n \quad \text{and} \quad \langle A\delta n \rangle = F_0^a \delta N.$$

II. Basic equations and their principal properties in the theory with the magnetic field

The developed formalism, especially the obtained forms of the Green functions, are the starting point to a coherent generalization of the LMC theory [22] in case when the external strong magnetic field is regarded. It is obvious that all the vertex equations keep their forms, however, detailed calculations induce some complications connected with mutual transpositions of the Green and vertex functions which are the matrices. Consequently, each of the generalized L,M,N,O kernels will have a few matrix forms, adequantely to the performed transpositions. Therefore, detailed considerations are usually carried under additional assumptions and for the select systems, e.g., for $k = 0$ (the NMR conditions) [83, 85, 93]. Moreover, renormalizing impact of the Fermi liquid interaction, especially on the strong total magnetic field, makes an important problem in the generalized LMC theory (being connected, e.g., with the Larmor frequency and the Knight shift, cf. [22, 63, 93, 104]) which cannot be appropriately constructed by means of the linear approximation. In Section 41 we give a precise generalization of the theory for the normal Fermi liquid. It illustrates the essence of the problem. At present a static case of the theory is developed for the neutral and charged Fermi systems with the pure S-pairing (^{3}He-^{4}He mixtures and superconductors) and for the neutral Fermi systems with the pure P-pairing for an arbitrary non-unitary state (superfluid ^{3}He). We also assume that the systems under consideration can be slightly inhomogeneous. The problem specified in this way constitutes non-linear development of the theory towards the strong magnetic field. The obtained results allow us to examine principal properties of the system under discussion. We also show that some assumptions taken in the spirit of the Landau quasiparticle concept lead to very interesting results. The formulated approach gives the possibility to include the particle-hole asymmetry. However, this effect is neglected until it delievers qualitatively new results. So we assume the full symmetry of a description and we restrict ourselves to the quasiparticles above the Fermi sphere.

8. Neutral BCS system [51]

We employ results of Sections 4 and 5. According to the accepted assumptions the pairing interaction is of the form

$$\underline{\hat{V}} = - g_0 \sigma^y \sigma'^y . \tag{1}$$

Before we pass to the principal considerations, let us notice that since the partial derivatives of the expression have the following properties

$$\frac{\partial E_{\pm}}{\partial \xi} = \operatorname{sgn}(E \pm h) \frac{\xi}{E} , \tag{2}$$

$$\frac{\partial E_\pm}{\partial \Delta} = \operatorname{sgn}(E \pm h)\,\frac{\Delta}{E}\,, \tag{3}$$

$$\frac{\partial E_\pm}{\partial h} = \pm\operatorname{sgn}(E \pm h), \tag{4}$$

an arbitrary differentiable function f(x) must fulfil the following equations

$$\frac{1}{\Delta}\cdot\frac{\partial f(E\pm)}{\partial \Delta} = \frac{1}{\xi}\cdot\frac{\partial f(E\pm)}{\partial \xi}\,, \tag{5}$$

$$\frac{\partial f(E\pm)}{\partial h} = \pm\,\frac{E}{\xi}\cdot\frac{\partial f(E\pm)}{\partial \xi}\,. \tag{6}$$

Now substituting the anomalous Green function into Eq. (5.4) and performing the appropriate integrations, the gap equation reduces to the form

$$\Delta = \frac{1}{4} g_0 \nu(0) \int_{-\xi_0}^{\xi_0} d\xi \left(\mathrm{th}\,\frac{E_+}{2T}\cdot\frac{\partial E_+}{\partial \Delta} + \mathrm{th}\,\frac{E_-}{2T}\,\frac{\partial E_-}{\partial \Delta}\right). \tag{7}$$

It is easy to notice that Eq. (7) can be rewritten in the more compact form

$$\Delta = \frac{1}{2} T g_0 \nu(0)\,\frac{\partial}{\partial \Delta}\int_{-\xi_0}^{\xi_0} d\xi\,\ln\left(\mathrm{ch}\,\frac{E_+}{2T}\,\mathrm{ch}\,\frac{E_-}{2T}\right). \tag{8}$$

Using Eq. (5) we obtain one more equivalent form

$$\Delta = \frac{1}{4} g_0 \nu(0)\,\Delta \int_{-\xi_0}^{\xi_0} \frac{d\xi}{E}\left(\mathrm{th}\,\frac{E+h}{2T} + \mathrm{th}\,\frac{E-h}{2T}\right). \tag{9}$$

The obtained equations allow us to find the relationship between the magnetic field, temperature and energy gap. As it may be noted the energy gap is an even function of the magnetic field. According to Eqs. (35.14) and (9) the magnetic field does not deform the energy gap at $T = 0$ as long as $h < \Delta$. This is due to the fact that there are no free quasi-particles and the magnetic field does not interact with the Cooper pairs until it starts destroying them. That is why at $T = 0$ the weak and medium magnetic fields cannot cause ony effects in BCS system.

Let us consider now the paramagnetic magnetization for the discussed system. We act in the similar way as before, but now we insert the normal Green function into Eq. (5.2) and after some integrations and transformations and using Eq. (4) we get

$$M = \frac{1}{4} \mu_B \nu_0 \int_{-\infty}^{+\infty} d\xi \left(\mathrm{th} \frac{E_+}{2T} \frac{\partial E_+}{\partial h} + \mathrm{th} \frac{E_-}{2T} \frac{\partial E_-}{\partial h} \right) \tag{10}$$

and can transform it again to the compact form

$$M = \frac{1}{2} \mu_B \nu_0 T \frac{\partial}{\partial h} \int_{-\infty}^{+\infty} d\xi \ln \left(\mathrm{ch} \frac{E_+}{2T} \mathrm{ch} \frac{E_-}{2T} \right) \tag{11}$$

where the last expression should be understood in the formal way, i.e.,

$$\frac{\partial}{\partial h} \int_{-\infty}^{+\infty} d\xi \equiv \lim_{\Xi \to \infty} \frac{\partial}{\partial h} \int_{-\Xi}^{+\Xi} d\xi \, . \tag{12}$$

On the other hand, using Eq. (16), we can rewrite (10) in the form

$$M = \frac{1}{4} \mu_B \nu_0 \int_{-\infty}^{+\infty} d\xi \left(\mathrm{th} \frac{E+h}{2T} - \mathrm{th} \frac{E-h}{2T} \right) . \tag{13}$$

The obtained equations express the dependence of the paramagnetic magnetization on $\underline{h}$ in the strong-magnetic-field approach. Let us consider now the obtained results in some specified limit.

1^0 The limit $T = 0$

After applying Eqs. (35.14)-(35.16) the paramegnetic magnetization reduces to the form

$$M = \mu_B \nu_0 \sqrt{h^2 - \Delta^2} \; \theta(h - \Delta) . \tag{14}$$

According to the obtained result the paramagnetic magnetization appears only if $h > \Delta$. Thus, in the case being discussed it must be investigated together with the gap equation (9) [50, 55].

2^0 The medium h limit

We consider the magnetization at non-zero temperatures assuming that $\Delta > h$. Applying some trigonometric rules we transform Eq. (13) to the form

$$M = \frac{1}{2} \mu_B \nu_0 \, \mathrm{sh} \frac{h}{T} \int_{-\infty}^{+\infty} \frac{d\xi}{\mathrm{ch} \frac{E}{T} + \mathrm{ch} \frac{h}{T}} , \tag{15}$$

which after the series expansion takes the form

$$M = \mu_B \nu_0 T \; \mathrm{sh}\,\frac{h}{T} \sum_{j=0}^{\infty} (-1)^j \mathrm{ch}^j \frac{h}{T} \cdot \int_0^{\infty} dx \; \mathrm{ch}^{-(j+1)} \sqrt{x^2 + (\Delta/T)^2} \; . \tag{16}$$

However, by applying (15), which turns out to be more convenient for our calculations, and computing the suitable derivatives we derive the paramagnetic magnetization up to the third order in h in the form

$$M = \mu_B \nu_0 Y_0 \, h + \frac{1}{12} \mu_B \nu_0 \, (2Y_0 - 3Y_{,2}) \frac{h^3}{T^2} \; . \tag{17}$$

Taking into account the Fermi liquid interaction (Section 7) in linear approximation we obtain the well-known expressions [82, 85]

$$h = \frac{\mu_B H}{1+b_0 Y_0} \quad \text{and} \quad \chi = \frac{\mu_B^2 \nu_0 Y_0}{1+b_0 Y_0} \; , \tag{18}$$

which combined with (17) give

$$M = \frac{\mu_B^2 \nu_0 Y_0}{1+b_0 Y_0} H + \frac{\mu_B^4 \nu_0}{12 T^2} \cdot \frac{2Y_0 - 3Y_{,2}}{(1+b_0 Y_0)^4} H^3 \tag{19}$$

and can be employed in experimental investigations. We emphasize that the vector of the paramagnetic field is always parallel to the external magnetic field direction and no other additional direction is distinguished in the system. Thus the spin susceptibility tensor $(\chi_{ij} = \partial M_i / \partial H_j)$ is proportional to the Kronecker delta.

In order to complete our considerations we enclose the following equation expressing the number of quasi-particles N (Eq. (51))

$$N = \nu_0 \int_0^{\infty} d\xi \left[1 - T \frac{\partial}{\partial \xi} \ln \left(\mathrm{ch}\,\frac{E_+}{2T} \, \mathrm{ch}\,\frac{E_-}{2T} \right) \right] , \tag{20}$$

from which we derive

$$N = \nu_0 T \ln \left(\mathrm{ch}\,\frac{\Delta+h}{2T} \, \mathrm{ch}\,\frac{\Delta-h}{2T} \right) + \nu_0 T 2 \ln^2 \tag{21}$$

as a function of Δ, T and h. Equation (20) together with Eq. (11) can be employed to define the quasiparticle distribution function and we obtained it in the form [50] (Fig. 1)

$$\hat{n} = \frac{1}{4} \left\{ \sigma^0 \left[1 - \frac{1}{2} \left(\mathrm{th}\,\frac{E+h}{2T} + \mathrm{th}\,\frac{E-h}{2T} \right) \frac{\xi}{E} \right] + \sigma^z \left(\mathrm{th}\,\frac{E+h}{2T} - \mathrm{th}\,\frac{E-h}{2T} \right) \right\} . \tag{22}$$

In most cases we consider it only for $\xi \geqslant 0$.

The results obtained above can be applied immediately to the charged systems in the paramagnetic approach.

9. Superconducting system - the generalized Gorkov approach [54]

We employ the results contained in Sections 4 and 5. Substituing the anomalous Green function into (5.4) and making some transformations the gap equation reduces to the form

$$1 = \frac{\nu(0)g_0}{16} \int_{-\omega_D}^{\omega_D} \frac{d\xi}{E} \int_{-1}^{+1} dx \left(\mathrm{th}\,\frac{E+h+xp_0v_s}{2T} + \mathrm{th}\,\frac{E-h+xp_0v_s}{2T} + \mathrm{th}\,\frac{E+h-xp_0v_s}{2T} + \mathrm{th}\,\frac{E-h-xp_0v_s}{2T} \right) \tag{1}$$

where the other integral expresses the averaging over spherical angles and can be easily computed. Then we obtain

$$\frac{2\,p_0\,v_s}{\nu(0)g_0T} = \int_0^{\omega_D} \frac{d\xi}{E} \ln \frac{\mathrm{ch}\,\frac{E+h+p_0v_s}{2T}\;\mathrm{ch}\,\frac{E-h+p_0v_s}{2T}}{\mathrm{ch}\,\frac{E+h-p_0v_s}{2T}\;\mathrm{ch}\,\frac{E-h-p_0v_s}{2T}} \,. \tag{2}$$

It is easy to observe that the obtained espression is invariant while changing the sign of h or v_s. Let us consider now the total number of quasiparticles, the average current and the paramagnetic field. After substituting the normal Green function of the superconducting state and making appropriate transformations we obtain

$$N = \frac{1}{4} \int_{-1}^{1} dx \int_{-\mu}^{+\infty} d\xi\; \nu(\xi)\,[1 + g_+(\xi, x)] \,, \tag{3}$$

$$j = e\,N\underline{v}_s - eN_n\underline{v}_s \,, \tag{4}$$

$$H_p = \frac{1}{4\,\mu_B^*} \int_{-1}^{1} dx \int_{-\infty}^{+\infty} d\xi\; g_-(\xi, x) \tag{5}$$

where

$$g_\pm(\xi,x) = \frac{1}{4}\left(1 + \frac{\xi}{E}\right)\left(\mathrm{th}\,\frac{E+h+xp_0v_s}{2T} \pm \mathrm{th}\,\frac{E-h+xp_0v_s}{2T}\right)$$

$$\mp \frac{1}{4}\left(1-\frac{\xi}{E}\right)\left(\operatorname{th}\frac{E+h-xp_0v_s}{2T} \pm \operatorname{th}\frac{E-h-xp_0v_s}{2T}\right) \tag{6}$$

and

$$N_n = \frac{3N}{4\,p_0v_s}\int_{-1}^{1} dx\; x \int_{-\infty}^{+\infty} d\xi \quad g_+(\xi,x) \tag{7}$$

can be identified with the number of the normal (uncoupled) quasiparticles.

In order to calculate the paramagnetic field H_p and the number of the normal quasi-particles we have to notice that owing to the symmetry the parts of the integrals (5) and (7) contaning the term ξ/E vanish. Hence we have

$$H_p = \frac{1}{16\;\mu_B^*}\int_{-1}^{1} dx \int_{-\infty}^{+\infty} d\xi \left(\operatorname{th}\frac{E+h+xp_0v_s}{2T} - \operatorname{th}\frac{E-h+xp_0v_s}{2T}\right.$$

$$\left. + \operatorname{th}\frac{E+h-xp_0v_s}{2T} - \operatorname{th}\frac{E-h-xp_0v_s}{2T}\right), \tag{8}$$

$$N_n = \frac{\nu(0)p_0}{8m\;v_s}\int_{-1}^{1} dx\; x \int_{-\infty}^{+\infty} d\xi \left(\operatorname{th}\frac{E+h+xp_0v_s}{2T} + \operatorname{th}\frac{E-h+xp_0v_s}{2T}\right.$$

$$\left. - \operatorname{th}\frac{E+h-xp_0v_s}{2T} - \operatorname{th}\frac{E-h-xp_0v_s}{2T}\right) \tag{9}$$

where H_p is directed according to the external magnetic field. It is easy to verify that the obtained functions have the following symmetry properties

$$\begin{aligned} H_p(h,v_s) &= -\,H_p(-\,h,v_s) = H_p(h,-v_s), \\ N_n(h,v_s) &= N_n(-h,v_s) = N_n(h,-v_s)\;. \end{aligned} \tag{10}$$

Hence the paramagnetic field and the normal current cannot depend on the directions of $\underline{v}_s$ and $\underline{h}$, respectively. After computing the first integral in Eq. (8) it reduces to the form

$$H_p = \frac{T}{2p_0v_s\;\mu_B^*}\int_0^{\infty} d\xi\; \ln \frac{\operatorname{ch}\dfrac{E+h+p_0v_s}{2T}\;\operatorname{ch}\dfrac{E-h-p_0v_s}{2T}}{\operatorname{ch}\dfrac{E+h-p_0v_s}{2T}\;\operatorname{ch}\dfrac{E-h+p_0v_s}{2T}}\;. \tag{11}$$

Unfortunately, such procedure cannot be repeated directly for Eq. (9) and integration by parts leads to divergent integral expressions. However, using Eq. (8) we can rewrite Eq. (9) in the form

$$\frac{N_n}{3N} + \frac{\mu_B^* H_p h}{p_0^2 v_s^2} = \frac{1}{4p_0^3 v_s^3} \int_0^\infty d\xi \int_{h-p_0 v_s}^{h+p_0 v_s} dz\, z \left(\mathrm{th}\, \frac{E+z}{2T} - \mathrm{th}\, \frac{E-z}{2T}\right) . \quad (12)$$

10. Superfluid systems with the P-wave-pairing (^3He) [51]

We employ Eqs. (4.9)-(4.11) and Section 5. The full interaction in the particle-particle channel is taken in the form (cf. Sections 6, 40, 42)

$$V_{ij} = -3g_1\left[(1+\tfrac{2}{5}\alpha)\delta_{ij}\delta_{kn} - \tfrac{3}{5}\alpha(\delta_{in}\delta_{jk} + \delta_{ik}\delta_{jn})\right]\hat{p}_k\hat{p}_n' . \quad (1)$$

Let us begin our investigation with the consideration of the possible partial derivatives of the expressions (4.10). They have the forms

$$\frac{\partial E_\pm}{\partial \xi} = \frac{1}{E_\pm}\left[\xi \pm \frac{(2\xi \underline{h} - \Delta^2 \underline{l})\underline{h}}{\sqrt{(\Delta^2\underline{l} - 2\xi\underline{h})^2 + 4\Delta^2|\underline{d}\underline{h}|^2}}\right], \quad (2)$$

$$\frac{\partial E_\pm}{\partial h_j} = \frac{1}{E_\pm}\left[h_j \pm \frac{(2\xi h_j - \Delta^2 l_j)\xi + \Delta^2(\underline{d}\underline{h}^* d_j + \underline{h}\underline{d} d_j^*)}{\sqrt{(\Delta^2\underline{l} - 2\xi h)^2 + 4\Delta^2|\underline{d}\underline{h}|^2}}\right], \quad (3)$$

$$\frac{\partial E_\pm}{\partial d_j^*} = \frac{\Delta^2}{2E_\pm}\left[d_j \pm \frac{i\Delta^2(\underline{l}\times\underline{d})_j - 2i\xi(\underline{h}\times\underline{d})_j + 2\underline{h}\underline{d}h_j}{\sqrt{(\Delta^2\underline{l} - 2\xi\underline{h})^2 + 4\Delta^2|\underline{d}\underline{h}|^2}}\right], \quad (4)$$

and

$$\frac{\Delta}{2}\frac{\partial E_\pm}{\partial \Delta} = d_j^* \frac{\partial E_\pm}{\partial d_j^*} \quad (5)$$

where the square root in the denominator can be performed in the form

$$E_+^2 - E_-^2 = 2\sqrt{(\Delta^2\underline{l} - 2\xi\underline{h}^2) + 4\Delta^2|\underline{h}\underline{d}|^2} . \quad (6)$$

We also have

$$E_\pm^2 - E^2 - h^2 = \pm\frac{1}{2}(E_+^2 - E_-^2) . \quad (7)$$

Substituting the anomalous Green function into Eq. (5.4) and using above equations we derive the gap equation in the form

$$\Delta^2 d_i = \frac{1}{2}\left\langle \underline{V}_{ij} \int_{-\xi_0}^{\xi_0} d\xi \left(\operatorname{th}\frac{E_+}{2T}\frac{\partial E_+}{\partial d_j^*} + \operatorname{th}\frac{E_-}{2T}\frac{\partial E_-}{\partial d_j^*}\right)\right\rangle \tag{8}$$

which can be also transformed into a more compact form

$$\Delta^2 d_i = -T\left\langle \underline{V}_{ij}\frac{\partial}{\partial d_j^*}\int_{-\xi_0}^{\xi_0} d\xi \ln\left(\operatorname{ch}\frac{E_+}{2T}\operatorname{ch}\frac{E_-}{2T}\right)\right\rangle, \tag{9}$$

which is analogous to (8.8). Although the further considerations can be continued if the dipole forces are regarded, they are neglected to make our considerations more convenient and explicit. In such a case Eq. (9) reduces to the form

$$\Delta^2 d_{i\alpha} = 3\nu_0 g_1 T\left\langle \hat{p}_\alpha \frac{\partial}{\partial d_1^*}\int_{-\xi_0}^{\xi_0} d\xi \ln\left(\operatorname{ch}\frac{E_+}{2T}\operatorname{ch}\frac{E_-}{2T}\right)\right\rangle, \tag{10}$$

hence, using Eqs. (5) and (7), we obtain

$$\Delta = \frac{1}{2}\nu_0 g_1 T\frac{\partial}{\partial\Delta}\left\langle\int_{-\xi_0}^{\xi_0} d\xi \ln\left(\operatorname{ch}\frac{E_+}{2T}\operatorname{ch}\frac{E_-}{2T}\right)\right\rangle, \tag{11}$$

and

$$\Delta^2\langle \underline{l}\rangle = -i\nu_0 g_1 T\left\langle \underline{d}^* \times \frac{\partial}{\partial \underline{d}}\int_{-\xi_0}^{\xi_0} d\xi \ln\left(\operatorname{ch}\frac{E_+}{2T}\operatorname{ch}\frac{E_-}{2T}\right)\right\rangle . \tag{12}$$

Equation (11) is almost the same as Eq. (8.8) and can be used to derive the external-magnetic-field and temperature dependence of the energy gap, whereas Eq. (12) allows us to prove that the vector $\underline{l}$ is an odd function of the magnetic field $\underline{h}$ and it vanishes if h tends to zero. That conclusion is obtained automatically if we notice that the quasiparticle energy $E_{\pm}$ is invariant during simultaneous alteration of the signs of magnetic field $\underline{h}$ and the non-unitary state vector $\underline{l}$.

Let us pass now to another point of our considerations, i.e., to the derivation of the superfluid-system average magnetization. Inserting the derived form of the normal Green function into Eq. (5.2) and applying Eqs. (3), (6) and (7), after some algebra we obtain

$$M_i = \frac{1}{2}\, \mu_B T \left\langle \frac{\partial}{\partial h_i} \int_{-\infty}^{+\infty} d\xi \nu\, (\xi) \ln \left(\mathrm{ch}\, \frac{E_+}{2T}\, \mathrm{ch}\, \frac{E_-}{2T} \right) \right\rangle \tag{13}$$

where the assumed form of the state density allows us to take into account all the effects which arise due to the particle-hole asymmetry. According to Eq. (12) the right-hand-side part of Eq. (13) depends on two macroscopic directions $\underline{h}$ and $\langle \underline{1} \rangle$ and one microscopic direction $\underline{d}$. Then, the total magnetization is determined by those directions and need not be parallel to the vector $\underline{h}$. In spite of the full analogy in forms obtained for superconducting and superfluid systems the inner structure of Eq. (13) is far more complicated than that of Eq. (8.11), and the former equation is investigated only for small values of $\underline{h}$. We also precede them with the introduction of the following indication

$$f_{\pm}(\underline{h}) = \ln \mathrm{ch}\, \frac{E_{\pm}}{2T}\, . \tag{14}$$

Then we have

$$\frac{\partial f_{\pm}(\underline{h})}{\partial h_i} = \frac{\mathrm{th}\, \frac{E_{\pm}}{2T}}{2T} \cdot \frac{\partial E_{\pm}}{\partial h_i}\, ,$$

and (15)

$$\frac{\partial^2 f_{\pm}(\underline{h})}{\partial h_i\, \partial h_j} = \frac{\mathrm{th}\, \frac{E_{\pm}}{2T}}{2T} \cdot \frac{\partial^2 E_{\pm}}{\partial h_i \partial h_j} + \frac{\mathrm{ch}^{-2}\, \frac{E_{\pm}}{2T}}{4T^2} \cdot \frac{\partial E_{\pm}}{\partial h_i} \cdot \frac{\partial E_{\pm}}{\partial h_j}$$

and putting $h = 0$ we have

$$E^0_{\pm} = \sqrt{\xi^2 + \Delta^2(|\underline{d}|^2 \pm |\underline{1}|)}\, ,$$

$$\frac{\partial E^0_{\pm}}{\partial h_i} = \mp \frac{\xi}{E^0_{\pm}}\, \hat{1}_i\, ,$$

$$\frac{\partial^2 E^0_{\pm}}{\partial h_i\, \partial h_j} = - \frac{\xi^2}{(E^0_{\pm})^3}\, \hat{1}_i \hat{1}_j + \frac{1}{E^0_{\pm}} \left[\delta_{ij} \pm \frac{2\xi^2(\delta_{ij} - \hat{1}_i \hat{1}_j) + \Delta^2(d_i d^*_j + d^*_i d_j)}{\Delta^2 |\underline{1}|} \right] \tag{16}$$

where $\hat{1}_i (= 1_i/|\underline{1}|)$ is the unit vector along the $\underline{1}$ direction. Using Eqs. (15) and (16) the magnetization (13) in the linear approximation reduces to the form

$$M_i = \frac{1}{4}\mu_B\Delta^2\left\langle l_i \int\limits_{-\infty}^{+\infty}\frac{d\xi}{E}\,\mathrm{th}\,\frac{E}{2T}\cdot\frac{\partial\nu}{\partial\xi}\right\rangle + \mu_B\nu_0\langle r_{ij}h_j\rangle\,, \tag{17}$$

and

$$r_{ij} = \delta_{ij} - \frac{d_i d_j^* + d_i^* d_j}{2|\underline{d}|^2}\left(1 - \int\limits_0^\infty \frac{d\xi}{2T}\,\mathrm{ch}^{-2}\,\frac{E}{2T}\right), \tag{18}$$

$$R_{ij} = \langle r_{ij}\rangle$$

where in order not to exceed the accuracy of the theory, we also restrict ourselves to the first term of the expansion in $|\underline{l}|$. The first term of magnetization depends linearly only on the vector $\underline{l}$, and according to the previous remarks it also depends lineary on $\underline{h}$, hence the magnetization must vanish if the external magnetic field is excluded. Moreover, this term appears only if the particle-hole asymmetry is admitted. While deriving the other term we consequently neglect the particle-hole asymmetry. The presented results are obtained within the frame of the self-consistent formalism, where the magnetic field affects the equilibrium state by dipole-dipole interaction effects. From Eq. (18), applying (7.17), we can find the static spin susceptibility for an arbitrary state. For the states being discussed by Leggett [84] as the best ground-state condidates when the dipole-dipole interaction is included, the following results are obtained (cf. [1, 39, 85, 127, 135, 149]).

1° The isotropic-BW state [15]

$$\underline{d} = \frac{1}{4}(-\hat{p}_x + \sqrt{15}\,\hat{p}_y)\hat{x} + \frac{1}{4}(-\hat{p}_y - \sqrt{15}\,\hat{p}_y)\hat{y} + \hat{p}_z\hat{z},$$

$$|\underline{d}|^2 = 1,\quad \underline{l} = 0,\quad r_{ij} = \delta_{ij} - (1-Y_0)d_i d_j\,,$$

$$R_{ij} = \delta_{ij}\left(\frac{2}{3} + \frac{1}{3}Y_0\right),$$

hence

$$\chi_{ij} = \mu_B^2\nu_0\,\frac{\frac{2}{3}(1+b_2Y_0)+\frac{1}{3}Y_0(1+b_2)}{\frac{2}{3}(1+b_0)(1+b_2Y_0)+\frac{1}{3}(1+b_0Y_0)(1+b_2)}\,\delta_{ij}\,.$$

2° The axial-ABM state [11, 12]

$$\underline{d} = \sqrt{\tfrac{3}{2}}(\hat{p}_x + i\hat{p}_2)\hat{y}, \quad |\underline{d}|^2 = \tfrac{3}{2}(1 - \hat{p}_y^2) ,$$

$$\underline{1} = 0, \quad r_{zj} = \delta_{zj} ,$$

$$R_{ij} = (\delta_{ij} - \delta_{iy}\delta_{jy}) + \delta_{iy}\delta_{jy}Y_{0,2} ,$$

hence

$$\chi_{ij} = \mu_B^2 \nu_0 \left[\frac{1}{1+b_0}(\delta_{ij} - \delta_{iy}\delta_{jy}) + \frac{Y_{0,2}}{1+b_0 Y_{0,2}}\delta_{iy}\delta_{jy}\right].$$

3° The planar-2D state [7]

$$\underline{d} = \sqrt{\tfrac{3}{2}}\,\hat{z} \times \hat{p}, \quad |\underline{d}|^2 = \tfrac{3}{2}(1 - \hat{p}_z^2), \quad \underline{1} = 0, \quad r_{zj} = \delta_{zj} ,$$

$$R_{ij} = \tfrac{1}{2}(\delta_{ij} + \delta_{iz}\delta_{jz}) + \tfrac{1}{2}(\delta_{ij} - \delta_{iz}\delta_{jz})Y_{0,2} ,$$

hence

$$\chi_{ij} = \mu_B^2 \nu_0 \left[\frac{\tfrac{1}{2}(1+Y_{0,z})}{1+\tfrac{1}{2}b_0(1+Y_{0,2})}(\delta_{ij} - \delta_{iz}\delta_{jz}) + \frac{1}{1+b_0}\delta_{iz}\delta_{jz}\right] .$$

4° The polar-1D state

$$\underline{d} = \sqrt{3}\,\hat{p}_z\hat{y}, \quad |\underline{d}|^2 = 3\hat{p}_z^2, \quad \underline{1} = 0, \quad r_{zj} = \delta_{zj} ,$$

$$R_{ij} = (\delta_{ij} - \delta_{iy}\delta_{jy}) + \delta_{iy}\delta_{jy}Y_{0,1} ,$$

hence

$$\chi_{ij} = \mu_B^2 \nu_0 \left[\frac{1}{1+b_0}(\delta_{ij} - \delta_{iy}\delta_{jy}) + \frac{Y_{0,1}}{1+b_0 Y_{0,1}}\delta_{iy}\delta_{jy}\right] . \tag{19}$$

It must be noticed that in the cases 2°-4° we have restricted ourselves to the molecular field approximation. However, taking into account the fact that the external magnetic field is chosen along the z-axis we should concentrate on the χ_{zz}-component of the spin susceptibility tensors, which is derived exactly and possesses the identical following form

$$\chi_{zz} = \frac{\mu_B^2 \nu_0}{1 + b_0} . \tag{20}$$

The presented results (19)-(20) differ from those obtained when the dipole -dipole interaction is neglected, because now the axes of the coordinate system are rotated in a special way. It causes that the standard value of the static spin susceptibility for the 1D-state is given by χ_{yy}-component.

The magnetization investigations can, of course, be executed up to an arbitrarily chosen order of $\underline{h}$. The problem becomes simpler if we restrict ourselves to the unitary states ($\underline{l} = 0$). Then, the quasi-particle energy reduces to the form [52]

$$E_{\pm} = \left[\left(\sqrt{\xi^2 + \Delta^2 |d_z|^2} \pm h\right)^2 + \Delta^2 (|\underline{d}|^2 - |d_z|^2)\right]^{1/2} \tag{21}$$

and

$$E_+(h) = E_-(-h).$$

It allows us to state that the integrands in Eqs. (9), (11) and (13) become the even functions of the magnetic field. The obtained form of the magnetization makes possible the investigations of the nonlinear magnetization of the superfluid Fermi liquids for arbitrary equlibrium--state systems. The number of quasi-particles N, which is the last point of our considerations, can be expressed in the form analogous to Eq. (8.20) if we replace the quasi-particle energy according to (4.10) and keep the average over spherical angles. Then after some calculations we get

$$N = \nu_0 T \left\langle \ln \left(\mathrm{ch} \frac{\Delta_+}{2T} \, \mathrm{ch} \frac{\Delta_-}{2T} \right) \right\rangle + \nu_0 T 2\ln 2 \tag{22}$$

where

$$\Delta_{\pm} = \left[\Delta^2 |\underline{d}|^2 + h^2 \pm \Delta \sqrt{\Delta^2 \underline{l}^2 + 4 |\underline{h}\underline{d}|^2}\right]^{1/2} .$$

The obtained result (22), identically as for (8.21) under assumption that the number of quasi-particles participating in superconductivity or superfluidity is constant and independent of temperature in the absence of magnetic field, allows us to formulate the following relation

$$\Delta_0 \langle |\underline{d}| \rangle = 2T \left\langle \ln \mathrm{ch} \frac{\Delta |\underline{d}|}{2T} \right\rangle + T 2\ln 2 \tag{23}$$

where $\Delta_0 \equiv \Delta(T = 0)$ and $|\underline{d}| \equiv 1$ for BCS state. Hence, for isotropic states energy gap as a function of temperature is derived in the form

$$\Delta = 2T \ln \left(\exp \frac{\Delta_0 - T2\ln 2}{2T} + \sqrt{\exp \frac{\Delta_0 - T2\ln 2}{T} - 1} \right), \tag{24}$$

which in the limit cases reduces to the forms

$$\Delta = 2\sqrt{\Delta_0 T_c}\sqrt{1 - \frac{T}{T_c}} \quad \text{if} \quad T_c - T \ll T_c \tag{25}$$

and

$$\Delta = \Delta_0 - 2Te^{-\frac{\Delta_0}{T}} \quad \text{if} \quad T \ll T_c \tag{26}$$

where we also employed the relation

$$\Delta_0 = T_c 2 \ln 2, \tag{27}$$

which can be obtained from Eq. (24) by putting $T = T_c$, and $\Delta = 0$. Let us remark that the above relation is in good conformity with the experimental data obtained, e.g., for gallium. Obviously, our assumption is very elementary, the obtained results show however the proper trends of the system being discussed. Moreover, taking into account that $\varepsilon_n = (2n + 1)\Pi T$ we note that the Fermi liquid interaction can renormalize the temperature and the new effective temperature is of the form

$$T^{eff} = kT^{ex} \tag{28}$$

where T^{ex} denotes the real temperature of the system. The coefficient k is connected with the Fermi liquid interaction and expresses the individual properties of the system. Substituting (28) into (27) we find it in the form

$$k = \frac{\Delta_0}{T_c^{ex}\, 2 \ln 2}, \tag{29}$$

which allows us to define its value by means of the experimental data. Comparing the obtained results with the BCS-theory results we state a good agreement between the two forms of the energy gap in phase transition vicinity (Eq. 25) if the temperature is replaced by effective temperature. The coefficient k derived from Eq. (28) is equal to 1.27.

Part Two

FERMI SYSTEMS IN STRONG MAGNETIC FIELDS

III. Superconductors

11. Preliminary remarks

The subject of our primary interests is connected with the type-II superconductors. We can determine the basic properties of the superconducting systems by using some characteristic quantities, i.e., the Ginzburg-Landau parameter $\varkappa$ and the penetration depth λ or the coherence distance ξ_c. Moreover, we identify the value of the critical magnetic field for which the energy gap (superconductivity) vanishes with H_{c^2} and we apply some results of the Ginzburg-Landau theory. Then we have [35, 38, 92, 124, 129 130]

$$H_{c^2} = H_c \sqrt{2}\, \varkappa \,, \tag{1}$$

$$\lambda = \left(\frac{m}{4 \Pi N_s e^2} \right)^{1/2} , \quad \xi_c = \frac{\lambda}{\varkappa} \,, \tag{2}$$

and since

$$\Psi(T, H) \sim \Delta(T, H) \sim N_s^{1/2} \,, \tag{3}$$

we can write

$$\left[\frac{\Delta}{\Delta(0)} \right]^2 = \eta \frac{N_s}{N} \tag{4}$$

where the factor η should be of the order unity. Moreover, we have

$$H_c(T) = H_c(0) \left[1 - \left(\frac{T}{T_c}\right)^2 \right] \tag{5}$$

and

$$H_c(0) = \sqrt{4 \pi \nu (0)} \quad \Delta(0). \tag{6}$$

Since for type-II superconductors the relation

$$H_{c^2} > H_c \quad \text{implies} \quad \varkappa > 1/\sqrt{2} \,, \tag{7}$$

then the latest relation can be exploited to settle the type of superconductivity in case the value of the Ginzburg-Landau parameter $\varkappa$ is known. The quantities $\varkappa$ and N_s can be always derived within the frame of the formulated approaches. Then, by applying (2) the penetration depth λ and the coherence distance ξ_c can also be found. Let us note that Eq. (4) can be used to test the correctness of every formalism.

In the developed approaches not all intermediate phenomena, such as the electron-phonon interactions and electromagnetic interactions connected with the motion of the charged fermions, which lead to the Cooper's pair creation are taken into account. We assume only that the considered quasiparticles are coupled in the singlet spin state. However the superconductivity is the collective phenomenon where the moving electrons are locked into a phase-coherent state. Some of the left out interactions can also cause the renormalization of the fundamental magnitudes expressing the properties of Cooper's pairs. Therefore considering the paramagnetic model we assume that the ignored effects renormalize the value of Bohr's magneton in case of quasiparticles coupled in Cooper's pairs. This assumption is quite justified on the grounds of the Ginzburg-Landau theory, thus - based on the evaluation of the free energy - we find the following relation

$$\frac{\mu_B^*}{\mu_B} = \left| \frac{\chi_d^0}{\chi_p} \right|^{\frac{1}{2}} \tag{8}$$

where χ_d^0 $(= -1/4\pi)$ is the diamagnetic susceptibility of an ideal superconductor in the Meissner state and χ_p is the paramagnetic susceptibility of the normal Fermi liquid (cf. [50]).

Let us note that the introduced assumption is consistent with theory of the normal Fermi liquid and that the approach to the superconducting state from the normal phase is well-defined in the phase transition point (cf. [35, 62, 92, 124]).

12. A general outline of the paramagnetic approach [55]

The present formalism allows us to consider the phenomenon of superconductivity in the interior of the system volume without defining the system surface. The total magnetic field is introduced to the calculations by means of the paramagnetic terms. The interaction of the magnetic field with electric-charged quasiparticles is compensated by the renormalization of Bohr's magneton. Let us now specify the principal ideas of the paramagnetic theory. The paramagnetic theory refers to the type-II superconductors in the mixed state. The quasiparticles under consideration are situated nearby the Fermi surface. We assume that there exists a full particle-hole symmetry in the system. It allows us to restrict our considerations to the quasiparticles above the Fermi surface. In order to discuss the achievements of the paramagnetic theory we restrict ourselves to the homogeneous and zero-temperature case. The principal result which permits us to construct the paramagnetic theory is the equation of the internal paramagnetic field (cf. Eq. (8.14))

$$H_p = \sqrt{H_T^2 - \left(\frac{\Delta}{\mu_B^*}\right)^2} \quad \theta\left(H_T - \frac{\Delta}{\mu_B^*}\right) . \tag{1}$$

This field is generated by discoupled and polarized quasiparticles which appear in the vicinity of the Fermi surface in the strong magnetic field ($H > H$). Such an assumption allows us to explain the process of pair destruction as follows. Cooper's pair stops to exist when the strong magnetic field flips the spin of one quasiparticle and both quasiparticles have the same spins directed according to the magnetic field. Hence the paramagnetic field appears. Let us note that the number of discoupled (normal) quasi-particles is proportional to the paramagnetic field, thus we have

$$N_n = 2\nu(0)\,\mu_B^*\,H_p = 2\,\nu(0)\cdot m_p \tag{2}$$

where m_p is the paramagnetic field expressed in the energy scale. The distribution function of quasiparticles in the superconducting state has the form (Fig. 1)

$$n_\alpha = \frac{1}{4}\begin{cases} 1 - \frac{\xi}{E} & \text{if} \quad |\xi| > m_p \\ 2\,\delta_{1\alpha} & \text{if} \quad |\xi| < m_p \end{cases} \tag{3}$$

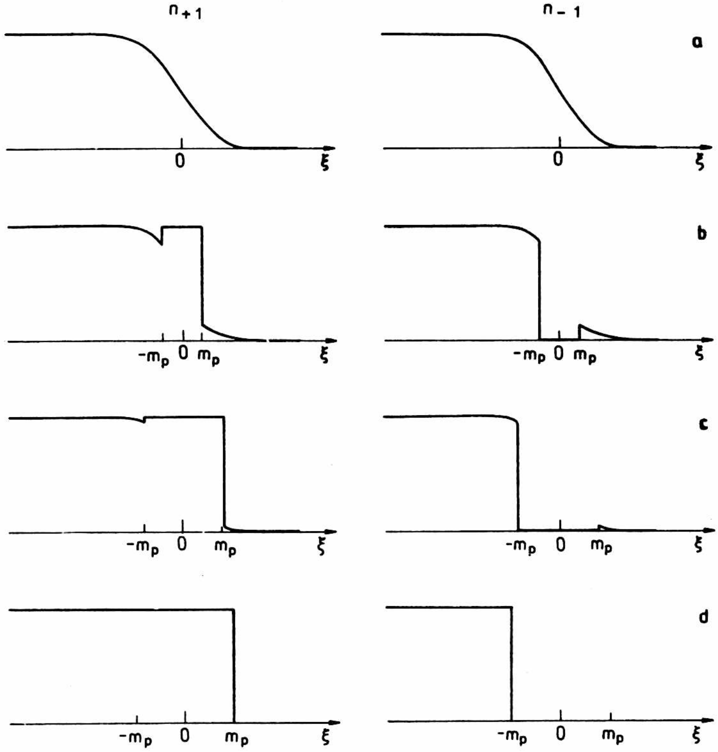

Fig. 1. The distribution function of quasiparticles with spin "up" n_{+1} and "down" n_{-1}, in the presence of the strong magnetic field for the type-II superconductors at T = 0. a) $H < H_{c1}$, the Meissner state, all quasiparticles are coupled in Cooper's pairs; b) and c), $H_{c1} < H < H_{c2}$, the mixed state, there exist only the quasiparticles with spin "up" near the Fermi surface; they cannot create Cooper's pairs; all other quasiparticles are still coupled; d) $H > H_{c2}$, the normal state

where $\alpha = \pm 1$ defines the position of the spin, i.e., "up" or "down", respectively, and $E = \sqrt{\xi^2 + \Delta^2}$. Applying (3) we can state that the number of superfluid quasiparticles is given by the expression

$$N_s = \nu(0)\,\mu_B^*\left[H_T - \sqrt{H_T^2 - \left(\frac{\Delta}{\mu_B^*}\right)^2}\,\right] = \nu(0)\,\mu_B^*\,(H_T - H_p) \qquad (4)$$

and the total number of quasiparticles

$$N = N_S + N_n \qquad (5)$$

is equivalent to the relation

$$\Delta(0) = \mu_B^*(H_T + H_p) \tag{6}$$

which is in agreement with electrostatics laws [36] and is identical with the gap equation (14.2). Such properties of the developed formalism allow us to consider space-homogeneous as well as space-inhomogeneous in the local limit superconducting states, although the condition of inhomogeneity is the most important assumption of other approaches [2, 3, 17, 29, 30, 35, 88, 89, 124, 138]. The equations of the paramagnetic approach keep their forms in both limits, however, in the local limit, the suitable quantities Δ, H_T, H_p and H_d become slow-varying functions of the position $\underline{R}$ (or the vector of inhomogeneity $\underline{q}$, cf. [37, 124]) and their average values have physical meaning, only. It will cause slight modification of results in comparison with some obtained in the homogeneous limit, i.e., there should appear an additional factor of the order of the unity connected with the averaging over configuration space. As it is shown below the space-homogeneous system constitutes an ideal type-II superconductor. Therefore to consider the real systems we must include the effects of inhomogeneity.

13. Equations of the paramagnetic theory [55]

We apply Eqs. (7.15), (8.9) and (8.13). We emphasize once more that in the local limit the quantities H_T, H_P and Δ are slow-varying functions of position $\underline{R}$, whereas the quantities H, $\Delta(T)$, μ_B^* and the Fermi liquid interaction are always space-homogeneous. Therefore the above equations become the microscopic equations. It causes that now we cannot divide the both sides of the gap equation by Δ to eliminate it. The macroscopic quantities are obtained after averaging over configuration space hence the macroscopic gap equation is of the form

$$\left\langle \Delta^2 \right\rangle_{\underline{R}} = \frac{1}{2}\, \nu(0)\, g_0 \left\langle \Delta^2 \int_0^{\omega_0} \frac{d\,\xi}{E} \left(\mathrm{th}\, \frac{E + \mu_B^* H_T}{2T} + \mathrm{th}\, \frac{E - \mu_B^* H_T}{2T} \right) \right\rangle_{\underline{R}} \tag{1}$$

In order to be able to consider the magnetic properties of superconducting systems we have to give the definition of the diamagnetic magnetization. We assume that the total magnetic field is composed of the paramagnetic and diamagnetic fields. Hence, the microscopic diamagnetic field is of the form

$$H_M = H_T - H_P \tag{2}$$

and the average diamagnetic magnetization can be expressed in the form

$$\langle M \rangle_{\underline{R}} = -\frac{1}{4\pi} \langle H_M \rangle_{\underline{R}} \tag{3}$$

Let us remark that the diamagnetic magnetization is proportional to the number of superfluid quasiparticles defined by Eq. (12.4).

14. A space-homogeneous case [50]

The space-homogeneous case, when $T = 0$, makes it possible to solve the problem in the largest range. Employing the relation [3, 22, 35, 96, 124]

$$1 = \nu(0)g_0 \ln \frac{2\omega_D}{\Delta(0)} \tag{1}$$

we reduce Eq. (6) to the form

$$\mu_B^*(H_p + H_T) = \Delta(0). \tag{2}$$

Equations (13.5), (13.8) and (2) form the complete set of non-linear equations for which two qualitatively different types of solutions are possible (cf. Section 43). As the first one, we consider the Meissner state, i.e., the case when the energy gap is constant and vanishes for the fixed value of the magnetic field H_c such that $H_c \leqslant H_{c1}$. The dependence of the energy gap and paramagnetic field on the external magnetic field can be described by the following equations

$$\Delta = \begin{cases} \Delta(0) & \text{if} \quad H < H_c \leqslant H_{c1}, \\ 0 & \text{if} \quad H \geqslant H_c, \end{cases} \tag{3}$$

and

$$H_p = \begin{cases} 0 & \text{if} \quad H < H_c, \\ \dfrac{H}{1+F_0^a} & \text{if} \quad H \geqslant H_c. \end{cases} \tag{4}$$

The diamagnetic magnetization is of the form

$$M = \chi_d^0 \begin{cases} H & \text{if} \quad H < H_c, \\ 0 & \text{if} \quad H \geqslant H_c. \end{cases} \tag{5}$$

The value of the external magnetic field H_c for which the energy gap vanishes can be derived from the Gibbs free energy evaluation and we get

$$\mu_B^* H_c = \sqrt{\frac{1}{2}(1 + F_0^a)}\, \Delta(0). \tag{6}$$

Let us consider now the other case when Δ tends to zero in the continuous way and it vanishes when $H = H_{c2}$. Since for $H < H_{c1}$ the Meissner state is realized, we discuss below only the mixed state when $H_{c1} \leqslant H \leqslant H_{c2}$. After some calculations we obtain the following functions

$$H_p = \frac{H - H_{c1}}{F_0^a - 1}, \tag{7}$$

$$\Delta^2 = \Delta^2(0)\, \frac{F_0^a + 1}{F_0^a - 1} \left(1 - \frac{H}{H_{c2}}\right) \tag{8}$$

where the critical magnetic fields have the forms

$$H_{c1} = \frac{\Delta(0)}{\mu_B^*} \quad \text{and} \quad H_{c2} = \frac{\Delta(0)(1 + F_0^a)}{2\mu_B^*}. \tag{9}$$

Since the type-II superconductivity is realized when $H_{c1} < H_{c2}$, we obtain the following condition

$$F_0^a > 1. \tag{10}$$

Let us assume now that for an ideal superconductor we can write

$$H_{c2} = H_c \sqrt{2}\, \varkappa, \qquad H_{c1} = \frac{H_c}{\sqrt{2}\, \varkappa} \tag{11}$$

and

$$H_c = \sqrt{4\pi \nu(0)}\, \Delta(0). \tag{12}$$

Then from Eqs. (9) we get

$$\varkappa = \frac{\sqrt{1 + F_0^a}}{2} \tag{13}$$

and

$$\mu_B^* = \frac{\varkappa}{\sqrt{2\pi \nu(0)}}. \tag{14}$$

Note that according to the obtained relations we have

$$\mu_B^* H_{c2} = 2\Delta(0)\, \varkappa^2. \tag{15}$$

Applying Eqs. (2), (7), (8) and (13.2) we obtain (cf. [17])

$$H_M = \left[\frac{\Delta}{\Delta(0)}\right]^2 \quad H_{c1} = \frac{2(H_{c2} - H)}{F_0^a - 1} \tag{16}$$

and hence inserting (13) we have (Fig.2)

$$H_M = \frac{H_{c2} - H}{2\varkappa^2 - 1} . \tag{17}$$

In order to calculate the Gibbs free energy difference we employ Eqs. (5.7) and (1), (2) where $H_T + H_P$ as a function of Δ and H can be derived from the following relation

$$\mu_B^*(H_T + H_P) = 2p + \frac{q\Delta^2}{\mu_B^*(H_T + H_P)} \tag{18}$$

where

$$p = \frac{\mu_B^* H}{F_0^a + 1} , \qquad q = \frac{F_0^a - 1}{F_0^a + 1} .$$

After some calculations we obtain

$$\Delta G = -\frac{\nu(0)}{2q}\left(\sqrt{p^2 + q\Delta^2} - p\right)^2 \tag{19}$$

and hence

$$\Delta G = -\frac{1}{4\pi}\,\frac{(H_{c2} - H)^2}{F_0^a - 1} = -\frac{1}{8\pi}\,\frac{(H_{c2} - H)^2}{2\varkappa^2 - 1} . \tag{20}$$

Now we state that ΔG is always negative if $F_0^a > 1$ ($\varkappa > 1/\sqrt{2}$) and $H < H_{c2}$, and $\Delta G = 0$ if $H = H_{c2}$. For $H = H_{c1}$ the Gibbs free energy difference reduces to the form

$$\Delta G = -\frac{H_{c1}^2}{8\pi}(2\varkappa^2 - 1) \tag{21}$$

and it tends to zero if $\varkappa \to 1/\sqrt{2}$. Moreover, we have

$$\frac{\partial \Delta G}{\partial H} = -M . \tag{22}$$

The same result (20) can be also obtained in the typical way. Then

$$\Delta G = -\frac{H_c^2}{8\pi} - \int_0^H M\,dH = -\frac{1}{8\pi}(H_c^2 - H_{c1}^2) - \int_{H_{c1}}^H M\,dH \tag{23}$$

where we replace

$$\frac{1}{2}\,\nu(0)\Delta^2(0) \quad \text{by} \quad \frac{H_c^2}{8\pi}\,.$$

Note that if we put $H = H_{c1}$, we get the relation

$$\Delta G(H_{c1}) = -\frac{1}{8\pi}\left(\frac{H_{c2}^2}{2\varkappa^2} - H_{c1}^2\right) \tag{24}$$

which can be employed to derive H_{c1} when ΔG is obtained by means of the formula (5.7). Now, of course, $H_{c1} = H_{c2}/2\varkappa^2$. The same problem can be considered also in dependence on the magnetic induction B. Then, after applying the following relation

$$H - B = H_T - H_P \tag{25}$$

in analogy to the previous results, we obtain (cf. [17])

$$H_P = \frac{B}{F_0^a + 1}\,, \tag{26}$$

$$\Delta^2 = \Delta^2(0)\left(1 - \frac{B}{H_{c2}}\right), \tag{27}$$

$$H_M = \frac{2(H_{c2} - B)}{F_0^a + 1} = \frac{H_{c2} - B}{2\varkappa^2}\,, \tag{28}$$

$$\Delta F = -\frac{1}{4\pi}\,\frac{(H_{c2} - B)^2}{F_0^a + 1} = -\frac{1}{8\pi}\,\frac{(H_{c2} - B)^2}{2\varkappa^2}\,, \tag{29}$$

and if B tends to zero Eq. (29) reduces to the form

$$\Delta F = -\frac{H_c^2}{8\pi}\,. \tag{30}$$

Moreover, the following relations are fulfilled

$$\frac{\partial \Delta F}{\partial B} = -M\,, \tag{31}$$

$$\Delta F = -\frac{H_c^2}{8\pi} - \int_0^B M\,dB\,, \tag{32}$$

and

$$\Delta G = \Delta F + 2\pi M^2. \tag{33}$$

It is worth reminding that the obtained results concern the mixed state only when $H_{c1} \leqslant H \leqslant H_{c2}$. Therefore, now ΔG as well as ΔF vanish when H or B achieve their critical values $H_{c2} = B_{c2}$. Moreover, we state the full consistence of the obtained results. However, the Helmholtz free energy difference does not imposes any restrictions on the Ginzburg-Landau parameter, therefore it possesses small practical meaning. In this way the problem of when the particular states are realized is solved univocally. The state with the energy gap existing for the magnetic fields $H > H_{c1}$ is more stable than the one with the vanishing energy gap, because its Gibbs free energy is lower. Such a state can only be realized when the inequality (10) is fulfilled, since in the opposite case this more preferable mixed state is forbidden.

15. A space-inhomogeneous case - the local limit [55]

a) The zero-temperature limit

In order to consider a real superconductor we must take into account the inhomogeneity of a system. We again assume that the energy gap is a continuous and decreasing function of the magnetic field which vanish in the whole volume coincidently when $H = H_{c2}$. Thus, all other quantities become the space-homogeneous, hence the local limit approximation is well-defined at least in the vicinity of the point H_{c2}. Therefore we can treat the effects of inhomogeneity as a small perturbation of the homogeneous state and apply the perturbation method. Then we can write

$$\mu_B^*(H_T + H_P) = p(1 + r\sqrt{1 - \varepsilon}) \tag{1}$$

where

$$r = \frac{1 + h}{1 - h}, \qquad \varepsilon = s\left(1 - \frac{\Delta_{\underline{R}}^2}{\Delta^2}\right)$$

and

$$h = 1 - \frac{H}{H_{c2}}, \qquad s = \frac{4h}{(1 + h)^2},$$

Δ is given by Eq. (14.8) and the energy gap $\Delta_{\underline{R}}$ has the same properties as the wave function of Cooper's pair. Note that since h fulfils the condition $0 \leqslant h \leqslant 1$, hence $0 \leqslant s \leqslant 1$.

Employing Eqs. (8.9), (8.13), (13.2), (14.18 - 19) we can write

$$\langle H_M \rangle_{\underline{R}} = \left\langle \frac{\Delta_{\underline{R}}^2}{(\mu_B^*)^2(H_T + H_P)} \right\rangle_{\underline{R}}, \tag{2}$$

$$\Delta G = -\frac{1}{2}\,\nu(0)\,q\left\langle\left[\frac{\Delta^2_{\underline{R}}}{\mu^*_B(H_T + H_P)}\right]^2\right\rangle_{\underline{R}} \tag{3}$$

and, moreover, $\langle B\rangle_{\underline{R}}$, $\langle H_T\rangle_{\underline{R}}$, $\langle H_P\rangle_{\underline{R}}$ can be found from the relations

$$\langle B\rangle_{\underline{R}} = H - \langle H_M\rangle_{\underline{R}}\,, \tag{4}$$

$$\langle H_T\rangle_{\underline{R}} = \frac{1}{2}\,\langle H_T + H_P + H_M\rangle_{\underline{R}}\,, \tag{5}$$

$$\langle H_P\rangle_{\underline{R}} = \frac{1}{2}\,\langle H_T + H_P - H_M\rangle_{\underline{R}}\,. \tag{6}$$

In order to derive the above quantities we have to solve the gap equation (13.1) which now can be reduced to the form

$$\left\langle \Delta^2_{\underline{R}} \ln \frac{\Delta(0)}{\mu^*_B\,(H_T + H_P)}\right\rangle_{\underline{R}} = 0. \tag{7}$$

After some calculations in the third order of the perturbation method we obtain

$$\langle H_M\rangle_{\underline{R}} = \frac{H_{c2} - H}{\beta_2(2\varkappa^2 - 1)}\;Y, \tag{8}$$

$$\Delta G = -\frac{1}{8\pi}\;\frac{(H_{c2} - H)^2}{\beta_2(2\varkappa^2 - 1)}\;Z\,, \tag{9}$$

$$\langle H_T + H_P\rangle_{\underline{R}} = \frac{H}{2\varkappa^2} + \frac{H_{c2} - H}{\beta_2\;2\varkappa^2}\;Y \tag{10}$$

where Y and Z are polynomials and have the forms

$$Y = 1 + \frac{3}{2}\,h\sigma_3 + 8h^2\sigma_3 + \frac{9}{2}\,h^2\sigma^2_3 - \frac{10}{3}\,h^2\sigma_4,$$

$$Z = 1 + h\sigma_3 + 4\,h^2\sigma_3 + \frac{9}{4}\,h^2\sigma^2_3 - \frac{5}{3}\,h^2\sigma_4,$$

and

$$\sigma_n = \frac{\beta_n}{\beta_2^{\,n-1}} - 1, \qquad \beta_n = \frac{\left\langle \Delta^{2n}_{\underline{R}}\right\rangle_{\underline{R}}}{\left\langle \Delta^2_{\underline{R}}\right\rangle^n_{\underline{R}}},$$

After applying Section 33, we state that $\sigma_n \geqslant 0$ and $\beta_n \geqslant 1$ and the equalities are attained for the homogeneous system only. It is easy to verify that the following condition is satisfied

$$\frac{\partial \Delta G}{\partial H} = - \langle M \rangle_{\underline{R}} \tag{11}$$

and hence the polynomials Y and Z must fulfil the relation

$$Y = Z + \frac{1}{2} h \frac{\partial Z}{\partial h} . \tag{12}$$

We cannot derive the factors σ_n in the frame of the presented approach. However, employing the results of Abrikosov's theory [2] with respect to the strongly localized Cooper's pairs we obtain

$$\frac{\beta_n}{\beta_2^{n-1}} = \frac{2^{\frac{n-1}{2}}}{\sqrt{n}} \tag{13}$$

where β_2 derived for the triangular or square lattice is equal to, respectively, (of. [17])

$$\beta_\Delta = 1.1596, \qquad \beta_\Box = 1.1803. \tag{14}$$

Employing the above results and Eq. (14.24) we can derive H_{c1} as a function $\varkappa$. In Figure 4 there is presented the function ξ_Δ where

$$\xi(\varkappa) = 2\varkappa^2 \frac{H_{c1}}{H_{c2}} . \tag{15}$$

Analysing this function we state that $\xi(1/\sqrt{2}) = 1$ and that for $\varkappa \gtrsim 0.8$ it can be approximated by the linear function of the following form

$$\xi(\varkappa) = A\varkappa + B \tag{16}$$

where

$$A = 2\left[\left(1 - \frac{1.3670}{\beta_2}\right)\right]^{1/2} , \qquad B = \frac{1.1966}{\beta_2}$$

so

$$\begin{aligned} \xi_\Delta &= 0.1975\varkappa + 1.0319, \\ \xi_\Box &= 0.2709\varkappa + 1.0138. \end{aligned} \tag{17}$$

The obtained interpolations give correct results only for the medium values of $\varkappa$ when

$$\left(1 - \frac{H_{c1}}{H_{c2}}\right)^3 << 1. \tag{18}$$

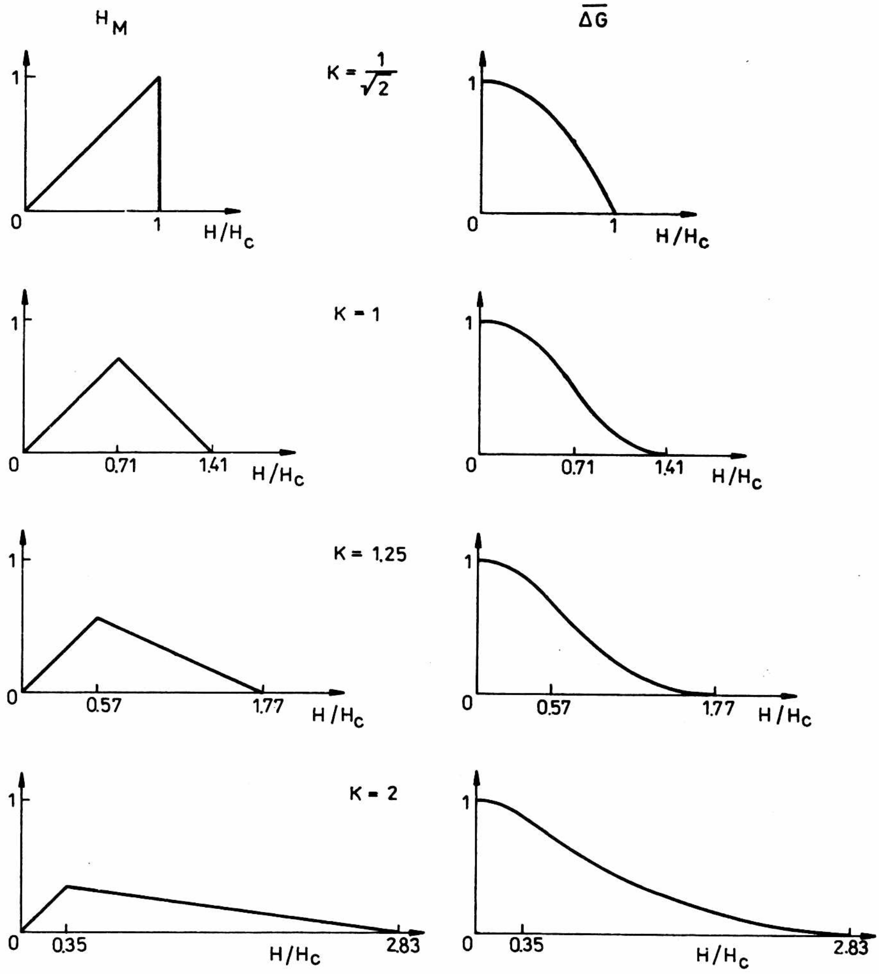

Fig. 2. The diamagnetic fields and Gibbs free energy differences of the ideal type-II superconductors

In order to illustrate the obtained results we give some examples of the diamagnetic fields and Gibbs free energies defined for a few fixed values of $\varkappa$. In Figure 2 the diamagnetic fields and Gibbs free energy differences of ideal type-II superconductors are presented for $\varkappa = 1/\sqrt{2}$, $\varkappa = 1$, $\varkappa = 1.25$ and $\varkappa = 2$. The case $\varkappa = 1/\sqrt{2}$ constitutes the limit case of type-II superconductors. Note that the graphs of the diamagnetic fields and Gibbs free energy differences constructed for the Meissner and mixed states intersect at the same points $H_{c1} = H_c/\sqrt{2}\,\varkappa$. It causes that the diamagnetic field is a continuous function of the external

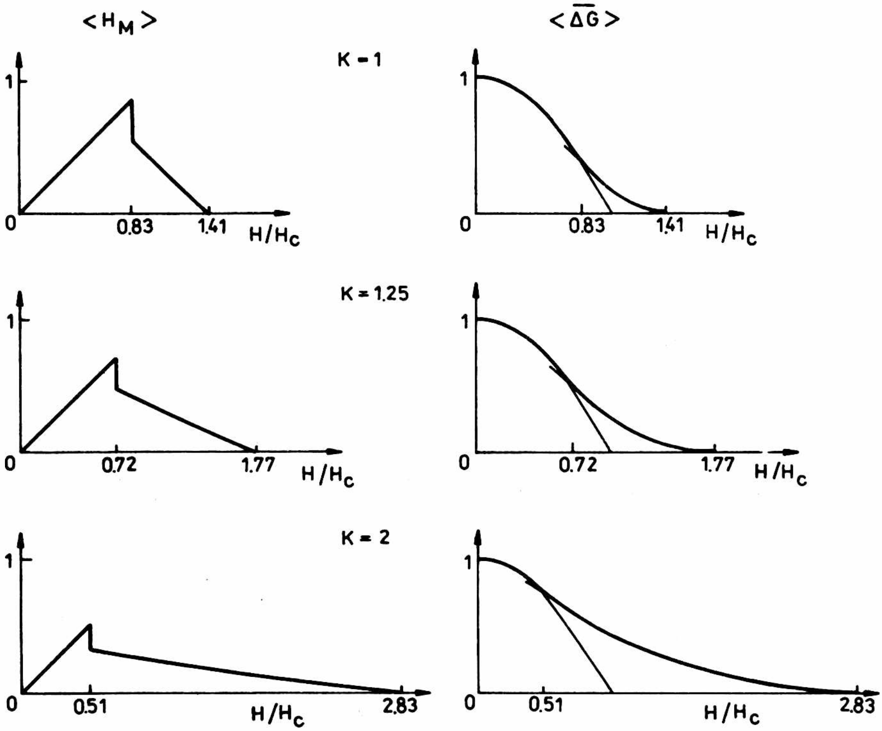

Fig. 3. The diamagnetic fields and Gibbs free energy differences of the real type-II superconductors, $\beta_2 = \beta_\Delta$

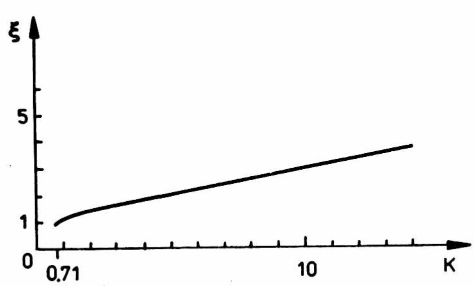

Fig. 4. The function $\xi_\Delta = 2\varkappa^2\, H_{c1}/H_{c2}$

magnetic field. It is also a linear function in the mixed state. In Figure 3 the diamagnetic fields and Gibbs free energy differences of real (i.e., inhomogeneous in the mixed state) type-II superconductors are presented for $\varkappa = 1$, $\varkappa = 1.25$, $\varkappa = 2$. The limit cases $\varkappa = 1/\sqrt{2}$ coinicide. The diamagnetic field in the mixed state is no longer a linear function of the external magnetic field. However, the flexion up of the curve is almost invisible in scale of the figure since such curve lies close to the straight line obtained in the homogeneous case in the whole range. Analysing the graphs of the Gibbs free energy differences in the Meissner and mixed states we notice that the curves intersect at the point $H > H_c/\sqrt{2}\,\varkappa$ which can be identified with H_{c1}. (Note that the mixed state cannot appear for $H < H_c/\sqrt{2}\,\varkappa$ whereas the Meissner state can be realized exclusively). It causes that the phase transition from the Meissner state to the mixed state should be of the first order as it is shown in the graphs. The influence of temperature will cause the softening of the diamagnetic field leap, which is observed experimentally.

In Figure 4 we present the function $\xi(\varkappa)$ for the triangular lattice. In the homogeneous case $\xi(\varkappa) = 1$. Now $\xi(1/\sqrt{2}) = 1$ and for about $\varkappa \gtrsim 0.8$ it can be approximated with the linear function according to Eq. (17).

b) The Ginzburg-Landau limit

In the presented paramagnetic theory we still consider the quasiparticles concentrated near the Fermi surface. Their total number is almost constant within the whole interval of temperatures discussed. At temperature increasing from zero to T_c there appear some uncoupled quasiparticles in the superconducting system. Thereby the external magnetic field can infiltrate the system. The paramagnetic magnetization appears at the weak magnetic fields [51]. However, this phenomenon does not destroy Cooper's pairs until $H < H_c$ or H_{c1}. The number of the coupled quasiparticles is fixed by the square of the wave function of Cooper's so it is proportional to the energy gap in the square, i.e., $N_s \sim \Delta^2$. The energy gap is a slow-decreasing function of temperature at a large range of temperatures and it tends rapidly to zero in the vicinity of T_c thereby the number of the superfluid quasiparticles substantially decreases. In this region (if the superconductivity is expected to exist) the quantities Δ and $\mu_B^* H_T$ become small in comparison with T_c. We however assume that the critical total magnetic field (expressed in energy scale) divided by T_c becomes a small quantity of second order, i.e., (cf. [35] and Section 21)

$$\frac{\mu_B^* H_{c2}(T)}{T} \sim \left[\frac{\Delta(T)}{T}\right]^2 << \frac{\Delta(T)}{T} \tag{19}$$

Therefore, we have to treat the total magnetic field as a perturbing term in the temperature-dependent gap equation and hence the effects of inhomogeneity should be of the same order. Let us consider now the paramagnetic field and gap equation. After expanding the integrands in a power series the above equations reduce to the forms (cf. Section 39)

$$H_p = H_T \left[1 - \frac{7\zeta(3)}{4\pi^2 T^2} \right] \Delta^2 \quad , \tag{20}$$

$$\Delta \ln \frac{T}{T_c} = \Delta \left\{ \frac{7\zeta(3)}{8\pi^2 T^2} \left[\Delta^2 + 2 (\mu_B^*)^2 H_T^2 \right] \right.$$

$$\left. - \frac{93\zeta(5)}{128 \pi^4 T^4} \left[\Delta^4 + 8\Delta^2 (\mu_B^*)^2 H_T^2 \right] + \frac{635 \zeta(7)}{1024 \pi^6 T^6} \Delta^6 \right\} \tag{21}$$

where we restrict ourselves to the sixth order terms of Δ/T ($\mu_B^* H_T \sim (\Delta/T)^2$) in the gap equation (cf. [95, 102]). We also give Eq. (7.15) which now reduces to the form

$$H_T = \frac{H}{F_0^a + 1} + \frac{F_0^a H_{c2}}{(F_0^a + 1)^2} \frac{7\zeta(3)}{\pi^2} \frac{\Delta^2}{4 T^2} \tag{22}$$

where we put the critical values of the magnetic field in order to keep the assumed accuracy of calculations. According to Eq. (19) we cannot assume that Δ vanishes for H_{c2} (cf. [92]). However, we can assume that Eq. (21) becomes an identity if we put $H = H_{c2}(T)$.

Comparing the terms of the same order in Eq. (21) we obtain

$$\ln \frac{T_c}{T} = \frac{7 \zeta(3)}{8\pi^2 T_c^2} \Delta^2(T) \tag{23}$$

and

$$\frac{7 \zeta(3)}{4\pi^2 T_c^2} (\mu_B^*)^2 H_{Tc2}^2 = \frac{93 \zeta(5)}{128 \pi^4 T_c^4} \Delta^4(T) \tag{24}$$

where we omit other insignificant relations. Taking into consideration Eqs. (22) and (23) from Eq. (24) we get (cf. [95, 120, 138])

$$H_{c2}(T) = \alpha 2 \sqrt{2} H_c(0) \left(1 - \frac{T}{T_c}\right) \varkappa \tag{25}$$

where number factor is

$$\alpha = \frac{e^C}{7\zeta(3)} \sqrt{\frac{186\zeta(5)}{7\zeta(3)}} = 1.013 \tag{26}$$

and could be identified with the unity, keeping the accuracy of the developed theory. We can also take into account that in the temperature regime under consideration we have [35]

$$H_c(T) = H_c(0)\, 2\left(1 - \frac{T}{T_c}\right) = H_c(0)\left[1 - \left(\frac{T}{T_c}\right)^2\right] , \tag{27}$$

hence

$$H_{c2}(T) = \alpha\sqrt{2}\,\varkappa H_c(T) \tag{28}$$

and, moreover,

$$\left(\frac{d\, H_{c2}(T)}{d\, T}\right)_{T_c} = -\, 17.92\, \varkappa\sqrt{\nu(0)} , \tag{29}$$

The presented method allows us to avoid the introduction of the additional parameters (relaxation ratios) and can be applied also for pure systems (cf. [88, 138]). Employing the above results we can consider the dependence of the system on the magnetic field when H is close to H_{c2}. Note that according to (22) appearing effects are of the fourth order. Applying Eq. (20) we find the following formula of the diamagnetic field

$$H_M = \frac{7\,\zeta(3)}{4\pi^2\, T^2}\;\frac{H_{c2}\,\Delta^2}{F_0^a + 1} . \tag{30}$$

Substituting Equation (22) into Eq. (21) and taking into account Eqs. (23) and (24) and remarks given above we get the following equation

$$\begin{aligned} H_{c2} - H &= \beta_2 \langle H_M \rangle_{\underline{R}} \\ &\times\left[F_0^a - \frac{93\zeta(5)}{98\,\zeta^2(3)}\,(F_0^a + 1) + \frac{635\zeta(7)}{2604\zeta(3)\zeta(5)}\,(F_0^a + 1)\right] \end{aligned} \tag{31}$$

and hence we obtain the diamagnetic magnetization in the form

$$\langle M \rangle_{\underline{R}} = -\,\frac{1}{4\,\pi}\;\frac{H_{c2} - H}{\beta_2(2.06\,\varkappa^2 - 1)} . \tag{32}$$

In order to refine the obtained results we can introduce Maki's [88] notation, then

$$\varkappa_1 = 1.013\,\varkappa , \qquad \varkappa_2 = 1.016\,\varkappa . \tag{33}$$

Moreover, employing Eqs. (22) and (14.25) we find the average magnetic induction, paramagnetic field and total magnetic field in the forms

$$\langle B \rangle_{\underline{R}} = H - \langle H_M \rangle_{\underline{R}} , \tag{34}$$

$$\langle H_P \rangle_{\underline{R}} = \frac{\langle B \rangle_{\underline{R}}}{F_0^a + 1}\,, \tag{35}$$

$$\langle H_T \rangle_{\underline{R}} = \frac{H}{F_0^a + 1} + \frac{F_0^a}{F_0^a + 1} \langle H_M \rangle_{\underline{R}}\,. \tag{36}$$

The obtained results reveal analogical properties in both discussed limits, though they differ in some details. However, if the diamagnetic magnetization is to be negative, additional conditions for the parameter $\varkappa$ are obtained.

© The presented formalism of superconductivity allows us to explain all processes taking place in the mixed state. The presented approach refers to the systems which can be described in the Fermi liquid terms. Thus one should restrict itself to the almost isotropic systems. However, in the static limit only F_0^a parameter of the Fermi liquid interaction modifies the results. This parameter defines the mean value of spin exchange quasiparticle interaction and can be identified with the intraatomic exchange integral in Hubbard model [140]. So, it can be correctly defined for the other systems with "non-spherical" Fermi surface. Moreover, the presented formalism can be easily developed by permitting the particle-hole asymmetry [101, 102] and by including a pure pairing interaction into isotropic systems in a more general form dependent on the momentum vector (not only its direction) (cf. [114]). It can be elaborated according to the prescription given in [124].

16. The generalized Gorkov approach [54]

The presented formalism constitutes the generalization of some other approaches when the Fermi liquid interaction is included. Moreover, we assume now that the paramagnetic and orbital terms can be considered as quantities of the same order (cf. [37, 54, 80, 95]).

The results of Section 9 have been employed here. We complete the above equations with the Fermi liquid interactions effects. Applying the formalism developed in Section 7 assuming the quasi-normal Green function in the form (4.17) we find the suitable relations in the form

$$v_s = V_s - \frac{1}{3} F_1^s\, v_s\, \frac{N_n}{N}\,, \tag{1}$$

$$H_T = H - F_0^a H_p \tag{2}$$

where the quantities v_s and H_T, being now renormalized by Fermi liquid interaction in opposition to V_s and H, appear effectively in all the expressions considered below. It is worth noticing that because of the inclusion of the Fermi liquid interaction in a quite general form all harmonics of the interaction give the renormalizing contribution to the form of the quasi-normal Green functions which results from the angular dependence of the zeroth Green function. Therefore, all higher Landau parameters should in fact be neglected in the presented considerations. Moreover in order to consider the obtained results in detail we restrict ourselves to two standard limits.

a) The zero-temperature limit

In the zero-temperature limit, after applying the relations (35.14, 15), all integrals defined in Section 9 can be computed explicitly and from (9.2), (9.11) we obtain

$$\Delta = \Delta(0) \qquad \text{if} \quad p_0 v_s + h < \Delta ,$$

$$(p_0 v_s - h) \ln \frac{\Delta}{\Delta(0)} - \sqrt{(p_0 v_s + h)^2 - \Delta^2} + (p_0 v_s + h) \ln \frac{p_0 v_s + h + \sqrt{(p_0 v_s + h)^2 - \Delta^2}}{\Delta(0)} = 0$$
$$\text{if} \quad |p_0 v_s - h| < \Delta \leqslant p_0 v_s + h ,$$

$$\operatorname{sgn}(p_0 v_s - h) \left[|p_0 v_s - h| \ln \frac{|p_0 v_s - h| + \sqrt{(p_0 v_s - h)^2 - \Delta^2}}{\Delta(0)} - \sqrt{(p_0 v_s - h)^2 - \Delta^2} \right] + (p_0 v_s + h) \ln \frac{p_0 v_s + h + \sqrt{(p_0 v_s + h)^2 - \Delta^2}}{\Delta(0)} - \sqrt{(p_0 v_s + h)^2 - \Delta^2} = 0$$
$$\text{if} \quad \Delta \leqslant |p_0 v_s - h| \tag{3}$$

and

$$H_p = 0 \qquad \text{if} \quad p_0 v_s + h < \Delta ,$$

$$H_p = \frac{1}{4 \mu_B^*} \left[\frac{p_0 v_s + h}{p_0 v_s} \sqrt{(p_0 v_s + h)^2 - \Delta^2} - \frac{\Delta^2}{p_0 v_s} \ln \frac{p_0 v_s + h + \sqrt{(p_0 v_s + h)^2 - \Delta^2}}{\Delta} \right]$$
$$\text{if} \quad |p_0 v_s - h| < \Delta \leqslant p_0 v_s + h ,$$

$$H_p = \frac{1}{4 \mu_B^*} \left[\frac{p_0 v_s + h}{p_0 v_s} \sqrt{(p_0 v_s + h)^2 - \Delta^2} - \frac{|p_0 v_s - h|}{p_0 v_s} \sqrt{(p_0 v_s - h)^2 - \Delta^2} - \frac{\Delta^2}{p_0 v_s} \ln \frac{p_0 v_s + h + \sqrt{(p_0 v_s + h)^2 - \Delta^2}}{|p_0 v_s - h| + \sqrt{(p_0 v_s - h)^2 - \Delta^2}} \right]$$
$$\text{if} \quad \Delta \leqslant |p_0 v_s - h| \tag{4}$$

and

$$N_n = 0 \qquad \text{if} \quad p_0 v_s + h < \Delta$$

$$\frac{2N_n}{N} + \frac{6(\mu_B^*)^2 H_p H_T}{p_0^2\, v_s^2} = \left[\left(1 + \frac{h}{p_0\, v_s}\right)^2 - \left(\frac{\Delta}{p_0\, v_s}\right)^2\right]^{3/2}$$

$$\text{if} \quad |p_0 v_s - h| < \Delta \leqslant p_0 v_s + h,$$

$$\frac{2N_n}{N} + \frac{6(\mu_B^*)^2 H_p H_T}{p_0^2\, v_s^2} = \left[\left(1 + \frac{h}{p_0 v_s}\right)^2 - \left(\frac{\Delta}{p_0 v_s}\right)^2\right]^{3/2}$$

$$+ \operatorname{sgn}(p_0 v_s - h)\left[\left(1 - \frac{h}{p_0\, v_s}\right)^2 - \left(\frac{\Delta}{p_0 v_s}\right)^2\right]^{3/2}$$

$$\text{if} \quad \Delta \leqslant |p_0 v_s - h|. \tag{5}$$

In order to derive H_{c2} we put $\Delta = 0$, then we obtain

$$\left| p_0 v_s - h \right|^{\frac{1}{2}\left(1 - \frac{h}{p_0 v_s}\right)} (p_0 v_s + h)^{\frac{1}{2}\left(1 + \frac{h}{p_0 v_s}\right)} = \tfrac{1}{2}\, e\, \Delta(0), \tag{6}$$

$$H_p = H_T\,, \tag{7}$$

$$N_n = N \tag{8}$$

and assuming that

$$p_0 v_s = \mu_B^*\, H\, x \tag{9}$$

after applying Eqs. (1) and (2) we obtain the following relations

$$p_0 v_s = \frac{\mu_B^* H}{1 + \frac{1}{3} F_1^s} \qquad \text{and} \qquad H_T = \frac{H}{1 + F_0^a}\,. \tag{10}$$

Substituting them into (6) we find the critical magnetic field in the form

$$H_{c2} = \frac{\bar{\Delta}(0)e}{2\,\bar{\mu}_B^*}\,(1 + F_0^a)\left|1 - \frac{1}{c}\right|^{\frac{1}{2}(c-1)} \left(1 + \frac{1}{c}\right)^{-\frac{1}{2}(c+1)} \tag{11}$$

where

$$c = \frac{1 + \frac{1}{3} F_1^s}{1 + F_0^a}\,,$$

In order to derive the Ginzburg-Landau parameter we assume that the generalized effective Bohr's magneton can be written in the form (14.14). Then we obtain

$$\varkappa_1^2 = \frac{e}{4}(1 + F_0^a)\left|1 - \frac{1}{c}\right|^{\frac{1}{2}(c-1)} (1 + \frac{1}{c})^{-\frac{1}{2}(c+1)} \qquad (12)$$

where we introduced Maki's notation [88].

Let us consider now the obtained expression in relation to the magnetic properties of the normal system. Assuming that the total magnetization is the sum of the Pauli paramagnetism and the Landau diamagnetism we can specify the following cases.

If the normal system reveals the strong diamagnetism, then $F_0^a \gg 1 + \frac{1}{3}F_1^s$ and Eq. (12) reduces to the form (Fig. 5a)

$$\varkappa_1 = \frac{1}{2}\sqrt{e(1 + \frac{1}{3}F_1^s)}. \qquad (13)$$

If the magnetic properties of the normal system are comparable with the free electron gas, then $\frac{1}{3}F_1^s \approx F_0^a$ and Eq. (12) reduces to the form

$$\varkappa_1 = \frac{1}{2}\sqrt{\frac{e}{2}}\left[(1 + \frac{1}{3}F_1^s)(1 + F_0^a)\right]^{1/4} \qquad (14)$$

and if the normal system shows the strong paramagnetism, or if there exist heavy fermions in the system, then $\frac{1}{3}F_1^a \gg 1 + F_0^a$ and Eq. (12) reduces to the form

$$\varkappa_1 = \frac{1}{2}\sqrt{(1 + F_0^a)}. \qquad (15)$$

The latest result is identical with the results achieved in Sections 14 and 15, thus the approaches are entirely consistent for heavy fermions system (cf. [89, 123, 138]).

b) The Ginzburg-Landau limit

In order to investigate the discussed system in the region nearby T_c we consider Eqs. (9.1), (9.8) and (9.9). Assuming that the following quantities Δ, h and $p_0 v_s$ are small in comparison with T, which is close to the phase transition temperature T_c, we can expand all the integrands in a power series neglecting the higher order terms. Then from Eqs. (9.1), (9.8) and (9.9) we obtain, respectively, (cf, [95])

$$\ln \frac{T_c}{T} = \frac{7\zeta(3)}{8\pi^2 T_c^2}\left(\Delta^2 + 2h^2 + \frac{2}{3} p_0^2 v_s^2\right)$$

$$- \frac{93\zeta(5)}{128\pi^4 T_c^4}\left(\Delta^4 + 8\Delta^2 h^2 + \frac{8}{3}\Delta^2 p_0^2 v_s^2\right.$$

$$\left. + \frac{8}{3} h^4 + \frac{8}{15} p_0^4 v_s^4 + \frac{49}{9} h^2 p_0^2 v_s^2\right), \tag{16}$$

$$H_p = H_T\left[1 - \frac{7\zeta(3)}{4\pi^2 T_c^2}\Delta^2\right], \tag{17}$$

$$N_n = N\left[1 - \frac{7\zeta(3)}{4\pi^2 T_c^2}\Delta^2\right] \tag{18}$$

where the applied relations are defined in Section 39.

Transforming Equation (18) we obtain the relation

$$\left[\frac{\Delta}{\Delta(0)}\right]^2 = \frac{4e^{2C}}{7\zeta(3)}\frac{N_s}{N} = 1.51\frac{N_s}{N}, \tag{19}$$

thus $\eta = 1.51$.

Equation (16) defires the temperature dependence of the energy gap in the presence of the perturbing magnetic field, comparing the terms of the same order we obtain

$$\ln \frac{T_c}{T} = \frac{7\zeta(3)}{8\pi^2 T_c^2}\Delta^2 \tag{20}$$

and

$$\frac{7\zeta(3)}{4\pi^2 T_c^2}\left(h^2 + \frac{1}{3} p_0^2 v_s^2\right) = \frac{93\zeta(5)}{128\pi^4 T_c^4}\Delta^4 \tag{21}$$

where other insignificant relations are omitted. On the other hand, each term of the series expansion is obtained under assumption that the other terms disappear, then the quantities h and v_s can be identified with their critical values which express themselves by means of the critical magnetic field H_{c2}. (The total magnetic field h and the superfluid velocity should be considered jointly). Taking into cosideration Eqs. (9), (10) and (20) from Eq. (21) we get (cf. Eqs. (15.26, 27))

$$H_{c2}(T) = \alpha\sqrt{2}\,H_c(0)\left(1 - \frac{T}{T_c}\right)\frac{(1 + \frac{1}{3}F_1^s)(1 + F_0^a)}{\sqrt{(1 + \frac{1}{3}F_1^s)^2 + \frac{1}{3}(1 + F_0^a)^2}}\,. \tag{22}$$

Hence, we find the Ginzburg-Landau parameter in the form $(\alpha = 1)$

$$\varkappa_1^2 = \frac{\sqrt{3}}{4}\;\frac{(1 + \frac{1}{3}F_1^s)(1 + F_0^a)}{\sqrt{3(1 + \frac{1}{3}F_1^s)^2 + (1 + F_0^a)^2}}\,. \tag{23}$$

Estimating the obtained expression (23), in the limits as before, we have (Fig.5b)

$$\varkappa_1 = \frac{1}{2}\sqrt{\sqrt{3}\,(1 + \tfrac{1}{3}F_1^s)} \qquad \text{if} \quad F_0^a \gg 1 + \tfrac{1}{3}F_1^s, \tag{24}$$

$$\varkappa_1 = \frac{1}{2}\left[\frac{\sqrt{3}}{2}\sqrt{(1 + \tfrac{1}{3}F_1^s)(1 + F_0^a)}\right]^{1/2} \qquad \text{if} \quad \tfrac{1}{3}F_1^s \approx F_0^a, \tag{25}$$

$$\varkappa_1 = \frac{1}{2}\sqrt{(1 + F_0^a)} \qquad \text{if} \quad \tfrac{1}{3}F_1^s \gg 1 + F_0^a. \tag{26}$$

According to Eqs. (24)-(26) and (13)-(15) the Ginzburg-Landau parameter $\varkappa$ is a function of the Landau parameters F_1^s and F_0^a. Its value can decrease when temperature increases from zero to T_c, and we have

$$\frac{\varkappa(T_c)}{\varkappa(0)} = \sqrt{\frac{\sqrt{3}}{e}} = 0.80 \qquad \text{if} \quad F_0^a \gtrsim \frac{1}{3}F_1^s \tag{27}$$

and

$$\frac{\varkappa(T_c)}{\varkappa(0)} = 1 \qquad \text{if} \quad \frac{1}{3}F_1^s \gg 1 + F_0^a \tag{28}$$

which are in good agreement with the results obtained in other approaches (cf. [35]). So, we can state that temperature can slightly modify the type of superconductivity only in the first and second discussed cases, and that it does not influence the type of superconductivity in case of the heavy fermion systems.

In conclusion, let us remark that the first and second cases coincide, thus only the heavy fermion case ($m^* = m(1 + \frac{1}{3}F_1^s)$) is particularly distinguished. Moreover, in case when the Fermi liquid interaction is excluded ($F_s^1 = F_0^a = 0$) the type-I superconductivity can be realized in the system only. This result confirms the advantage of the considered formalism in opposition to the other Green function approaches neglecting the paramagnetic effects (cf. [30, 31, 37, 57, 89, 92, 120, 124, 132, 138]). However, in order to take stock of this approach let us compare the results obtained in zero-temperature limit with the Gibbs free energy difference. Applying the formula (5.7) and Eqs. (1), (2), (4), (5), and (14.1) in the small Δ limit ($\Delta \longrightarrow 0$) from Eq. (4) we obtain (cf. Fig. 6)

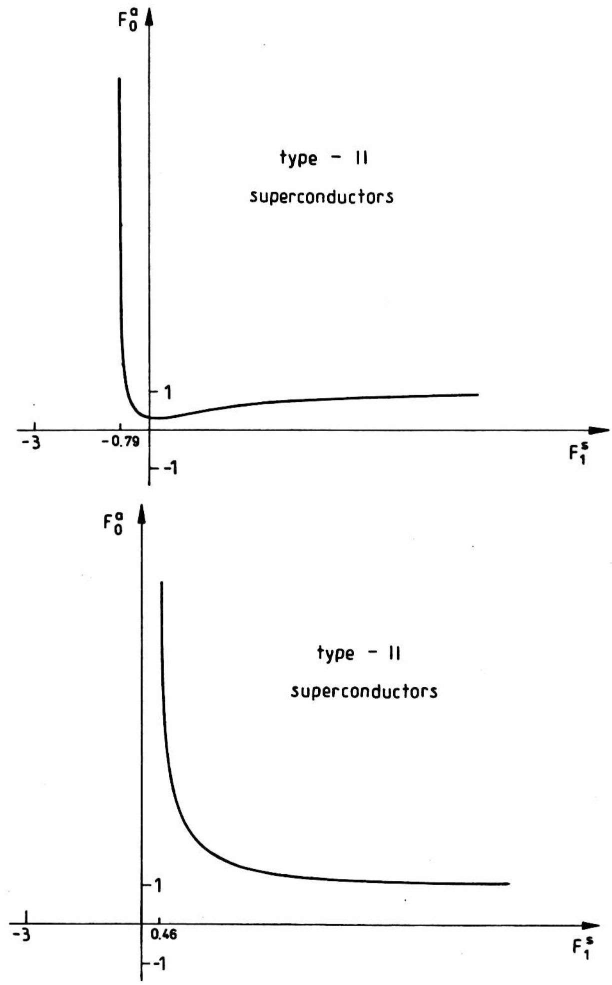

Fig. 5. The curve $\varkappa_1 = 1/\sqrt{2}$ in the plane of the Landau parameters F_1^s and F_0^a. The type-II superconductivity is realized in the region above the curve. a) The case T = 0. The minimum is achieved at the point F_1^s = 0.31, F_0^a = 0.33. For F_1^s = 0, F_0^a = 0.35. b) The case $T \lesssim T_c$

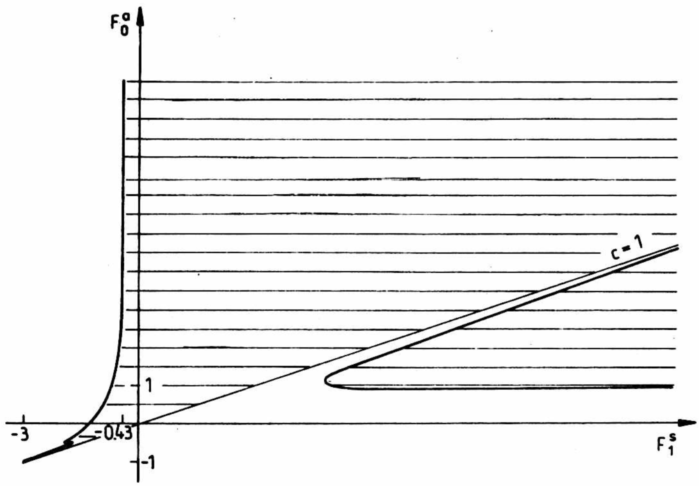

Fig. 6. The plot of the function $\Sigma = 0$. The type-II superconductivity is realized in the lined regions ($\Delta G < 0$). The curve from the right possesses the vertex at the point $F_1^s = 4.84$, $F_0^a = 1.13$ and the minimum at the point $F_1^s = 7.74$, $F_0^a = 0.92$. For $F_1^s = 5.13$, $F_0^a = 1$. The leap visible on the curve from the left happens in the region contained between the points $F_1^s = -1.85$, $F_0^a = -0.53$ and $F_1^s = -1.57$, $F_0^a = -0.50$. For $F_1^s = -1.28$, $F_0^a = 0$. The asymptote is of the form $F_0^a = 0.325\, F_1^s$

$$\Delta G = -\nu(0)\,\frac{\langle \Delta^4 \rangle_{\underline{R}}}{8(\mu_B^*)^2\, H_{c2}^2}\,\Sigma \tag{29}$$

where

$$\Sigma = \frac{(1+\frac{1}{3}F_1^s)^2}{1-c^2}\left\{1 + L - \frac{1}{1+\frac{1}{3}F_1^s}\left[L - \frac{1}{2}\ln^2 a\,(1-c^2)\,(1-c)\right]\right\},$$

$$L = (1-c^2)\left[6 - 6c\ln a + \frac{1}{2}c^2\ln^2 a\left(3 + \frac{1}{c^2}\right)\right]$$

and

$$a = \frac{c+1}{|c-1|}.$$

In the two limits specified previously Eq. (29) reduces to the forms

$$\Delta G = -\frac{1}{2}\nu(0)\,\frac{\langle\Delta^4\rangle_{\underline{R}}}{e^2\,\Delta^2(0)}\;\frac{1+\frac{7}{3}F_1^s}{1+\frac{1}{3}F_1^s} \quad \text{if} \quad F_0^a \gg 1+\frac{1}{3}F_1^s \tag{30}$$

and

$$\Delta G = -\frac{1}{2}\nu(0)\,\frac{\langle\Delta^4\rangle_{\underline{R}}}{\Delta^2(0)}\;\frac{F_0^a-1}{1+F_0^a} \quad \text{if} \quad F_1^s \gg 1+F_0^a. \tag{31}$$

Hence ΔG is negative if $F_1^s > -0.43$ or $F_0^a > 1$, respectively. Since the type-II superconductivity should be realized if ΔG is negative, we state that in the generalized case the obtained results (12) and (29) are not coherent. This situation can be explained if we include Maki's classification [88]. Then Equation (12) defines $\varkappa_1$, whereas from Eq. (29) $\varkappa_2$ can be derived. Note that $\varkappa_2$ tends to $\varkappa_1$ only in the heavy fermion limit.

17. Conclusions

Due to the inclusion of the Fermi liquid interaction the critical magnetic field H_{c2} and hence the Ginzburg-Landau parameter $\varkappa_1$ become the functions of the Landau parameters which can be estimated correctly in the normal state.

The obtained results and the present classification make is possible to predict the superconductor properties of some metals (elements or alloys) on the grounds of their normal static spin susceptibilities or electronic paramagnetic resonance (EPR) (cf. [113]), i.e., one can exactly foresee the type of the possible superconductivity, but one cannot determine whether the superconductivity will appear at all. Let us remark that although the present formalism concerns the superconductors with the almost spherical Fermi surface, nevertheless, some of their aspects should be revealed in a large scale. For example, pressure, defects or impurities can modify the Fermi liquid interaction, hence they should change some superconducting properties of the system, such as the type of superconductivity or values of the critical field. On the other hand, the same factors (pressure, defects and impurities) can also modify the value of the energy gap $\Delta(0)$ and hence the critical magnetic fields (cf. [111, 129-131]).

For these reasons the type of superconductivity is the property of the system determined already in the normal state and it can be predicted as well as framed by means of the research on paramagnetic properties of the normal system. As to the alkali metals it is known that F_1^s is always positive, whereas F_0^a is negative (Na and K) [116], thus only the type--I superconductivity can be realized. So far, however, the phase transition (the first order) to the superconducting state has been observed only in Cs (and probably in Rb) [131]. Let us note that both the approaches are fully coherent merely for the heavy fermion system, i.e., if $F_1^s >> 1$. Moreover, we can state that the elimination of any type of magnetic effects is achieved when the suitable Landau parameter tends to infinity.

We remind that the parameter F_1^s modifies also the density of states by means of the effective mass.

Comparing the presented formalism with the other Green function approaches we can state that the inclusion of the Fermi liquid interaction is equiponderant to some other formal actions (connected, e.g., with impurities) which modify Gorkov's equations and lead to renormalization of the system parameters [37, 62, 80, 89, 124, 132, 138].

Although such parameters can be often eliminated from final equations they always exist in some intermediate relations in an implicit manner (cf. [2, 3, 17, 35, 96, 124, 129, 130]). Thus, our assumptions about renormalization of Bohr's magneton are entirely justified. Moreover, the presented actions exploit only the existing parameters of the Fermi liquid theory, therefore such an approach should find broad application, e.g., in EPR and NMR investigations when the Larmor frequency and Knight shift are to be derived.

IV. Superfluid ^{3}He

18. Stable states in the strong magnetic field [52]

Stability of individual phases of the superfluid ^{3}He is directly connected with their magnetic properties. However, because of the complex form of the order parameter for the system with the P-pairing all the investigations of ^{3}He are carried out in the weak magnetic field for the initially postulated stable or metastable phases [22, 39, 85, 121, 135, 151]. Since general forms of equations for arbitrary non-unitary states and non-zero temperatures in the presence of the strong magnetic field are too complicated to be solved in an analytic way, we restrict our study to the unitary states and zero-temperature limit. The imposed restrictions are minor since they leave large possibilities to choose the ground state from among the unitary states which are the main candidates to become stable [34, 58, 66, 73, 85]. We again consider the equations of magnetization and order parameter (energy gap). The considered equations are variational derivatives of Gibbs free energy and form the system of nonlinear integral equations which can be precisely solved. The obtained solutions allow us to derive the magnetization and order parameter as the function of the external magnetic field and the Fermi liquid interaction parameters. Since the considered integral equations are nonlinear there appear some bifurcation points at which the solutions split (Section 43). However, stable phases are determined by the supreme values of energy gap, which is always confirmed by the evaluation of the Gibbs free-energy difference.

In the presented consideration we fix the direction of the external magnetic field and denote it by " $\parallel$ " (parallel). The perpendicular directions are denoted by "$\perp$".

In order to derive explicit forms of magnetization and gap equations we have to compute a few characteristic integrals.

Since the computational methods are analogous in both cases we discuss them in detail, deriving the magnetization only. The applied forms of elliptic integrals are to be found in Sec. 38. All other appearing integrals can be computed in an elementary way, thus they are given without comments.

a) Paramagnetic magnetization and gap equations

The parameter of an arbitrary unitary state can be introduced in the form

$$\hat{\Delta} = \Delta \underline{d}\, \sigma_i\, \sigma^y \tag{1}$$

where $\underline{d}$ is the complex vector being the linear function of the unit vector $\hat{p}$, i.e., $d_i = d_{i\alpha}\, \hat{p}_\alpha$, and satisfies the condition

$$\underline{I} = \underline{d} \times \underline{d}^* = 0, \tag{2}$$

Δ is the energy gap which is a function of the external magnetic field only. The assumed restrictions allow us to write the following relation

$$\hat{\Delta}\hat{\Delta}^+ = \Delta^2 |\underline{d}|^2 \tag{3}$$

where the obtained expression is proportional to the unit matrix. Moreover, we have introduced the following symbols

$$\Delta_T = \Delta|\underline{d}|, \qquad \Delta_\perp = \Delta|\underline{d}_\perp| , \qquad \Delta_\parallel = \Delta|\underline{d}_\parallel| \tag{4}$$

where $\underline{d}_\perp$ and $\underline{d}_\parallel$ are the components of the vector $\underline{d}$, perpendicular and parallel to the direction of external magnetic field, respectively, and

$$|\underline{d}|^2 = |\underline{d}_\perp|^2 + |\underline{d}_\parallel|^2 . \tag{5}$$

By virtue of the above assumptions and results inserted in Section 10, the paramagnetic magnetization can be expressed in the form

$$M = \frac{\mu_B \nu_0}{2} \left\langle \int_0^\infty d\xi \left[\frac{\sqrt{\xi^2 + \Delta_\parallel^2} + h}{\sqrt{(\sqrt{\xi^2 + \Delta_\parallel^2} + h)^2 + \Delta_\perp^2}} - \frac{\sqrt{\xi^2 + \Delta_\parallel^2} - h}{\sqrt{(\sqrt{\xi^2 + \Delta_\parallel^2} - h)^2 + \Delta_\perp^2}} \right] \right\rangle . \tag{6}$$

Substituting

$$x = \sqrt{\xi^2 + \Delta_\parallel^2} , \tag{7}$$

the expression (6) reduces to the form

$$M = \frac{\mu_B \nu_0}{2} \left\langle \int_{\Delta_\parallel}^{\infty} \frac{dx\, x(x+h)}{\sqrt{x^2 - \Delta_\parallel^2}\sqrt{(x+h)^2 + \Delta_\perp^2}} + \int_{-\Delta_\parallel}^{-\infty} \frac{dx\, x(x+h)}{\sqrt{x^2 - \Delta_\parallel^2}\sqrt{(x+h)^2 + \Delta_\perp^2}} \right\rangle \tag{8}$$

where we have also replaced x by $-x$ in the other integral in order to compute only one type of the integral in the appropriate limits. Integrating (8) by parts, we get

$$M = \frac{\mu_B \nu_0}{2} \left\langle \sum_{\beta=0}^{1} \frac{(x+h)\sqrt{x^2 - \Delta_\parallel^2}}{\sqrt{(x+h)^2 + \Delta_\perp^2}} \Bigg|_{(-1)^\beta \Delta}^{(-1)^\beta \infty} - \Delta_\perp^2 \sum_{\beta=0}^{1} \int_{(-1)^\beta \Delta_\parallel}^{(-1)^\beta \infty} dx \frac{\sqrt{x^2 - \Delta_\parallel^2}}{\left[(x+h)^2 + \Delta_\perp^2\right]^{3/2}} \right\rangle . \tag{9}$$

The value of the first sum, which can be easily calculated if we replace x by $1/y$ and use the de l'Hospital's rule, is equal to zero. The non-vanishing part of Eq. (9) can be presented in the form

$$M = -\frac{\mu_B \nu_0}{2} \left\langle \Delta_\perp^2 \sum_{\beta=0}^{1} \left\{ \int_{(-1)^\beta \Delta_\parallel}^{(-1)^\beta \infty} \frac{dx}{\sqrt{(x^2 - \Delta_\parallel^2)\left[(x+h)^2 + \Delta_\perp^2\right]}} - \int_{(-1)^\beta \Delta_\parallel}^{(-1)^\beta \infty} \frac{dx(\Delta_T^2 + h^2 + 2hx)}{\left[(x+h)^2 + \Delta_\perp^2\right]\sqrt{(x^2 - \Delta_\parallel^2)\left[(x+h)^2 + \Delta_\perp^2\right]}} \right\} \right\rangle \tag{10}$$

which permits us to remark that general form of the magnetization can be expressed solely by means of elliptic integrals. In order to solve the specified problem and to simplify the subsequent calculations we introduce the following denotations

$$
\begin{aligned}
N &= (\Delta_T^2 + h^2)^2 - 4h^2\,\Delta_\parallel^2 ,\\
P &= \Delta_T^2 + h^2 - \sqrt{N} ,\\
Q &= \Delta_T^2 + h^2 + \sqrt{N} ,\\
R &= -\Delta_T^2 + h^2 + \sqrt{N} ,\\
S &= \Delta_T^2 - h^2 + \sqrt{N} .
\end{aligned} \tag{11}
$$

Let us note that the defined expressions are always positive for arbitrary structures of the order parameter. Now we substitute

$$x = \frac{-Q}{2h(t+1)}\left(1 + t\,\frac{P}{Q}\right) . \tag{12}$$

Then the following relations are valid,

$$dx = -\frac{\sqrt{N}}{h}\,\frac{dt}{(t+1)^2} , \tag{13}$$

$$\sqrt{(x^2 - \Delta_\parallel^2)\left[(x+h)^2 + \Delta_\perp^2\right]} = \frac{\sqrt{NQS}}{2h^2(t+1)^2}\sqrt{\left(1 - t^2\,\frac{P}{Q}\right)\left(1 + t^2\,\frac{R}{S}\right)} , \tag{14}$$

$$(x+h)^2 + \Delta_\perp^2 = \frac{S\sqrt{N}}{2h^2(t+1)^2}\left(1 + t^2\,\frac{R}{S}\right) . \tag{15}$$

Substituting equations (12)-(15) into Eq. (10) we obtain

$$
\begin{aligned}
M = -\frac{\mu_B \nu_0}{2}\Bigg\langle \Delta_\perp^2 \sum_{\beta=0}^{1}\Bigg[&\frac{2h}{\sqrt{QS}} \int_{(-1)^\beta\sqrt{\frac{Q}{P}}}^{-1} \frac{dt}{\sqrt{\left(1-t^2\,\frac{P}{Q}\right)\left(1+t^2\,\frac{R}{S}\right)}} \\
&- \frac{4h^3}{S\sqrt{QS}} \int_{(-1)^\beta\sqrt{\frac{Q}{P}}}^{-1} \frac{dt\,(t^2-1)}{\left(1+t^2\,\frac{R}{S}\right)\sqrt{\left(1-t^2\,\frac{P}{Q}\right)\left(1+t^2\,\frac{R}{S}\right)}}\Bigg]\Bigg\rangle .
\end{aligned} \tag{16}
$$

Since the functions under integrals are even functions of the variable t, we can reduce the limits of integration. Hence, we obtain

$$M = \mu_B \nu_0 \left\langle \frac{2h\,\Delta_\perp^2}{\sqrt{QS}} \left[\int_0^1 \frac{dt}{\sqrt{\left(1 - t^2 \frac{P}{Q}\right)\left(1 + t^2 \frac{R}{S}\right)}} \right.\right.$$

$$\left.\left. - \frac{2h^2}{S} \int_0^1 \frac{dt(t^2 - 1)}{\left(1 + t^2 \frac{R}{S}\right)\sqrt{\left(1 - t^2 \frac{P}{Q}\right)\left(1 + t^2 \frac{R}{S}\right)}} \right] \right\rangle, \tag{17}$$

and introducing the new variable

$$z = \sqrt{1 - t^2 \frac{P}{Q}}, \tag{18}$$

Eq. (17) reduces to the form

$$M = \mu_B \nu_0 \left\langle N^{1/4}(1 - k^2) \left[\int_{z_0}^{-1} \frac{dz}{(1-k^2z^2)\sqrt{(1-z^2)(1-k^2z^2)}} \right.\right.$$

$$\left.\left. - \int_{z_0}^{1} \frac{dz}{\sqrt{(1 - z^2)(1 - k^2z^2)}} \right] \right\rangle \tag{19}$$

where

$$k^2 = \frac{QR}{4h^2\sqrt{N}}, \qquad z_0 = \sqrt{1 - \frac{P}{Q}},$$

and $k^2 \leqslant 1, \quad z_0 \leqslant 1.$

While examining the first integral of Eq. (19) we show that it can be expressed in a simpler way

$$\int_{z_0}^{1} \frac{dz}{(1-k^2z^2)\sqrt{(1-z^2)(1-k^2z^2)}} = \frac{1}{1-k^2} \left[k^2 \frac{z_0\sqrt{1-z_0^2}}{\sqrt{1-k^2z_0^2}} \right.$$

$$\left. + \int_{z_0}^{1} \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} - k^2 \int_{z_0}^{1} \frac{dz\,z^2}{\sqrt{(1-z^2)(1-k^2z^2)}} \right], \tag{20}$$

hence Eq. (19) reduces to the form

$$M = \mu_B \nu_0 \left\langle N^{1/4} k^2 \left[\frac{z_0\sqrt{1-z_0^2}}{\sqrt{1-k^2z_0^2}} + \int_{z_0}^{1} \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} \right.\right.$$

$$- \int_{z_0}^{1} \frac{dz z^2}{\sqrt{(1 - z^2)(1 - k^2 z^2)}} \Bigg] \Bigg\rangle . \tag{21}$$

Introducing the next variable φ defined by the equation

$$\sin\varphi = z \tag{22}$$

we obtain finally

$$M = \mu_B \nu_0 \Bigg\langle N^{1/4} \Bigg[\frac{R}{2hN^{1/4}} - (1 - k^2) \int_{\varphi_0}^{\pi/2} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2\varphi}}$$

$$+ \int_{\varphi_0}^{\pi/2} d\varphi \sqrt{1 - k^2 \sin^2\varphi} \Bigg] \Bigg\rangle \tag{23}$$

where

$$\varphi_0 = \arc\sin\,(2N^{1/4}/\sqrt{Q}).$$

Both integrals are the so-called elliptic integrals. Applying the symbols introduced by Legendre we can rewrite Eq. (23) in the form (cf. Section 38)

$$M = \mu_B \nu_0 \Bigg\langle N^{1/4} \Bigg\{ \frac{R}{2hN^{1/4}} - (1 - k^2)\,[F(k) - F(\varphi_0, k)]$$

$$+ [E(k) - E(\varphi_0, k)] \Bigg\} \Bigg\rangle . \tag{24}$$

Keeping all general conditions specified above, the gap equations reduce to the form

$$\Delta d_i = \frac{3}{4} g_1 \nu_0 \hat{p}_\alpha \Bigg\langle \hat{p}_\alpha \Bigg[\int_0^{\xi_p} d\xi \frac{\sqrt{\xi^2 + \Delta_\parallel^2}\,\Delta d_i + h\Delta_\parallel \delta_{i\parallel}}{\sqrt{\xi^2 + \Delta_\parallel^2}\sqrt{(\sqrt{\xi^2 + \Delta_\parallel^2} + h)^2 + \Delta_\perp^2}}$$

$$+ \int_0^{\xi_p} d\xi \frac{\sqrt{\xi^2 + \Delta_\parallel^2}\,\Delta d_i - h\Delta_\parallel \delta_{i\parallel}}{\sqrt{\xi^2 + \Delta_\parallel^2}\sqrt{(\sqrt{\xi^2 + \Delta_\parallel^2} - h)^2 + \Delta_\perp^2}} \Bigg] \Bigg\rangle . \tag{25}$$

Since the parameter ξ_p fulfils the condition $\xi_p \gg \Delta$ it can be put equal to infinity in all convergent integrals. In our calculations this principle is automatically applied. Multiplying equation (25) by Δd_i^* and averaging over spherical angles we obtain

$$\Delta^2 = \frac{g_1 \nu_0}{4} \left\langle \int_0^{\xi_p} d\xi \frac{\sqrt{\xi^2 + \Delta_\parallel^2}\, \Delta_T^2 + h\Delta_\parallel^2}{\sqrt{\xi^2 + \Delta_\parallel^2}\sqrt{(\sqrt{\xi^2 + \Delta_\parallel^2} + h)^2 + \Delta_\perp^2}} \right.$$

$$\left. + \int_0^{\xi_p} d\xi \frac{\sqrt{\xi^2 + \Delta_\parallel^2}\, \Delta_T^2 - h\Delta_\parallel^2}{\sqrt{\xi^2 + \Delta_\parallel^2}\sqrt{(\sqrt{\xi^2 + \Delta_\parallel^2} - h)^2 + \Delta_\perp^2}} \right\rangle . \tag{26}$$

Dividing equation (25) by Δ (that is permitted when $\Delta \neq 0$) we obtain the equation which refers to the structure of the order parameter only.

Because of the rich structure of the energy gap form the value of energy gap and its structure can be considered independently. Let us remark that the integrals over ξ do not disturb the structure of Eq. (25). The order parameter equation can be always reconstructed from the simpler Eq. (26). Such a possibility allows us to restrict ourselves to the integration of Eq. (26).

All the transformations carried out in this section are the same as the former ones. Thus, we restrict ourselves to the presentation of some intermediate stages. At first we transform Eq. (26) to the form

$$\Delta^2 = \frac{g_1 \nu_0}{4} \left\langle \sum_{\beta=0}^{1} \left[\Delta_T^2 \int_{(-1)^\beta \Delta_\parallel}^{(-1)^\beta x_p} \frac{dx\,x}{\sqrt{x^2 - \Delta_\parallel^2}\sqrt{(x+h)^2 + \Delta_\perp^2}} \right. \right.$$

$$\left. \left. + \Delta_\parallel^2 h \int_{(-1)^\beta \Delta_\parallel}^{(-1)^\beta x_p} \frac{dx}{\sqrt{x^2 - \Delta_\parallel^2}\sqrt{(x + h)^2 + \Delta_\perp^2}} \right] \right\rangle , \tag{27}$$

which again contains one type of the integrands. Applying the following relation

$$\frac{d}{dx} \ln\left[\sqrt{x^2 - \Delta_\parallel^2} + \sqrt{(x + h)^2 + \Delta_\perp^2}\right]$$

$$= \frac{x+h}{\sqrt{(x^2 - \Delta_\parallel^2)\left[(x+h)^2 + \Delta_\perp^2\right]}} + \frac{h}{\Delta_T^2 + h^2 + 2hx} \left[1 - \sqrt{\frac{(x+h)^2 + \Delta_\perp^2}{x^2 - \Delta_\parallel^2}} \right] . \tag{28}$$

Equation (27) reduces to the form

$$\Delta^2 = \frac{g_1 \nu_0}{4} \left\langle \sum_{\beta=0}^{1} \left\{ \Delta_T^2 \ln(\sqrt{x^2 - \Delta_\parallel^2} + \sqrt{(x + h)^2 + \Delta_\perp^2}) \right|_{(-1)^\beta \Delta_\parallel}^{(-1)^\beta x_p} \right.$$

$$+\Delta_T^2 h\left[\int_{(-1)^\beta\Delta_\parallel}^{(-1)^\beta x_p}\frac{dx\left[(x+h)^2+\Delta_\perp^2\right]}{(\Delta_T^2+h^2+2hx)\sqrt{(x^2-\Delta_\parallel^2)\left[(x+h)^2+\Delta_\perp^2\right]}}\right.$$

$$\left.-\int_{(-1)^\beta\Delta_\parallel}^{(-1)^\beta x_p}\frac{dx}{\Delta_T^2+h^2+2hx}\right]-\Delta_\perp^2 h\int_{(-1)^\beta\Delta_\parallel}^{(-1)^\beta\infty}\frac{dx}{\sqrt{(x^2-\Delta_\parallel^2)\left[(x+h)^2+\Delta_\perp^2\right]}}\Bigg\}\Bigg\rangle, \qquad (29)$$

whence after putting $h = 0$ and dividing by Δ_0^2 we can obtain the equation

$$1 = g_1\nu(0)\left\langle |d_0|^2(\ln 2\xi_p - \ln\Delta_{0T})\right\rangle \qquad (30)$$

where the symbol "0" denotes the quantities appearing in the absence of a magnetic field. We would be in agreement with the existing estimations if we assumed that the Balian-Werthamer state appeared in the absence of a magnetic field and zero-temperature limit. We extend, however, the problem under consideration assuming that the order parameter fulfils one of the following relations:

$$|\underline{d}_0| = 1 \qquad (31)$$

or

$$|\underline{d}_0| = |\underline{d}|. \qquad (32)$$

Then, other states the structure of which is constant and does not fluctuate with the magnetic field can exist. The conditions (31)-(32) and Eq. (30) allow us to rewrite Eq. (29) in the form

$$\Bigg\langle\Delta_T^2(\ln\Delta_{0T}-\frac{1}{4}\ln N)$$

$$+\frac{1}{2}h\sum_{\beta=0}^{1}\Bigg\{\Delta_T^2\left[\int_{(-1)^\beta\Delta_\parallel}^{(-1)^\beta x_p}\frac{dx\left[(x+h)^2+\Delta_\perp^2\right]}{(\Delta_T^2+h^2+2hx)\sqrt{(x^2-\Delta_\parallel^2)\left[(x+h)^2+\Delta_\perp^2\right]}}\right.$$

$$\left.-\int_{(-1)^\beta\Delta_\parallel}^{(-1)^\beta x_p}\frac{dx}{\Delta_T^2+h^2+2hx}\right]-\Delta_\perp^2\int_{(-1)^\beta\Delta_\parallel}^{(-1)^\beta\infty}\frac{dx}{\sqrt{(x^2-\Delta_\parallel^2)\left[(x+h)^2+\Delta_\perp^2\right]}}\Bigg\}\Bigg\rangle=0, \qquad (33)$$

whence, using Eqs. (9)-(13), we obtain

$$\Bigg\langle\Delta_T^2(\ln\Delta_{0T}-\frac{1}{4}\ln N)$$

$$+ \Delta_t^2 \lim_{\gamma\to 0^+}\left[\sqrt{\frac{S}{Q}}\int_{1-\gamma}^{0}\frac{dt}{t^2-1}\cdot\frac{1+t^2\frac{R}{S}}{\sqrt{\left(1-t^2\frac{P}{Q}\right)\left(1+t^2\frac{R}{S}\right)}}-\int_{1-\gamma}^{0}\frac{dt}{t^2-1}\right]$$

$$- \Delta_\perp^2\frac{2h^2}{\sqrt{SQ}}\int_1^0\frac{dt}{\sqrt{\left(1-t^2\frac{P}{Q}\right)\left(1+t^2\frac{R}{S}\right)}}\Bigg\rangle = 0 \tag{34}$$

where we have used again the fact that integrands have turned out to be an even function of t. Now, applying (18) and (22) we obtain, respectively:

$$\Bigg\langle\Delta_T^2(\ln\Delta_{0T}-\frac{1}{4}\ln N)+\Delta_T^2 h z_0 N^{-1/4}\lim_{\gamma\to 0^+}\left[\sqrt{1-k^2z_0^2}\int_{z_0-\gamma}^{1}\frac{dzz}{(z_0^2-z^2)\sqrt{1-z^2}}\right.$$

$$\left.-z_0\int_{z_0-\gamma}^{1}\frac{dz}{z_0^2-z^2}\sqrt{\frac{1-k^2z^2}{1-z^2}}\right]+\Delta_\perp^2 h^2 N^{-1/4}\int_{z_0}^{1}\frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}\Bigg\rangle=0 \tag{35}$$

and

$$\Bigg\langle\Delta_T^2(\ln\Delta_{0T}-\frac{1}{4}\ln N)$$

$$-\Delta_T^2 h N^{-1/4}\int_{\varphi_0}^{\pi/2}\frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}+l\sin\varphi}$$

$$+\Delta_\perp^2 h N^{-1\ 4}\int_{\varphi_0}^{\pi/2}\frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}\Bigg\rangle = 0 \tag{36}$$

where

$$l^2 = \frac{\Delta_\parallel^2}{\sqrt{N}}\,.$$

Equation (36) analogously to Eq. (24) can be written in the form (cf. Section 38)

$$\langle\Delta_T^2(\ln\Delta_{0T}-\frac{1}{4}\ln N)-\Delta_T^2 h N^{-1/4}\left[G(l,k)-G(\varphi_0,l,k)\right]$$

$$+\Delta_\perp^2 h N^{-1/4}\left[F(k)-F(\varphi_0,k)\right]\rangle = 0. \tag{37}$$

The obtained results allow us to reconstruct the form of the structural gap equation. Hence, we get

$$\left\langle \hat{p}_\alpha \left\{ d_i(\ln \Delta_{OT} - \frac{1}{4} \ln N) - d_i h N^{-1/4} [G(l,k) - G(\varphi_0, l, k)] \right.\right.$$
$$\left.\left. + (d_i - d_\parallel \delta_{i\parallel}) h N^{-1/4} [F(k) - F(\varphi_0, k)] \right\} \right\rangle = 0. \qquad (38)$$

This is the way the magnetization and gap equations are derived.

b) Stable states

The parameter h expresses the value of the total magnetic field and can be defined by means of the relation (cf. Section 7)

$$h = \mu_B H - \frac{1}{\mu_B \nu_0} \sum_{l=0}^{\infty} F_l^a \langle P_l(\hat{p}\hat{p}')m(\hat{p}')\rangle \qquad (39)$$

where $m(p)$ is defined by the equation

$$M = \langle m(\hat{p})\rangle . \qquad (40)$$

Each of the following equations (24), (37)-(39) is a function of the quantities Δ, $d_{i\alpha}$ and H and a few additional parameters Δ_0, $|\underline{d}_0|^2$ and F_l^a. While examining the specified equations and the condition (2) we find that they constitute a closed system of nonlinear integral equations which should be solved simultaneously. A precise solution of this system would allow us to derive the order parameter, i.e., Δ and $d_{i\alpha}$, as the functions of the strong external magnetic field. The results would determine the phases of the superfluid ^{3}He in the magnetic fields. However, such solutions can be obtained solely by a numerical computation. That is why we consider some limiting cases below.

1° The weak magnetic field limit, i.e., $h \ll \bar{\Delta}_T$

The introduced parameters reduce to the forms

$$\sqrt{N} = \bar{\Delta}_T^2 + \frac{\bar{\Delta}_\perp^2 - \bar{\Delta}_\parallel^2}{\bar{\Delta}_T^2} h^2,$$

$$k^2 = \frac{\bar{\Delta}_\perp^2}{\bar{\Delta}_T^2}\left(1 + \frac{2\bar{\Delta}_\parallel^2}{\bar{\Delta}_T^4} h^2\right),$$

$$l^2 = \frac{\bar{\Delta}_\parallel^2}{\bar{\Delta}_T^2}\left(1 - \frac{\bar{\Delta}_\perp^2 - \bar{\Delta}_\parallel^2}{\bar{\Delta}_T^4} h^2\right),$$

$$R = \frac{2\bar{\Delta}_\perp^2}{\bar{\Delta}_T^2} h^2,$$

$$z_0 = 1 - \frac{\Delta_\parallel^2}{\Delta_T^4} h^2.$$

$$\varphi_0 = \frac{\pi}{2} - \frac{\Delta_\parallel}{\Delta_T^2} h \tag{41}$$

where we apply the formulae

$$\varphi_0 = \arcsin z_0 = \frac{\pi}{2} - \frac{1}{2} \arcsin 2z_0\sqrt{1-z_0^2} \approx \frac{\pi}{2} - z_0\sqrt{1-z_0^2}. \tag{42}$$

Inserting equations (41) into Eqs. (24), (37) and (38) we obtain

$$M = \mu_B \nu_0 \left\langle \frac{\Delta_\perp^2}{\Delta_T^2} h \right\rangle , \tag{43}$$

$$\langle \Delta_T^2(\ln\Delta_{0T} - \ln \Delta_T)\rangle = 0, \tag{44}$$

and

$$\left\langle \hat{p}_\alpha \left[d_i(\ln \Delta_{0T} - \ln\Delta_T) + \frac{h^2}{\Delta_T^4} (\Delta_\parallel^2 d_i - \Delta_T^2 d_\parallel \ \delta_{i\parallel})\right] \right\rangle = 0. \tag{45}$$

Equation (43) is the well-known result which in connection with Eq. (30) allows us to derive static spin susceptibilities for the fixed states. Equation (44) proves that the energy gap is constant and independent of a magnetic field, provided that it is sufficiently weak.

Assuming that Eq. (44) allows us to neglect the first term in Eq. (45), we obtain

$$\left\langle \hat{p}_\alpha \frac{h^2}{\Delta_T^4} (\Delta_\parallel^2 d_i - \Delta_T^2 d_\parallel \ \delta_{i\parallel}) \right\rangle = 0. \tag{46}$$

Since the above equation cannot be fulfilled by BW state and is always fulfilled if one of the order parameter components $d_\parallel$ or $d_\perp$ vanishes, it can be suspected that in the discussed region the magnetic field induces the phase transition of the first type. Moreover, let us notice and emphasize the fact that for arbitrary states ($\Delta_T = \Delta_T(\hat{p})$) the condition $h \ll \Delta_T$ can be fulfilled completely if and only if $h = 0$ (see below).

2^0 The limit of the very strong magnetic field ($h \gg \Delta_T$)

The introduced parameters reduce to the forms:

$$\sqrt{N} = h^2 + \Delta_\perp^2 - \Delta_\parallel^2 ,$$

$$k^2 = 1,$$

$$l^2 = \frac{\Delta_{\|}^2}{h^2},$$

$$z_0 = \sin \varphi_0 = 1 - \frac{\Delta_{\|}^2}{2h^2}. \tag{47}$$

Inserting the obtained relations into Eqs. (24) and (37) and computing directly all integrals (they are of elementary type) we obtain

$$M = \mu_B \nu_0 \left\langle h \left(1 - \frac{\Delta_{\|}^2}{2h^2}\right)\right\rangle, \tag{48}$$

$$\left\langle \Delta_T^2 \left\{ \ln \Delta_{0T} - \ln\left[h \left(1 + \frac{\Delta_{\perp}^2 - \Delta_{\|}^2}{2h^2}\right)\right] - \left(1 - \frac{\Delta_{\perp}^2}{2h^2}\right) \ln\left[2\left(1 + \frac{\Delta_{\|}^2}{4h^2}\right)\right]\right\} + I \right\rangle = 0 \tag{49}$$

where

$$I = \Delta_{\perp}^2 \left(1 - \frac{\Delta_{\perp}^2 - \Delta_{\|}^2}{2h^2}\right) \int_{\varphi_0}^{\pi/2} \frac{d\varphi}{\cos\varphi},$$

and

$$I = \begin{cases} 0 & \text{if} \quad \underline{d}_{\perp} = 0, \\ \Delta_{\perp}^2 \left(1 - \dfrac{\Delta_{\perp}^2}{2h^2}\right) \ln\left(2 \dfrac{h}{\Delta_{\perp}}\right) & \text{if} \quad \underline{d}_{\|} = 0, \\ \infty & \text{in other cases.} \end{cases} \tag{50}$$

Then the sole state which can be in fact realized in the very strong magnetic field limit is the one-dimension state for which $\underline{d}_{\perp}$ vanishes. However, such a state can be considered precisely in an independent way.

The states with $\underline{d}_{\|} = 0$ are in general two-dimensional states and can be reduced, e.g., to the ABM or planar states, whereas the states with $\underline{d}_{\perp} = 0$ are one-dimensional and can be reduced, e.g., to the polar state.

3^0 In the two-dimensional limit ($\Delta_{\|} = 0$) we obtain

$$\sqrt{N} = h^2 + \Delta_{\perp}^2, \quad k^2 = 1,$$
$$l^2 = 0, \quad R^2 = 2h^2. \quad \varphi_0 = \frac{\pi}{2}, \tag{51}$$

hence

$$M = \mu_B \nu_0 \langle h \rangle , \tag{52}$$

$$\left\langle \Delta_\perp^2 \ln(\Delta_0 / \Delta_\perp) \right\rangle = 0. \tag{53}$$

From equations (39) and (52) we derive the magnetization in the form

$$M = \mu_B^2 \nu_0 \frac{H}{1 + F_0^a} \tag{54}$$

The obtained expression is identical with the one for the normal state (cf. Eq. (10.19)). In order to discuss the gap equation (53) we assume that (cf. Eq. (3))

$$\Delta_\perp^2 = \Delta^2 |\underline{d}_\perp|^2 \tag{55}$$

where (cf. Section 33)

$$|\underline{d}_\perp|^2 = \frac{3}{2} (1 - x^2) \quad \text{and} \quad x = \cos \vartheta .$$

Substituting (55) into the gap equation (53), it reduces to the form

$$\ln \eta_\perp = \frac{1}{2} \left\langle |\underline{d}_\perp|^2 \ln |\underline{d}_\perp|^2 \right\rangle \tag{56}$$

where

$$\eta_\perp = \frac{\Delta_0}{\Delta}$$

and after integration we obtain

$$\ln \eta_\perp = \frac{1}{2} \ln 6 - \frac{5}{6} . \tag{57}$$

Hence

$$\eta_\perp = \left[6 \exp(-\frac{5}{3}) \right]^{\frac{1}{2}} = 1.06. \tag{58}$$

Since the free-energy difference derived by means of the formula (5.7) in case when $H = 0$ is of the form

$$\Delta F_\perp = - \frac{1}{2} \nu(0) \frac{\Delta_0^2}{\eta_\perp^2} = -0.44 \nu(0) \Delta_0^2 , \tag{59}$$

we state that $\Delta F_\perp > \Delta F_{BW}$ (cf. [13, 15]).

4° In the one-dimensional limit ($\Delta_\perp = 0$) the following cases must be considered independently:

4° a) $h < \Delta_\parallel$. This condition need not be fulfilled in the whole momentum space since $\underline{d}_\parallel$ is a function of $\hat{p}$. Therefore we restrict our consideration to quasi-particles, creating Cooper's pairs from the subspace Φ in which the condition $h < \Delta_\parallel$ is fulfilled. Then, we obtain

$$\sqrt{N} = \Delta_{\parallel}^2 - h^2, \qquad k^2 = 0,$$

$$l^2 = \frac{\Delta_{\parallel}^2}{\Delta_{\parallel}^2 - h^2}, \qquad R = 0,$$

$$\sin \varphi_0 = \frac{\sqrt{\Delta_{\parallel}^2 - h^2}}{\Delta_{\parallel}} = \frac{1}{l}. \tag{60}$$

Hence, the average magnetization arising from quasi-particles from Φ--subspace equals

$$M_{\Phi} = 0, \tag{61}$$

and the gap equation in Φ-subspace is of the form ($h = \mu_B H$)

$$\left\langle \Delta_{\parallel}^2 \ln(\Delta_{0T}/\Delta_{\parallel}) \right\rangle_{\Phi} = 0, \tag{62}$$

thus it is independent of the external magnetic field for all values of $H < H_c = \Delta_0/\mu_B$.

4^0 b) $h > \Delta_{\parallel}$. Let us note that this condition is fulfilled in the whole momentum space for sufficiently strong magnetic field. If the magnetic field is too weak we have to consider the appropriate subspace. Now, we have

$$\sqrt{N} = h^2 - \Delta_{\parallel}^2, \qquad k^2 = 1,$$

$$l^2 = \frac{\Delta_{\parallel}^2}{h^2 - \Delta_{\parallel}^2}, \qquad R = 2h^2,$$

$$\sin \varphi_0 = \frac{\sqrt{h^2 - \Delta_{\parallel}^2}}{h}, \tag{63}$$

assuming that the inequality is fulfilled in the whole space, we obtain

$$M = \mu_B \nu_0 \left\langle \sqrt{h^2 - \Delta_{\parallel}^2} \right\rangle \tag{64}$$

and

$$\left\langle \Delta_{\parallel}^2 \ln\left[\Delta_{0T}/(h + \sqrt{h^2 - \Delta_{\parallel}^2})\right] \right\rangle = 0. \tag{65}$$

Investigating the obtained expressions in the limit $\sqrt{\langle \Delta_{\parallel}^2 \rangle} \ll \Delta_0$ we can compute again all the averages and derive the magnetization, the energy gap and the Gibbs free-energy difference as functions of the external magnetic field in the following forms

$$M = 2\mu_B^2 \nu(0) \frac{(\beta_{\parallel} F_0^a + 2 - \beta_{\parallel})H - 2H_{c\parallel}}{\beta_{\parallel}(1 + F^a)(F^a - 1)}, \tag{66}$$

$$\Delta(H) = \Delta_0 \sqrt{\frac{1 + F_0^a}{\beta_{\parallel}(F_0^a - 1)}} \sqrt{1 - \frac{H}{H_{c_{\parallel}}}} \,, \tag{67}$$

and

$$\Delta G_{\parallel} = -\frac{1}{2} \nu(0) \frac{\beta_{\parallel}(F_0^a - 1)}{1 + F_0^a} \frac{\Delta^4(H)}{\Delta_0^2}$$

$$= -\frac{1}{2} \nu(0)\Delta_0^2 \frac{1 + F_0^a}{\beta_{\parallel}(F_0^a - 1)} \left(1 - \frac{H}{H_{c_{\parallel}}}\right)^2 \tag{68}$$

where

$$H_{c_{\parallel}} = \frac{(1 + F_0^a)\Delta_0}{2\,\mu_B} \tag{69}$$

is the critical magnetic field, $\beta_{\parallel}$ being the average $\beta_{\parallel} = \langle |\underline{d}_{\parallel}|^4 \rangle$ fulfils the conditions $\beta_{\parallel} \geqslant 1$. Let us note that $\Delta G_{\parallel} < 0$ if $F_0^a > 1$ and $H < H_{c_{\parallel}}$. Moreover, the obtained results describe the phase transition of the second order and they are physical solely if the following condition is fulfilled

$$F_0^a > 1. \tag{70}$$

For that reason we have to consider the opposite case:

4^o c) $h > \Delta_{\parallel}$ and

$$F_0^a \leqslant 1. \tag{71}$$

Now, non-zero energy gap cannot exist and has to vanish rapidly for the chosen direction of the unit vector $\hat{p}$ at the point $H = \Delta_{\parallel}(\hat{p})/\mu_B$. Hence, we have

$$\Delta = 0 \tag{72}$$

and

$$M = \mu_B^2 \nu_0 \frac{H}{1 + F_0^a} \,. \tag{73}$$

The solution discussed in the case 4^oc is correct also for the case 4^ob, when two physical solutions are possible, because the point $h = \Delta_{\parallel}$ is a standard point of bifurcation. We neglect the other solution as metastable and impossible, since the system prefers solutions with smaller Gibbs free energy (larger energy gap).

Such a behaviour of the system determines the phase transition of the first order. For both the cases discussed above we have also assumed that

$$\langle \Delta_{\perp\,(\parallel)}^{2} \ln \Delta_{0T} \rangle = \langle \Delta_{\perp\,(\parallel)}^{2} \rangle \ln \Delta_{0} \tag{74}$$

where

$$\Delta_0 = \eta \sqrt{\langle \Delta_{0T}^2 \rangle} \; .$$

Then according to the imposed conditions (31) and (32) we have, respectively,

$$\eta = 1 \quad \text{for BW state,} \tag{75}$$

$$\eta > 1 \quad \text{for the other "constant" states,}$$

i.e., $\eta_{\perp} = 1.065$ and $\eta_{\parallel} = 1.241$.

Although the precise derivation of stable states can be carried out only by means of the numerical calculations, we try to make some estimations according to the obtained results. First, let us note that the state which for the fixed magnetic field possesses the highest value of the energy gap possesses also the lowest Gibbs free energy and is preferred by the system. Thus Cooper's pairs in the state described in 4^{0}b are always preferred by the system over the 4^{0}c-state pairs if the condition (70) is fulfilled. It should also be explained that $\sqrt{\langle \Delta_{\parallel}^{2} \rangle}$ must be small in comparison with Δ_0, since only Δ_0 and $H_{c\parallel}$ are constant and independent of Δ and H.

According to the remarks given above $\Delta_{\parallel}$ is a function of the unit vector $\hat{p}$ and the inequality $h < \Delta_{\parallel}$ can be realized only in some parts of the momentum space. Such a situation complicates our problem. However, in order to solve it we can consider the appropriate averages over the proper parts of the momentum space (spherical angles). Taking into account Eqs. (60), (64), (66) and (73) we can note that the magnetization of the one-dimensional state achieves the non-zero values already in the weak magnetic field, and, if for instance the condition (71) is fulfilled, then the Cooper's pairs exist in the subspace Φ only. Hence, the magnetization is of the form

$$\mathbf{M} = \frac{\mu_B^2 \nu_0}{1 + F_0^a} \int_{\Delta_{\parallel}(\hat{p})<h} d^3p, \tag{76}$$

which, e.g., for the so-called "polar" state ($\Delta_{\parallel} = \sqrt{3}\,\Delta\,|\cos\vartheta|$) reduces to the form

$$M = \mu_B^2 \nu_0 \frac{H^2}{HF_0^a + H_{c1}} \tag{77}$$

where

$$H_{cl} = \sqrt{3}\,\frac{\Delta}{\mu_B}\,(1 + F_0^a) \tag{78}$$

is the critical magnetic field in the case when the system passes to the normal state in its whole volume. While discussing the problem of the energy gap we note that though the energy gap is constant in the subspace Φ, this subspace is a decreasing function of the magnetic field and this fact must be expressed in the free energy terms. Hence, e.g., for the same "polar" state in the limit $H \to H_{cl}'$, we obtain

$$\Delta(H) = \frac{\mu_B H_{cl}}{\sqrt{3}(1 + F_0^a)^{3/2}}\sqrt{1 - \frac{H}{H_{cl}}}\,. \tag{79}$$

The analogous considerations can be carried out for the case when the condition (70) is fulfilled. They, however, are more complicated because Cooper's pairs exist also outside the subspace Φ and if it vanishes we obtain the case discussed in the point 4°b. In conclusion of this section we can state that the transition of superfluid ^{3}He to the normal phase in the strong magnetic field can be only of the second order. Moreover, the order parameter distinguishes three qualitatively different structures, i.e., three-, two-, one-dimensional structures, according to the conditions:

$$\begin{array}{llll} \text{3-D} & \underline{d}_\perp \neq 0 & \text{and} & \underline{d}_\parallel \neq 0, \\ \text{2-D} & \underline{d}_\perp \neq 0 & \text{and} & \underline{d}_\parallel = 0, \\ \text{1-D} & \underline{d}_\perp = 0 & \text{and} & \underline{d}_\parallel \neq 0, \end{array} \tag{80}$$

and the transitions between those three kinds of phases (if they exist at all) are of the first order (the qualitative change of the order parameter). However, the 3-dimensional state (BW) can exist in weak magnetic fields exclusively, whereas in strong magnetic fields only 1-dimensional state is permitted.

Applying the developed formalism to the superfluid ^{3}He we have to take into account the real value of the Landau parameter F_0^a. According to the data presented in [125] the value of F_0^a is negative within the whole range of the permitted pressures being $-0.76 \leqslant F_0^a \leqslant -0.67$. Then only the condition (71) is fulfilled, and the suitable behaviour of the system is determined.

ⓒ Although the above considerations are carried out in molecular field approximation they can be directly developed for the case when the higher Fermi liquid interaction harmonics are included. Then obtained results will be modified by the other Landau parameters and additional relations will appear.

19. The solution of the gap equation in the presence of dipole forces if $H = 0$ [43]

Now let us consider extra the system with pure P-pairing without a magnetic field when the dipole-dipole interaction is included (cf. [85, 88, 104, 126, 127]). The problem specified in this way allows us to determine equilibrium states which can be employed in linear response investigations. The inclusion of dipole forces causes the change of rotational symmetry of the problem. The system is no longer invariant under the separate rotations of spin and momentum variables. Thus, we have to express total interaction in the particle-particle channel in spherical tensor representation. Employing results of Sections 6, 33 and 41 we get

$$V_{ij} = -3f_1\Big[(1-2\alpha)\,B^{ik}_{00}\,B^{*jn}_{00} + (1+\alpha)\,B^{ik}_{1M}B^{*jn}_{1M} + \left(1-\frac{1}{5}\alpha\right)B^{ik}_{2M}B^{*jn}_{2M}\Big]\hat{p}_k\hat{p}'_n \tag{1}$$

where $\alpha = \nu(0)g_0/f_1$ and $\alpha \ll 1$. As we can note this interaction is split according to J and is independent of M which reflects the symmetry of the problem. Let us rewrite Eq. (1) in the form more convenient for the discussion of the gap equation:

$$V_{ij} = -3f_1\Big[\frac{1}{3}(1-2\alpha)\delta_{ik}\delta_{jn} + \frac{1}{2}(1+\alpha)(\delta_{ij}\delta_{kn} - \delta_{ik}\delta_{jn}) + \frac{1}{2}\left(1-\frac{1}{5}\alpha\right)\left(\delta_{ij}\delta_{kn} + \delta_{in}\delta_{jk} - \frac{2}{3}\delta_{ik}\delta_{jn}\right)\Big]\hat{p}_k\hat{p}'_n. \tag{2}$$

The gap equation has the form

$$d_{ik}\hat{p}_k = -\langle V_{ij}H(\hat{p})d_{jn}\hat{p}_n\rangle \tag{3}$$

where

$$\hat{\Delta}(\hat{p}) = \Delta(T)\sigma^k d_{kn}\hat{p}_n i\sigma^2,$$

$$H(\hat{p}) = \int_0^{\xi_p} d\xi\,\frac{\operatorname{th}(E/2T)}{E}, \qquad E = [\xi^2 + |\hat{\Delta}(\hat{p})|^2]^{1/2}. \tag{4}$$

In order to solve Eq. (3) within the frames of the applied approach, the form of the tensor d_{ik} must be known. It, however, can be constructed from the spherical tensor B^{ik}_{JM} according to the following ideas. The spherical tensors describe the possible two-particle states in the system with the full rotational symmetry. If the symmetry of the system is broken by distinction of one direction, the equilibrium state becomes a linear combination of spherical tensors with the fixed M which is still a good quantum number ($M = 0, \pm 1, \pm 2$). In this way the thermodynamic phases can be classified according to the quantum number M [41, 73].

(c) After distinguishing more than one direction in the system, e.g., inhomogeneity and the external strong magnetic field, no quantum number is a good quantum number and the equilibrium state should be constructed from all nine spherical tensors.

Below we consider the equilibrium state when $M = 0$, which is identified with B phase [15]. The tensor d_{ik} has the form

$$d_{ik} = a\delta_{ik} + b(\delta_{ik} - 3\hat{k}_i\hat{k}_k) + c\varepsilon_{ikn}\hat{k}_n \tag{5}$$

where $\hat{k}$ is a unit vector of the distinguished direction. If a, b and c are real numbers or they have the same phase, the state described above is unitary. Let us choose a, b, and c as real numbers; then the following conditions are fulfilled:

$$a^2 + 2b^2 + \frac{2}{3}c^2 = 1, \tag{6}$$

$$d_{ik}d^*_{in} = d_{ik}d_{in} = A\delta_{kn} + B\hat{k}_k\hat{k}_n \tag{7}$$

where

$$\begin{gathered} A = (a+b)^2 + c^2, \qquad B = 3b^2 - 6ab - c^2, \\ A + \frac{1}{3}B = 1, \qquad 0 \leqslant A \leqslant \frac{3}{2}. \end{gathered} \tag{8}$$

Now, if we estimate the free energy of the system in the state (5), $\Delta F \sim A^2 - 2A$, then for $A = 1 (B = 0)$ it is minimal, i.e., in the isotropic state. Solving equations 8, we obtain

$$a = \frac{1}{3}(2\cos\theta + 1), \quad b = \frac{1}{3}(\cos\theta - 1), \quad c = \mp \sin\theta, \tag{9a}$$

$$a = \frac{1}{3}(2\cos\theta - 1), \quad b = \frac{1}{3}(\cos\theta + 1), \quad c = \mp \sin\theta. \tag{9b}$$

The first solution can be interpreted as a rotation in momentum space about the $\hat{k}$ axis by an angle $\pm\theta$, respectively, and the second solution as the analogous improper rotation.

According to Leggett's considerations, the free energy of the equilibrium states (9) is minimized by dipole forces if the angle $\theta = \cos^{-1}(-1/4)$ in the first case (9a), and the angle $\theta = \cos^{-1}(1/4)$ in the second one (9b).

For the case of the isotropic gap $|(\hat{\Delta}(\hat{p})|^2 = \mathrm{const})$, $H(\hat{p}) = \mathrm{const} = h_0$ (cf. Section 33) and from Eq. (3) we get the following equalities:

$$a = a(1 - 2\alpha)h_0 f_1,$$

$$b = b(1 - \frac{1}{5}\alpha)h_0 f_1, \tag{10}$$

$$c = c(1 + \alpha)h_0 f_1.$$

These equalities cannot be fulfilled simultaneously if $\alpha \neq 0$, except the case when $\theta = 0$ for (9a) or $\theta = \pi$ for (9b). In order to solve this problem let us assume that the gap is slightly anisotropic ($A = 1 - \varepsilon$; $B = 3\varepsilon$) and has the form of $|\hat{\Delta}(\hat{p})|^2 = \Delta^2[1 - \varepsilon + 3(\hat{k}\hat{p})^2\varepsilon]$. Since ε is small, we can apply in our calculations the following expansion of the function $H(\hat{p}) \equiv H_\varepsilon(\hat{p})$:

$$H_\varepsilon(\hat{p}) = H_0 + H_1[3(\hat{k}\hat{p})^2 - 1]\varepsilon$$

where

$$H_1 = \Delta^2 \int_0^{\xi_p} \frac{d\xi'}{2E} \frac{\partial}{\partial E} \frac{\mathrm{th}(E/2T)}{E}.$$

Now, from Eq. (4) we obtain

$$a = f_1(1 - 2\alpha)[3h_2 a + (a - 2b)h_3], \tag{12a}$$

$$b = f_1(1 - \frac{1}{5}\alpha)[3h_2 b - (a - 2b)h_3], \tag{12b}$$

$$c = f_1(1 + \alpha)3h_2 c \tag{12c}$$

where h_2 and h_3 agree with the notation given in Section 33. From Equation (12c) we find that the following equality has to be fulfilled:

$$1 = 3h_2 f_1(1 + \alpha). \tag{13}$$

In order to examine the solution of Eqs. (12) we introduce the following notations:

$$a = a_0 + a_1, \qquad b = b_0 + b_1, \qquad c = c_0 + c_1$$

where a_0, b_0, c_0, are taken in the form defined by (9) and a_1, b_1, c_1 do not exceed the order of α.

Equations (12) at the limit $\alpha = 0$ can be fulfilled only if $c = 0$, $a - 2b = 0$, or $h_3 = 0$. If there appear the dipole forces, $a - 2b$ does not exceed α. Keeping the accuracy of our calculations we find that

Eqs. (12) are fulfilled if $a_0 = b_0 = 0$ ($c = \pm\frac{3}{2}$), i.e., in the ordinary 2D state. The condition $h_3 = 0$ can be fulfilled if $B = 0 (A = 1)$, i.e., in the isotropic state. Using Equation (13), after some calculations, from Eqs. (12a) and (12b) we obtain

$$\pm\frac{2}{5}\,\varepsilon H_1 = 3\alpha H_0 a_0, \tag{14a}$$

$$\mp\frac{2}{5}\,\varepsilon H_1 = \frac{6}{5}\,\alpha H_0 b_0. \tag{14b}$$

The plus and minus signs are connected with the two possibilities of the choice of a_0 and b_0. Equations (14) give us the relation $a_0 = -(2/5)b_0$, which can be fulfilled if and only if $\cos\theta = \mp 1/4$, respectively. We recall that these are Leggett's conditions of the free energy minimization. If we now insert $a_0 = \pm\frac{1}{6}$ or $b_0 = \mp 4/15$ into Eqs. (14) we obtain

$$\varepsilon = \frac{5}{4}\alpha H_0/H_1. \tag{15}$$

Returning to Eq. (11), we can rewrite it in the following form:

$$H(\hat{p}) = H_0\left[1 - \frac{5}{4}\alpha + \frac{15}{4}\alpha(\hat{k}\hat{p})^2\right]. \tag{16}$$

Substituting $H(\hat{p})$ given by (16) into the formula (13), we obtain

$$H_0 g_1(1 + \alpha/2) = 1. \tag{17}$$

The factors a, b, and c can be calculated from Eqs. (8). Taking into account that $a_0 = \pm 1/6$, $b_0 = \mp 5/12$, and $c_0 = (\pm)\sqrt{15}/4$, after some transformation we obtain

$$\begin{aligned} a &= \pm\tfrac{1}{6}(1 + 10\varepsilon + 15\delta), \\ b &= \mp(5/12)(1 - \tfrac{4}{5}\varepsilon - 3\delta), \\ c &= (\pm)(\sqrt{15}/4)(1 + \delta) \end{aligned} \tag{18}$$

where δ is an arbitrary number of the order of ε. The free energy of the state described by Eq. (18) depends on ε linearly. Hence, it has to increase or decrease for nonzero ε. Following Leggett's estimations [85], we see that the free energy will be modified only by its dipole part $\Delta F_{dip} \sim (6a^2 + 3b^2 - c^2) \sim -(0.1 - \varepsilon)$ and it will decrease if ε is negative. This occurs, according to Eq. (15), if $H_1 < 0$, since α and H_0 are positive. It seems to be of interest to find the relation H_0/H_1 and its dependence on temperature and ξ_p, though this relation is probably always negative. The free energy is independent of δ. Hence, the equilibrium state of the system is degenerated, or rather, it is not defined in a unique way. Such a consideration of the problem is most general although it can be also interpreted in terms of simultaneous rotation and dilatation [104, 126].

Let us return now to the interaction in the particle-particle channel. This interaction can be rewritten in a new form, equivalent to the previous one,

$$V_{ij} = -3f_1(1+\alpha)\,[\delta_{ij}\delta_{kn} - \gamma(\delta_{ij}\delta_{kn} + \delta_{ik}\delta_{jn} + \delta_{in}\delta_{jk})]\,\hat{p}_k\hat{p}'_n \quad (19)$$

where $\gamma = 3\bar{\alpha}/5(1+\bar{\alpha})$ and $\gamma \ll 1$. This expression can be treated as a kernel of the integral operator

$$\hat{\underline{V}} = -f_1(1+\alpha)(\hat{\underline{1}} - \gamma\hat{\underline{U}}), \quad (20)$$

If we remember that $\gamma \ll 1$, the inverse operator can be found immediately in the form

$$\hat{\underline{V}}^{-1} = -H_0(1-\alpha/2)(\hat{\underline{1}} + \gamma\hat{\underline{U}}) \quad (21)$$

where we also used Eq. (17). In Equation (21) we neglected quantities of the order of $\bar{\alpha}^n/f_1$ for $n \geqslant 2$. These kinds of dimensionless quantities will appear in further considerations. They satisfy the following relations:

$$H_0 \gg \alpha H_0 \gg \varepsilon \sim \varepsilon\alpha H_0 \sim \delta\,. \quad (22)$$

In order to be consistent with our previous calculations, the quantities of the order of ε should be treated as being negligibly small. Hence, the gap, if multiplied by a factor of the order of αH_0, should be replaced by

$$d_{ik} = \pm\frac{1}{6}\,\delta_{ik} \mp (5/12)(\delta_{ik} - 3\hat{k}_i\hat{k}_k) - [\pm(\sqrt{15}/4)]\,\varepsilon_{ikn}\hat{k}_n \quad (23)$$

in order not to surpass the accuracy of the equations.

Comparing the results of this and previous sections we state that the two-dimension state 2D is the best candidate to replace the BW state if the magnetic field is included (cf. [7]).

V. ^3He-^4He mixtures

20. Properties of ^3He-^4He mixtures

According to the theoretical investigations [112] in ^3He-^4He mixtures S-pairing is preferred for the concentrations $x \leqslant 5\%$, with a growing tendency to P-pairing for higher concentrations. The estimated values of the phase transition temperature $T_c \leqslant 5\cdot 10^{-6}$K at concentrations x = 1,3% and x = 5%. Moreover, the Landau parameter F_0^a becomes positive (it is negative for ^{3}He), although it is small $F_0^a \approx 0.09$. Nevertheless it should be a function of pressure. Such tendencies of ^{3}He-^{4}He mixtures allow us to consider them as neutral superfluids with the singlet- or triplet-state Coopers's pairs for low or high concentrations, respectively. Below we consider only neutral BCS systems which constitute the qualitatively new problem.

21. A neutral BCS system

We consider the problem applying the results contained in Sec. 8 in the case of the particle-hole symmetry. Since all vectorial quantities are parallel we neglect their vectorial structure and consider their positive values only. In case of the homogeneous BCS-system none of the used quantities depends on the direction of the momentum vector, hence the total magnetic field can be written in the form

$$H_T = H - F_0^a\, H_p. \tag{1}$$

We emphasize that F_0^a is the sole parameter of the Fermi liquid interaction which modifies the total magnetic field.

a) The zero-temperature limit [50]

In order to solve the specified problem in the analytic way we put T = 0. The reduced forms of the paramagnetic field and gap equation are analytic functions of H_T (Eqs. (12.1), (14.2)). Analogously as in Section 14 we state that the investigated problem possesses two qualitatively different solutions which have to be considered as two independent cases, i.e., (cf. Section 43):

I) when Δ is constant ($\Delta = \Delta_{\bar{0}}$) and it rapidly vanishes for $H = H_0 \leqslant H_1 \equiv \Delta_{\bar{0}}/\mu_B$, and

II) when Δ is constant ($\Delta = \Delta_{\bar{0}}$) for $H \leqslant H_1$ and it changes its value in a continuous way for $H \geqslant H_1$.

In the former case we have

$$\Delta = \begin{cases} \Delta_{\bar{0}} & \text{if} \quad H < H_{\bar{0}} \leqslant H_1, \\ \\ 0 & \text{if} \quad H \geqslant H_{\bar{0}} \end{cases} \tag{2}$$

and

$$M_p = \chi_p^0 \begin{cases} 0 & \text{if} \quad H < H_0, \\ \dfrac{H}{1 + F_0^a} & \text{if} \quad H \geqslant H_0. \end{cases} \tag{3}$$

Since the parameter H_0 defines the value of the magnetic field when the system approaches the normal state so it can be found by Gibbs free-energy comparison from the relation

$$\frac{1}{2} \nu(0) \Delta_0^2 = \frac{\nu(0)\mu_B^2 H_0^2}{1 + F_0^a} . \tag{4}$$

Hence we obtain

$$H_0 = H_1 \sqrt{\frac{1}{2} (1 + F_0^a)} \tag{5}$$

and because of $H_0 \leqslant H_1$ we get

$$F_0^a \leqslant 1 . \tag{6}$$

In this way we state that if the system possesses the type-I properties, the inequality (6) has to be fulfilled. In the latter case the gap equation reduces to the form

$$H_1 = H_p + H_T \tag{7}$$

and substituting this into Eq. (1) we can derive the paramagnetic magnetization for $H \geqslant H_1$ in the form

$$M_p = \chi_p^0 \frac{H - H_1}{F_0^a - 1} , \tag{8}$$

whence, we find

$$\Delta^2 = \frac{\mu_B^2 H_1}{F_0^a - 1} \left[(1 + F_0^a) H_1 - 2H \right] . \tag{9}$$

The demand of positive paramagnetic magnetization implies the following inequality

$$F_0^a > 1 \tag{10}$$

which ensures also that the energy gap becomes the decreasing function of the external magnetic field. On the other hand, Eq. (9) allows us to derive the value of the critical magnetic field H_2 for which the energy gap Δ vanishes. We find it in the form

$$H_2 = \frac{1}{2}(1 + F_0^a)H_1 \tag{11}$$

where the condition (10) ensures that $H_1 < H_2$.

In the limit case, when $F_0^a = 1$, the critical magnetic fields H_1 and H_2 become equal and the system shows the properties attributed to the first discussed case. Then the critical magnetic field H_c is equal to H_1.

In order to sum up the obtained results we emphasize that the energy gap can tend to zero in the continuous way only if the Landau parameter $F_0^a > 1$; then, the energy gap and the paramagnetic magnetization are expressed in the following forms

$$\Delta = \Delta_0 \begin{cases} 1 & \text{if} \quad H \leqslant H_1, \\ \left[\dfrac{(1 + F_0^a)H_1 - 2H}{(F_0^a - 1)H_1}\right]^{1/2} & \text{if} \quad H_1 \leqslant H \leqslant H_2, \\ 0 & \text{if} \quad H \geqslant H_2, \end{cases} \tag{12}$$

and

$$M_p = \chi_p^0 \begin{cases} 0 & \text{if} \quad H \leqslant H_1, \\ \dfrac{H - H_1}{F_0^a - 1} & \text{if} \quad H_1 \leqslant H \leqslant H_2, \\ \dfrac{H}{1 + F_0^a} & \text{if} \quad H \geqslant H_2. \end{cases} \tag{13}$$

Let us estimate now the Gibbs free energy difference for the magnetic fields H when $H_1 < H < H_2$. Employing equation (5.7) and Eqs. (13.18) we obtain in turn

$$\Delta G = -\frac{\nu(0)}{2q}\left(\sqrt{p^2 + q\Delta^2} - p\right)^2 \tag{14}$$

where

$$p = \frac{\mu_B H}{1 + F_0^a}, \qquad q = \frac{F_0^a - 1}{F_0^a + 1}$$

and

$$\Delta G = - 2\nu(0)\,\mu_B^2 \frac{(H_2 - H)^2}{(1 + F_0^a)(F_0^a - 1)} , \tag{15}$$

hence we state that ΔG is always negative if $F_0^a > 1$ and $H < H_2$, and $\Delta G = 0$ if $H = H_2$. For $H = H_1$ the Gibbs free-energy difference reduces to the form

$$\Delta G = - \frac{1}{2}\, \nu(0)\Delta^2(0) \frac{F_0^a - 1}{F_0^a + 1} . \tag{16}$$

The latest relation demonstrates that the Gibbs free-energy difference is equal to zero if $H = H_c$ and $F_0^a = 1$, i.e., for type-I systems. In this way we have proved that the neutral BCS-systems reveal the paramagnetic duality which is determined by the values of the Landau parameter F_0^a; if $F_0^a \leqslant 1$ the energy gap vanishes and the appearing paramagnetic magnetization attains the same value as for the normal system for the fixed value $H_0 (H_0 \leqslant H_1)$; if $F_0^a > 1$, then while the external magnetic field changes from H_1 to H_2 the energy gap decreases from the value Δ_0 to zero and the paramagnetic magnetization increases from zero to a value of the normal state. Since the other discussed state is more stable in comparison with the first one it is always realized if $F_0^a > 1$. Moreover we can assume that $\varkappa = \frac{1}{2}(1 + F_0^a)$ then $H_2 = H_c \varkappa$ and the following condition

$$\varkappa > 1 \tag{17}$$

is equivalent to Eq. (10).

b) The Ginzburg-Landau limit

1° In order to consider the system in this limit we can employ the results of Section 16b after putting $v_s = 0$ or F_1^s equal to infinity or results of Section 15b. In the formalism developed there and based on thermodynamic potential estimations it is assumed that in the Ginzburg-Landau limit the following relations have to be fulfilled

$$\frac{\bar{\mu}_B H}{T_c} < \frac{\bar{\mu}_B H_2(T)}{T_c} << \frac{\Delta(T)}{T_c} << 1 . \tag{18}$$

These relations allow us to treat the magnetic field as a small magnitude in comparison with the energy gap and to apply a perturbation method to solve the gap equation (15.21) or (16.16). Then we get

$$H_2(T) = \alpha H_1(0)\left(1 - \frac{T}{T_c}\right)(1 + F_0^a) \tag{19}$$

where

$$\alpha = \frac{e^c}{7\zeta(3)}\sqrt{\frac{186\,\zeta(5)}{7\,\zeta(3)}} = 1.01.$$

On the other hand, comparing the thermodynamic potential we obtain (cf. Eq. (4) and [35])

$$H_0(T) = \gamma H_1(0)\left(1 - \frac{T}{T_c}\right)\sqrt{1 + F_0^a} \tag{20}$$

where

$$\gamma = \frac{2\,e^c}{7\zeta(3)} = 1.23.$$

Now, just as in the zero-temperature case, we can assume that the type-II superfluidity is realized in a system when the critical magnetic field $H_2(T)$ fulfils the relation

$$H_0(T) < H_2(T). \tag{21}$$

This relation, after applying Eqs. (19) and (20), implies the following condition

$$F_0^a > \varepsilon - 1 \tag{22}$$

where

$$\varepsilon = \left(\frac{\gamma}{\alpha}\right)^2 = \frac{98\,\zeta^2(3)}{93\,\zeta(5)} = 1.47.$$

In the limit case, when $F_0^a = \varepsilon - 1 = 0.47$, we have $H_0(T) = H_2(T)$ and introducing $H_c(T)$ in the form

$$H_c(T) = \alpha\varepsilon H_c(0)\left(1 - \frac{T}{T_c}\right) \tag{23}$$

($\alpha\varepsilon = 1.48$) we can write

$$H_2(T) = H_c(T)\,\frac{1 + F_0^a}{\varepsilon}\,. \tag{24}$$

Hence, the $\varkappa$-parameter can be defined as follows

$$\varkappa(T_c) = \frac{1 + F_0^a}{\varepsilon} . \tag{25}$$

According to the obtained results (10) and (22) we state that the magnetic properties of ^3He-^4He mixtures can be altered (type-I systems become the ones of type-II) when temperature tends to T_c and F_0^a is contained within the interval $(\varepsilon - 1, 1)$. Moreover, after regard Eq. (25) we can apply the other results of Section 15b.

2^0 Let us emphasize that the same problem is usually treated in very formal way, while one assumes that the energy gap has to vanish ($\Delta = 0$) when $H = H_2$ [92, 95, 120]. Then employing the gap equation (15.24) and Eqs. (1), (15.20), (15.23) we obtain the following results

$$H_2(T) = \alpha H_1(0) \left(1 - \frac{T}{T_c}\right)^{1/2} (1 + F_0^a) \tag{26}$$

and

$$H_2(T) = \frac{\Delta(T)}{\mu_B} \frac{1 + F_0^a}{\sqrt{2}} . \tag{27}$$

However, now the critical exponent is equal to 0.5. This value being inconsistent with that obtained from termodynamic potential estimation. Therefore we cannot compare Eqs. (20) and (26) and separate the $\varkappa$-parameter.

In this way we prove that the results obtained in the former approach are consistent and coherent, whereas the results of the latter approach seem to be incorrect. Therefore the presented method becomes competitive to typical approaches [92, 95, 120] and we prefer it in our considerations, though in the case of superconducting systems we apply Eq. (15.27) instead of Eq. (23).

Part Three

LINEAR RESPONSES OF BCS AND BW SYSTEMS

The generalization of the Green function microscopic theory formulated by Larkin, Migdal [79] and Czerwonko [22] for non-zero temperature systems [39, 40], the correct derivation of the Maki and Ebisawa function [43, 47], the inclusion of the dipole-dipole interaction [43-45, 53] and the development of some special mathematical methods (Part Four) allow us to state new significant properties of the superfluid and superconducting systems, which appear in the linear response theory and which are due to the Fermi liquid interaction.

In Chapter VI we investigate the homogeneous BCS and BW systems with the full pairing interaction and we define the relations between the parameters of the pairing and Fermi liquid interactions.

In Chapters VII and VIII the phase B [4, 5, 8, 12, 15, 67, 85, 97, 107, 108, 137, 139] of superfluid ^{3}He is considered in the presence of the dipole-dipole interaction. Since we do not impose any restrictions on the frequency, wave vector, and temperature, we have to assume that all Landau parameters disappear, except a few first ones [40, 43]. The suitable equilibrium state is derived in Section 19.

VI. Influence of the Fermi liquid interaction on possible two-particle States [49]

22. Statement of the problem

The general rotational symmetry of superconducting and superfluid systems allows us to describe the Fermi liquid interaction by the expansion of Legendre polynomials where, as far as the appropriate factors, i.e., the Landau parameters are concerned, we assume only that they have to fulfil the Pomeranchuk [117] inequalities and become small for sufficiently high terms of the expansion. So far these restrictions are satisfied by the experimental data and the Landau parameters become negligibly small already for quite low expansion terms [139]. On the other hand, there exists one more quasiparticle interaction in the superfluid and superconducting systems: the pairing interaction. According to the same symmetry conditions the pairing interaction can be represented in the analogous way, i.e., by means of the Legendre polynomial series, where some parameters of the expansion have to be equal to zero according to Pauli exclusion principle.

In most of the cases discussed so far the authors restrict themselves to the one pairing harmonic responsible for the creation of the quasiparticle pairs in the ground state. Now, we consider superconducting and superfluid systems with fixed ground states of the BCS or BW type, respectively, in which the Fermi liquid and pairing interactions are included in quite general forms. Such a procedure is fully justified if we assume that the pairing energies of all other two-particle states are lower than energies for the BCS or BW state, respectively. Hence, there appear some restrictions imposed on the pairing interaction parameters relative to the ground state harmonic.

The energy spectrum of all possible two-particle states has to be placed between the ground state energy levels of the superfluid and normal liquids. Since the above two levels differ by Δ per quasi-particle, the excitations of the ground state pairs to other available two-particle states by the energy charges smaller than 2Δ become possible.

We investigate the possibilities of such excitations in the linear response approach.

It allows us to discuss the linear response of the system to the external mechanic perturbations of quasi-particle density. Our main purpose is to find the dependences between the Landau and pairing interaction parameters. We also give an interpretation of the obtained results with regard to experiments.

Our original considerations were carried out in the zero temperature limit since all Cooper's pairs are locked then into the ground (BCS or BW) state, respectively. In case of non-zero temperatures Cooper's pairs should occupy other higher localized permissible two-particle states. Therefore, the order parameter connected with the thermodynamic equilibrium state should be modified adequately. If, however, we neglect this effect [144] the results of [49] are valid for non-zero temperatures provided that the temperature form of the Maki and Ebisawa function is taken into account (Section 34). Suitable changes are performed below.

23. The system in BCS state

We consider the superconducting system in the BCS state at the homogeneous limit ($k = 0$) in the case when the full Fermi liquid and pairing interactions are included. Some rather superfical investigations of this system were performed by Larkin [78]. Now, together with the far more complicated superfluid BW system, we consider it in detail and discuss the results obtained.

The basic equations, necessary to deal with the problem under consideration, can be easily obtained from the equation given by Czerwonko [22] and have the form

$$\mathcal{T} = 1 + \langle \overset{s}{A}(-F\mathcal{T} + tFT)\rangle ,$$

$$T = \langle \Phi_1 [(t^2F + f_0^{-1})T - tF\mathcal{T}]\rangle \tag{1}$$

where $t = \omega/(2\Delta)$ is the reduced frequency.

Such a simple form of of equations can be achieved after using the symmetry properties of the initial equation. Some symbols used have more precisely determined meanings, as defined below. The normal vertex function is of the form

$$\mathcal{T} = \sum_{j=0}^{\infty} (2j + 1)\, \mathcal{T}_j P_j ,$$

and $\mathcal{T}_j = 0$ for all add j. The anomalous vertex function is of the form

$$T = \sum_{j=0}^{\infty} (2j + 1) T_j P_j ,$$

and $T_j = 0$ for all odd j. From the spin direct Fermi liquid interaction only its symmetric part $\overset{s}{A}$ is taken and Φ_1 is the spin antisymmetric pairing interaction (cf. Section 6).

We leave out an account of the autocorrelation function equation, since we are not going to use it immediately, but we still keep in mind that the poles of the autocorrelation and vertex functions are equivalent.

Using the symbols introduced we obtain from Eq. (1) after some calculations

$$\mathcal{T}_j = \delta_{0j} - a_j(\mathcal{T}_j - tT_j)F,$$

$$T_j = f_j(tT_j - \mathcal{T}_j)tF + f_j f_0^{-1} T_j . \tag{2}$$

The obtained equations are separated with respect to the quantum number j. This, of course, takes place only in the homogeneous limit. Still, in the quasi-homogeneous limit we can separate with a good accuracy the collective excitation spectra and obtain the collective excitation gaps from Eqs. (2). Since the excitations are connected with the poles of the appropriate vertex function it is sufficient to derive the main determinant of the above system of equations. Then we have

$$\left(\ln \frac{\Delta}{r_j} - t^2F\right)(1 + a_jF) + a_j t^2F^2 = 0 \tag{3}$$

where r_j can be interpreted as the value of the pairing energy per quasiparticle in the state with the fixed quantum number j. Moreover, $r_j < \Delta$ for $l \neq 0$ and $r_0 \equiv \Delta$.

We assume that the collective excitation spectrum in the quasi -homogeneous limit has the form

$$t^2 = t_0^2 + Ku^2 \tag{4}$$

where $u = kv/(2\Delta)$ is the reduced wave vector which makes the system to be inhomogeneous and $u \ll t_0$ but K can be of the order t_0^2. The value of t_0 should be found as a solution of the equation

$$t^2F(t) - [1 + a_jF(t) \ln]\frac{\Delta}{r_j} = 0. \tag{5}$$

Let us notice that if $a_j = 0$ and $r_j \longrightarrow \Delta$, then $t_0 \longrightarrow 0$ which is in agreement with the assumed conditions. Let us note furthermore that: if the gaps of the real physical excitation are smaller than 2Δ, then the value of t_0 has to fulfil the following inequalities

$$0 < t_0 < 1. \tag{6}$$

We exclude the equality on the righ-hand-side of Eq. (6) as leading to the ground state degeneracy.

The Landau parameter a_j has to satisfy the following Pomeranchuk [117] conditions:

$$-1 < a_j < \infty \tag{7}$$

for all j = 0, 1,2,... .

From Equation (5) we derive

$$a_j = \frac{t^2}{\ln \Delta/r_j} - \frac{1}{F(t)}, \tag{8}$$

and assuming that

$$a_j = A_j(t), \tag{9}$$

we can investigate Eq. (8) in relation to the inequalites (6) and (7). Then we obtain

$$\frac{dA_j(t)}{d(t^2)} = \frac{F^2(t) + G(t) \ln \frac{\Delta}{r_j}}{F^2(t) \ln \frac{\Delta}{r_j}}. \tag{10}$$

Since the functions $F(t)$ and $G(t)$ are always positive if $0 \leqslant t < 1$

(cf. Sections 34, 35), $A_j(t)$ is a monotonically increasing function within the whole interval considered. Hence the following inequalities are justified

$$A_j(0) = -\frac{1}{1-X} \leqslant A_j(t) \leqslant \left(\ln \frac{\Delta}{r_j}\right)^{-1} = A_j(1), \tag{11}$$

and they lead to the relation

$$-\frac{1}{1-X} < a_j < \left(\ln \frac{\Delta}{r_j}\right)^{-1}, \tag{12}$$

in which the Landau and pairing interaction parameters are combined. It is easily noted that the above restrictions are equivalent to the ones in Eq. (7) provided that r_j tends to Δ, and $T = 0$.

The existence of additional states, which has to be manifested in collective excitations with the gap, can have an essential meaning, e.g., for determining the heat capacity.

The value of the gap determines the energy levels of the two-particle excitations. According to the obtained results these levels are modified by Fermi liquid interaction and for a positive large enough Landau parameter the two-particle states cannot be created, whereas for all negative values of the Landau parameter such excitations become possible and their gap tends to the value $t_s \geqslant 0$ defined by the equation

$$A_j(t_s) = -1 \tag{13}$$

and $t_s > 0$ if $T > 0$, even for the Landau parameters taken on the border of stability, i.e., if they tend to minus one (cf. Section 35). Such a situation seems to be quite understandable, since the positive values of Landau parameters are connected with the repulsive part of the Fermi liquid interaction, whereas the negative values describe its attractive part. Due to the superposition of these two interactions the attractive effects connected with the pairing interaction can be totally neutralized by repulsive Fermi liquid interaction effects. Such a situation takes place if the gap assumes the value 2Δ, which in our symbols is equivalent to $t = 1$. All these above remarks allow us to formulate the following conclusions.

The collective excitations with the gap, which are connected with the transitions to other states, are possible only if there exists an appropriate harmonic in the pairing interaction and if the appropriate Landau parameter is either sufficiently small or negative. In the opposite case, i.e., for a strong repulsive Fermi liquid interaction if $a_j > (\ln\Delta / r_j)^{-1}$ the possible two-particle states become unstable and simply cannot appear.

As to experiment the above conclusion permits us to either exclude or include some pairing interaction harmonic. Namely if the value a_j is negative and, according to experimental evidence, there are no appropriate collective excitations with the gap, such a harmonic should be excluded from the pairing interaction by putting f_j equal to zero; if, however, there exists the appropriate collective excitation with the gap and a_j is positive, then using Eq. (12) we can estimate the parameter r_j and obtain

$$r_j > \Delta \exp\left(-\frac{1}{a_j}\right). \tag{14}$$

In such a situation the pairing interaction creates the additional two -particle states if the energy parameter r_j satisfies the inequalities

$$\Delta \exp\left(-\frac{1}{a_j}\right) < r_j < \Delta . \tag{15}$$

24. The system in BW state

According to the formalism developed by Czerwonko [22] in the homogeneous limit the system of initial equations has the form

$$\begin{aligned} \mathcal{T} &= 1 + \langle \overset{s}{A}(-F\mathcal{T} + tF\tau^n\hat{p}_n)\rangle , \\ \tau^n &= \langle \Phi_{-1}\left\{\left[(t^2-1)F + f_1^{-1}\right]\tau^n + F\tau^i\hat{p}_i\hat{p}_n - tF\hat{p}_n\mathcal{T}\right\}\rangle \end{aligned} \tag{1}$$

where, as before, $\mathcal{T}$ is the normal vertex function, τ^n is the anomalous vertex function which can be expressed in the form

$$\tau = T\hat{p}_n + \Theta\,\hat{k}_n. \tag{2}$$

Then

$$\mathcal{T} = \sum_{j=0}^{\infty} (2_j + 1)\mathcal{T}_j P_j ,$$

$$T = \sum_{j=0}^{\infty} (2_j + 1)T_j P_j ,$$

$$\Theta = \sum_{j=0}^{\infty} (2_j + 1)\Theta_j P_j$$

where $\mathcal{T}_j = T_j = 0$ for all odd j and $\Theta_j = 0$ for all even j. $\overset{s}{A}$ is the symmetric part of the spin-direct Fermi liquid interaction, Φ_{-1} is the spin symmetric part of the pairing interaction and $r_1 \equiv \Delta$ (cf. Section 6).

After substituting Eq. (2), and some algebra Eqs. (1) reduce to the forms

$$\mathcal{T} = 1 + \langle \overset{s}{A} [-F\mathcal{T} + tF(T + \Theta w')] \rangle \quad , \tag{3}$$

$$Tw + \Theta = \langle \Phi_{-1} [(t^2F + f_1^{-1})(Tw' + \Theta) + F\Theta(w'^2 + 1) - tF\mathcal{T}w'] \rangle, \tag{4}$$

$$T + \Theta w = \langle (\hat{p}\hat{p}') \Phi_{-1} [(t^2F + f_1^{-1})T$$

$$+ F\Theta w' - tF\mathcal{T}] \rangle + w \langle \Phi_{-1} \{ [(t^2 - 1) F + f_1^{-1}] \Theta \} \rangle \tag{5}$$

where $w = \hat{k}\hat{p}$ and Eqs. (4) and (5) are obtained from one initial equation due to its vector structure. Applying the recurrence formula (33.5) and using the linear independence of Legendre polynomials we obtain

$$(2l+1)(1+a_lF)\mathcal{T}_l + (2l+1)a_l tFT_l - l a_l tF\Theta_{l-1} - (l+1) a_l TF\Theta_{l+1} = \delta_{l0}, \tag{6}$$

$$l f_l tF\mathcal{T}_{l-1} + (l+1) f_l tF\mathcal{T}_{l+1} + l [1 - f_l (t^2F + f_1^{-1})] T_{l-1}$$

$$+ (l+1) [1 - f_l (t^2F + f_1^{-1})] T_{l+1} + (2l+1) \Big[1 - f_l (t^2F - F + f_1^{-1})$$

$$- f_l \frac{2l^2 + 2l - 1}{(2l-1)(2l+3)} F\Big] \Theta_l - f_l \frac{l(l-1)}{2l-1} F\Theta_{l-2} - f_l \frac{(l+1)(l+2)}{2l+3} F\Theta_{l+2} = 0, \tag{7}$$

$$[l f_{l-1} + (l+1) f_{l+1}] tF\mathcal{T}_l + \{(2l+1) - [l f_{l-1} + (l+1) f_{l+1}](t^2F + f_1^{-1})\} T_l$$

$$+ l \Big\{1 - \frac{F}{2l+1} [l f_{l-1} + (l+1) f_{l+1}] - f_{l-1} [(t^2-1)F + f_1^{-1}]\Big\} \Theta_{l-1}$$

$$+ (l+1) \Big\{1 - \frac{F}{2l+1} [f_{l-1} + (l+1) f_{l+1}] - f_{l+1} [(t^2-1)F + f_1^{-1}]\Big\} \Theta_{l+1} = 0. \tag{8}$$

To make the above system of equations more symmetric, the number l in Eq. (7) is replaced by $l + 1$, then

$$(l+1) f_{l+1} tF\mathcal{T}_l + (l+2) f_{l+1} tF\mathcal{T}_{l+2}$$

$$+ (l+1) [1 - f_{l+1} (t^2F + f_1^{-1})] T_l + (l+2) [1 - f_{l+1} (t^2F + f_1^{-1})] T_{l+2}$$

$$+ (2l+3) \Big[1 - f_{l+1} (t^2F - F + f_1^{-1}) - f_{l+1} \frac{2l^2 + 6l + 3}{(2l+1)(2l+5)}\Big] \Theta_{l+1}$$

$$- f_{l+1} \frac{l(l+1)}{2l+1} F\Theta_{l-1} - f_{l+1} \frac{(l+2)(l+3)}{2l+5} F\Theta_{l+3} = 0. \tag{9}$$

The system of equations obtained is far more complicated than the one given in the previous section. Before we begin to solve it we have to make some modifications in order to eliminate the cut-off parameter ξ_p. This will be done by eliminating the normal vertex function, inserting Υ_l derived from Eq. (6) into Eq. (9) and dividing the result by f_{l+1}. Then after some calculations we obtain

$$(l+1)\left(\ln\frac{\Delta}{r_{l+1}}-\frac{t^2F}{1+a_lF}\right)T_l+(l+2)\left(\ln\frac{\Delta}{r_{l+1}}-\frac{t^2F}{1+a_{l+2}F}\right)T_{l+2}$$
$$-\frac{l(l+1)}{2l+1}\left(1-\frac{t^2a_lF}{1+a_lF}\right)F\theta_{l-1}+\left\{(2l+3)\left[\ln\frac{\Delta}{r_{l+1}}-(t^2-1)F\right]\right.$$
$$\left.-\frac{(l+1)^2}{2l+1}\left(1-\frac{t^2a_lF}{1+a_lF}\right)F-\frac{(l+2)^2}{2l+5}\left(1-\frac{t^2a_{l+2}F}{1+a_{l+2}F}\right)F\right\}\theta_{l+1}$$
$$-\frac{(l+2)(l+3)}{2l+5}\left(1-\frac{t^2a_{l+2}F}{1+a_{l+2}F}\right)F\,\theta_{l+3}=-\frac{tF}{1+a_0F}\,\delta_{l0}. \tag{10}$$

As this procedure cannot be repeated for Eq. (8) we can only reduce it to the form

$$f_{l-1}\left\{l\left(\ln\frac{\Delta}{r_{l-1}}-\frac{t^2F}{1+a_lF}\right)T_l+\frac{1}{2l+1}\right.$$
$$\times\left[(2l+1)\ln\frac{\Delta}{r_{l-1}}-l\left(1-\frac{t^2a_lF}{1+a_lF}\right)F-(2l+1)(t^2-1)F\right]\theta_{l-1}$$
$$\left.-\frac{l(l+1)}{2l+1}\left(1-\frac{t^2a_lF}{1+a_lF}\right)F\,\theta_{l+1}\right\}$$
$$+f_{l+1}\left\{(l+1)\left(\ln\frac{\Delta}{r_{l+1}}-\frac{t^2F}{1+a_lF}\right)T_l\right.$$
$$-\frac{l(l+1)}{2l+1}\left(1-\frac{t^2a_lF}{1+a_lF}\right)F\theta_{l-1}+\frac{l+1}{2l+1}\left[(2l+1)\ln\frac{\Delta}{r_{l+1}}-(l+1)\left(1-\frac{t^2a_lF}{1+a_lF}\right)F\right.$$
$$\left.\left.-(2l+1)(t^2-1)F\right]\theta_{l+1}\right\}=-f_1\frac{tF}{1+a_0F}\,\delta_{l0}. \tag{11}$$

We need not be interested in the details of the rigth-hand-sides of Eqs. (10) and (11), since the main problem of our calculations is to derive the poles of vertex functions. While analyzing Eqs. (10) and (11) it can be noted that their structure allows us to rewrite them in the following way

$$R_{l+2} + S_l = X_l \tag{12}$$

and

$$f_{l-1}R_l + f_{l+1}S_l = f_1 X_l \tag{13}$$

where the symbols R_l, S_l and X_l can be easily identified by means of Eq. (11). The expression X_l is independent of the functions T_l and θ_l and it vanishes if $l > 0$. After inserting S_l derived from Eq. (12) into Eq. (13), the latter is reduced to the form

$$f_{l-1}R_l - f_{l+1}R_{l+2} = 0. \tag{14}$$

Since $f_{-1} \equiv 0$, we have $R_2 = 0$, whence

$$R_l = 0 \quad \text{and} \quad S_l = 0 \tag{15}$$

for all l. Let us rewrite Eqs. (15) in an explicit form. Then we have

$$l\left\{\left(\ln\frac{\Delta}{r_{l-1}} - \frac{t^2F}{1+a_lF}\right)T_l + \left[\ln\frac{\Delta}{r_{l-1}} - \frac{1}{2l+1}\left(1-\frac{t^2a_lF}{1+a_lF}\right)F + (1-t^2)F\right]\theta_{l-1}\right.$$
$$\left. - \frac{l(l+1)}{2l+1}\left(1-\frac{t^2a_lF}{1+a_lF}\right)F\theta_{l+1}\right\} = 0, \tag{16}$$

$$\left(\ln\frac{\Delta}{r_{l+1}} - \frac{t^2F}{1+a_lF}\right)T_l - \frac{l}{2l+1}\left(1-\frac{t^2a_lF}{1+a_lF}\right)F\theta_{l-1}$$
$$+ \left[\ln\frac{\Delta}{r_{l+1}} - \frac{l+1}{2l+1}\left(1-\frac{t^2a_lF}{1+a_lF}\right)F + (1-l^2)F\right]\theta_{l+1} = 0, \tag{17}$$

in which for the general case the number l can take all values from zero to infinity. In such a way we obtain an open system of equations which can be solved only simultaneously and this task being rather complicated. Despite this fact a special case of a small gap was investigated in the so-called acoustic limit [119].

In the following we limit ourselves to some particular cases which seem to be of interest from physical point of view, i.e., to the case where for a fixed even number n all parameters f_{n+k} disappear or to the case where only two parameters f_1 and f_{n-1} differ from zero. In the first case we have

$$f_1 \neq 0, \quad f_3 \neq 0, \ldots, f_{n-1} \neq 0, \quad f_{n+1} = 0, \quad f_{n+3} = 0, \ \ldots,$$

hence

$$T_n = 0, \quad T_n+2 = 0, \ldots,$$

and

$$\theta_{n+1} = 0, \quad \theta_{n+3} = 0, \dots . \tag{18}$$

Although the above assumptions make the system of Eqs. (16)-(17) closed, the solution of the problem is still very complicated. However, if we put $l = n$ from Eq. (16) we obtain

$$n\left[\ln\frac{\Delta}{r_{n-1}} - \frac{n}{2n+1}\left(1 - \frac{t^2 a_n F}{1+a_n F}\right)F + (1-t^2)F\right]\theta_{n-1} = 0, \tag{19}$$

and hence the following gap equation

$$\ln\frac{\Delta}{r_{n-1}} - \frac{n}{2n+1}\left[1 - \frac{t^2 a_n F(t)}{1+a_n F(t)}\right]F(t) + (1-t^2)F(t) = 0. \tag{20}$$

Proceeding in a similar way as in the previous section we derive

$$a_n = \frac{nt^2}{(2n+1)\ln\frac{\Delta}{r_{n-1}} + (n+1)(1-t^2)F(t)} - \frac{1}{F(t)}, \tag{21}$$

and assuming

$$a_n = A_n(t) \tag{22}$$

we find

$$\frac{dA_n(t)}{d(t^2)} = \Bigg\{ n(n+1)F^2(t)E_{\frac{n+1}{2n+1}}(t)$$

$$+ (2n+1)\ln\frac{\Delta}{r_{n-1}}\Big[nF^2(t) + 2(n+1)(1-t^2)F(t)G(t)$$

$$+ (2n+1)\ln\frac{\Delta}{r_{n-1}}G^2(t)\Big]\Bigg\}\left[(2n+1)\ln\frac{\Delta}{r_{n-1}} + (n+1)(1-t^2)F(t)\right]^{-2}F^{-2}(t). \tag{23}$$

The function $E_s(t)$ is defined and discussed in Sec. 35. Since the function $E_s(t)$ is positive and all other quantities are also positive to if $t < 1$, the derivative $dA_n(t)/d(t^2)$ is also positive for every n. Investigating the limits of the function $A_n(t)$ we obtain

$$-\frac{1}{1-X} = A_n(0) \leqslant A_n(t) \leqslant A_n(1) = \frac{n}{2n+1}\left(\ln\frac{\Delta}{r_{n-1}}\right)^{-1}. \tag{24}$$

Thus, the kind of excitations caused by the last pairing harmonic leads to the following inequalities

$$-\frac{1}{1-X} < a_n < \frac{n}{2n+1}\left(\ln\frac{\Delta}{r_{n-1}}\right)^{-1} . \tag{25}$$

The inequalities obtained are analogous to the ones given by Eqs. (23.12) and the remarks contained in the previous section are also valid.

Let us pass on now to the second case mentioned, i.e., of only one additional pairing harmonic which does not vanish. Then we have

$$f_1 \neq 0,\quad f_3 = 0,\ldots,f_{n-3} = 0,\quad f_{n-1} \neq 0,\quad f_{n+1} = 0,\ \ldots\ , \tag{26}$$

and according to Eqs. (16) $T_k = \theta_{k+1} = 0$ for all $k \geqslant n$. Since for $n \geqslant 6$ the system of equations obtained can be separated, the above problem should be considered for two independent cases. Firstly we consider the case for $n \geqslant 6$ and from Eqs. (16) and (17) we obtain

$$-\frac{t^2F}{1+a_0F}(T_0 + \theta_1) = 0, \tag{27}$$

$$-2\frac{t^2F}{1+a_2F}T_2 - \frac{2}{5}\left[2\left(1 - \frac{t^2a_2F}{1+a_2F}\right) - 5(1-t^2)\right]F\theta_1 - \frac{6}{5}\left(1 - \frac{t^2a_2F}{1+a_2F}\right)F\theta_3 = 0, \tag{28}$$

$$n\left[\ln\frac{\Delta}{r_{n-1}} - \frac{n}{2n+1}\left(1 - \frac{t^2a_nF}{1+a_nF}\right)F + (1-t^2)F\right]\theta_{n-1} = 0, \tag{29}$$

$$\left(\ln\frac{\Delta}{r_{n-1}} - \frac{t^2F}{1+a_{n-2}F}\right)T_{n-2} - \frac{n-2}{2n-3}\left(1 - \frac{t^2a_{n-2}F}{1+a_{n-2}F}\right)F\theta_{n-3}$$

$$+\left[\ln\frac{\Delta}{r_{n-1}} - \frac{n-1}{2n-3}\left(1 - \frac{t^2a_{n-2}F}{1+a_{n-2}F}\right)F + (1-t^2)F\right]\theta_{n-1} = 0. \tag{30}$$

Putting $l = 2$ in Eq. (17) and $l = n-2$ in Eq. (16) we ascertain that the obtained identities are correct if

$$T_2 = -\theta_3 \quad\text{and}\quad T_{n-2} = -\theta_{n-3}. \tag{31}$$

Hence, Equations (28) and (30) reduce to the expressions

$$-\frac{2}{5}\left[2\left(1 - \frac{t^2a_2F}{1+a_2F}\right)F - 5(1-t^2)F\right](T_2 + \theta_1) = 0 \tag{32}$$

and

$$\left[\ln\frac{\Delta}{r_{n-1}} - \frac{n-1}{2n-3}\left(1 - \frac{t^2a_{n-2}F}{1+a_{n-2}F}\right)F + (1-t^2)F\right](T_{n-2} + \theta_{n-1}) = 0. \tag{33}$$

Equation (32) does depend on the pairing parameters. Equation (29) is equivalent to Eq. (20), which is in agreement with our expectation. From equation (33) we obtain one more gap equation

$$\ln\frac{\Delta}{r_{n-1}} - \frac{n-1}{2n-3}\left[1 - \frac{t^2 a_{n-2}F(t)}{1+a_{n-2}F(t)}\right]F(t)+(1-t^2)F(t) = 0. \tag{34}$$

Although Equation (34) is analogous to Eq. (20) it nevertheless cannot be obtained by an appropriate change of the number n. The same proceeding as before yields

$$a_{n-2} = \frac{(n-1)t^2}{(2n-3)\ln\frac{\Delta}{r_{n-1}} + (n-2)(1-t^2)F(t)} - \frac{1}{F(t)}, \tag{35}$$

hence

$$-\frac{1}{1-X} < a_{n-2} < \frac{n-1}{2n-3}\left(\ln\frac{\Delta}{r_{n-1}}\right)^{-1}. \tag{36}$$

From the results given by Eqs. (29) and (33) we can conclude that there is a dispersion in the excitation spectrum. Owing to the rich spin structure and the properties of the angular momentum, the number of two-particle states created for a given type of the pairing interaction may be greater than one. Such new states are usually not degenerated, though it can occur that some of them are degenerated. The existence of additional two-particle states, which can be realized due to some excitations in the system with pure P-pairing, was investigated in [43--45]. With the reference to above papers we can notice that due to the lack of degeneracy and the dispersion of the energy levels connected with it there appears an energy gap in the collective excitation spectrum. So the additional states are situated at higher energy levels and the difference in energy is exploited to rebuild the spin-angular momentum structure of the ground state pairs. Since some excitations to the higher states are realized by means of special external fields, in the case studied some transitions can be forbidden. All these forbidden states are not good candidates for taking part in the equilibrium state, unless some extra external conditions are taken into account. The influence of the Landau parameters is the same as before, i.e., that the gap equations (20) and (34) are modified in such a way that the distances connected with the pairing energy and the dispersion of the state fluctuate simultaneously. It can also occur that the states disappear or become degenerated. Moreover, it is worth emphasizing that the transitions within the same split level result only in the rebuilt energy.

Let us now consider the last case, i.e., for n = 4. We have

$$f_1 \neq 0, \quad f_3 \neq 0, \quad f_5 = 0, \quad f_7 = 0, \quad \ldots,$$

then $T_k = \theta_{k+1} = 0$ for all $k \geqslant 4$ and Eqs. (16) and (17) reduce to the forms

$$-\frac{t^2 F}{1+a_0 F}\left(T_0 + \theta_1\right) = 0, \tag{38}$$

$$-2\frac{t^2}{1+a_2 F}T_2 - \frac{2}{5}\left[\left(1 - \frac{t^2 a_2 F}{1+a_2 F}\right) - 5(1-t^2)\right]F\theta_1$$
$$-\frac{6}{5}\left(1 - \frac{t^2 a_2 F}{1+a_2 F}\right)F\theta_3 = 0, \tag{39}$$

$$3\left(\ln\frac{\Delta}{r_3} - \frac{t^2 F}{1+a_2 F}\right)T_2 - \frac{6}{5}\left(1 - \frac{t^2 a_2 F}{1+a_2 F}\right)F\theta_1$$
$$+3\left[\ln\frac{\Delta}{r_3} - \frac{3}{5}\left(1 - \frac{t^2 a_2 F}{1+a_2 F}\right)F + (1-t^2)F\right]\theta_3 = 0, \tag{40}$$

$$4\left[\ln\frac{\Delta}{r_3} - \frac{4}{9}\left(1 - \frac{t^2 a_4 F}{1+a_4 F}\right)F + (1-t^2)F\right]\theta_3 = 0. \tag{41}$$

While analysing the structure of the derived equations we notice that the main determinant vanishes only when one of the following equations is fulfilled

$$\frac{t^2 F(t)}{1+a_0 F(t)} = 0, \tag{42}$$

$$\ln\frac{\Delta}{r_3} - \frac{4}{9}\left[1 - \frac{t^2 a_4 F(t)}{1+a_4 F(t)}\right]F(t) + (1-t^2)F(t) = 0, \tag{43}$$

$$\left[1 - \frac{t^2 a_2 F(t)}{1+a_2 F(t)}\right]\frac{t^2 F(t)}{1+a_2 F(t)} + \left\{\left[1 - \frac{t^2 a_2 F(t)}{1+a_2 F(t)}\right] - \frac{5}{2}(1-t^2)\right\}$$
$$\times\left[\ln\frac{\Delta}{r_3} - \frac{t^2 F(t)}{1+a_2 F(t)}\right] = 0. \tag{44}$$

Since Equation (42) is gapless and Eq. (43) is equivalent to Eq. (20) we consider only Eq. (44) which after some algebra reduces to the form

$$\frac{1}{1+a_2 F(t)}\left\{\ln\frac{\Delta}{r_3}\left[t^2\left(1+\frac{3}{5}a_2 F(t)\right) - \frac{3}{5}\left(1+a_2 F(t)\right)\right] + t^2(1-t^2)F(t)\right\} = 0, \tag{45}$$

hence

$$a_2 = \frac{5t^2}{3\ln\frac{\Delta}{r_3}} + \frac{5(t^2-\frac{3}{5})}{3(1-t^2)F(t)} . \tag{46}$$

Assuming

$$a_2 = A_2(t) \tag{47}$$

and differentiating the righ-hand-side of Eq. (46) we get that the derivative

$$\frac{dA_2(t)}{d(t^2)} = \frac{5}{3\ln\frac{\Delta}{r_3}} + \frac{2E_{3/5}(t)}{3(1-t^2)^2F^2(t)} \tag{48}$$

is a positive function of t if $0 \leqslant t < 1$.

Since

$$A_2(0) = -\frac{1}{1-X} \quad \text{and} \quad A_2(1) = +\infty \tag{49}$$

we do not obtain any extra connections this time. This result seems to be quite understandable, since the possible two-particle state is now a superposition of two other states which in the previous case were described independently by Eqs. (32) and (33), and the state connected with the P-pairing exists for the all values of the Landau parameter a_2. Thus in this case there exist only two additional two-particle states, which can be achieved by two-particle pairs during the perturbation with the weak and inhomogeneous definite symmetry external fields.

In conclusion, we ought to emphasize that the large number of two -particle states which arises from their rich interior-angular momentum structure becomes the reason for their mutual mixing and creation of the composite two-particle states which depend on many pairing interaction parameters. Such a situation is already determined by the forms of Eqs. (16) and (17).

The extra problem is connected with the high terms of the expansion. Since for the large value of l the parameters a_l and f_l must become small and thus r_l must become small, too, it may happen that one of the inequalities (25) is fulfilled and then there appear the shallow two-particle states (cf. Eq. (23.12)).

It may be noticed that although the two-particle states are created owing to spin-symmetric pairing interaction, their forms are determined by the interior spin-angular momentum structure and their existence decided by the Fermi liquid interaction. In the case of the spin-antisymmetric pairing interaction (which is found in all superconductors) the two-particle states have a poor spin-angular momentum structure and for the fixed pairing harmonic they can be realized only in one way. Moreover, all the possible two-particle states are independent; they do not mix and do not split. Precisely: One pairing harmonic can give only one single two-particle state, the bound energy of which is mofified by an appropriate Landau parameter. The predicted properties of ^3He-^4He mixtures in dependence on concentrations [112] allow us to expect broad application of the developed formalism.

VII. Spin oscillations of ^3He-B in the presence of dipole forces

25. The spin susceptibility tensor [43]

In order to calculate the spin susceptibility tensor χ_{ij} we apply the LMC theory [22, 79]. Since we do not impose any restrictions on the frequency, wave vector, and temperature, the kernels L,M,N,O are non -analytic functions of the variables kv and ω (cf. Sec. 37). For this reason the final results will contain an infinite number of Landau parameters if any restrictions are imposed on them. On the other hand, in the limiting cases where $\omega = 0$ or $k = 0$, or $T = 0$, these kernels become analytic functions of kv or ω, or kv and ω, respectively. This alone allows us to obtain the final results in a closed form without any restrictions on the Landau parameters, although the cases $\omega = 0$ and $T = 0$ demand some extra restrictions imposed on kv. If $kv \ll \omega$ and $T \neq 0$, the kernels L,M,N,O, become analytic functions alone of kv, and though in this limiting case any restrictions need not be imposed on the Landau parameters, the obtained results are valid for some frequencies only. Thus, in order to solve the problem we should assume that all the Landau parameters b_l are equal to zero for $l > l_0$. From the experiments we know (Section 30) that the main part of the information about the ^3He system is contained in the zeroth Landau parameter. The contributions of the remaining Landau parameters are smaller and cause only small quantitative alterations. That is why we confine ourselves to the case $l_0 = 0$. For the sake of symmetry we also assume that the wave vector is parallel to the unit vactor $\hat{k}$ introduced in Section 19. The basic equations necessary to solve such a problem have the following forms:

$$\mathcal{T}^i_j = \delta_{ij} + b_0 \left\langle (L-O)\mathcal{T}^i_j + 2O\mathcal{T}^k_j d_{km}\hat{p}_m d_{in}\hat{p}_n - 2M\varepsilon_{ikm}\tau^k_j d_{mn}\hat{p}_n \right\rangle , \qquad (1)$$

$$\tau^{i}_{j} = \langle V_{ik} \quad \{[N + O - H(\hat{p})] \; \tau^{k}_{j}$$

$$- 2OT^{m}_{j}d_{mn}\hat{p}_{n}d_{kr}\hat{p}_{r} - 2M\,\varepsilon_{kmn}\mathcal{T}^{n}_{j}d_{nr}\hat{p}_{r}\}\rangle \,, \tag{2}$$

$$\chi_{ij} = -\mu^{2}_{B}\nu_{0} \langle (L - O)\,\mathcal{T}^{i}_{j} + 2O\,\mathcal{T}^{k}_{j}d_{km}\hat{p}_{m}d_{in}\hat{p}_{n}$$

$$- 2M\,\varepsilon_{ikm}\,\tau^{k}_{j}d_{mn}\hat{p}_{n}\rangle \tag{3}$$

where $\mathcal{T}^{i}_{j}$ and τ^{i}_{j} are the normal and anomalous vertex functions.

Let us introduce the following notation:

$$\hat{q}_{i} = d_{im}\hat{p}_{m}, \quad w = \hat{k}\hat{p}, \quad \tau^{i}_{j} = \tau^{im}_{j}\,\hat{q}_{m} \tag{4}$$

and, according to Eq. (19.23), we have

$$d_{im}\hat{k}_{m} = \pm\,\hat{k}_{i}, \quad d_{ik}d_{im} = \delta_{km}, \quad w = \pm\,\hat{k}\hat{q}. \tag{5}$$

The average over $\hat{p}$ can be easily replaced by an average over $\hat{q}$ in Eqs. (1) and (3). This procedure allows us to estimate the tensor d_{ik} from these equations. Multiplying Equation (2) by $\underline{\hat{V}}^{-1}$, and taking account of the remarks given in Section 19, we can also eliminate d_{ik} from the right-hand side of the obtained equation. Now, the kernel of the operator $\underline{\hat{U}}$ has the form

$$U_{ij} = (\delta_{ij}\delta_{kn} + d_{ki}d_{nj} + d_{ni}d_{kj})\hat{q}_{k}\hat{q}'_{n} \,. \tag{6}$$

The expression $\mathcal{T}^{i}_{j}$ and $\mathcal{T}^{n}_{j}\hat{k}_{n}\hat{k}_{i}$ can be calculated from the transformed equation (1) via the application of the averaging formulas (Section 33). Next, substituting these expressions into the transformed equations (2) and (3) and performing all the averaging procedures, we obtain very complicated formulas. To simplify them, let us introduce several symbols. First, we introduce the 7-vector $\hat{T}$ of the third-rank tensors where

$$\begin{aligned}
\hat{T}_{1} &= (1/\sqrt{2})\varepsilon_{ikn}\hat{k}_{k}\hat{k}_{j}, \\
\hat{T}_{2} &= (1/\sqrt{3})\delta_{in}\hat{k}_{j}, \\
\hat{T}_{3} &= (1/\sqrt{6})(\delta_{in} - 3\hat{k}_{i}\hat{k}_{n})\hat{k}_{j}, \\
\hat{T}_{4} &= \tfrac{1}{2}(\varepsilon_{ijk}\hat{k}_{k}\hat{k}_{n} + \varepsilon_{kjn}\hat{k}_{k}\hat{k}_{i}), \\
\hat{T}_{5} &+ \tfrac{1}{2}(\delta_{ij}\hat{k}_{n} + \delta_{nj}\hat{k}_{i} - 2\hat{k}_{i}\hat{k}_{n}\hat{k}_{j}), \\
\hat{T}_{6} &= \tfrac{1}{2}(\delta_{ij}\hat{k}_{n} - \delta_{nj}\hat{k}_{i}), \\
\hat{T}_{7} &= \tfrac{1}{2}(\varepsilon_{ijk}\hat{k}_{k}\hat{k}_{n} - \varepsilon_{kjn}\hat{k}_{k}\hat{k}_{i}).
\end{aligned} \tag{7}$$

It is worth emphasizing that the tensor $\hat{T}_\alpha$ can be expressed via tensors B^{in}_{JM} (Section 40).

Hence, the tensors $\hat{T}_\alpha$ are connected with the excitation to the states (cf. [90, 103, 104, 127, 134])

$$
\begin{aligned}
&\hat{T}_1: \quad J = 1, \quad M = 0,\\
&\hat{T}_2: \quad J = 0, \quad M = 0,\\
&\hat{T}_3: \quad J = 2, \quad M = 0,\\
&\hat{T}_4: \quad J = 1, \quad M = \pm 1,\\
&\hat{T}_5: \quad J = 2, \quad M = \pm 1,\\
&\hat{T}_6: \quad J = 1, \quad M = \pm 1,\\
&\hat{T}_7: \quad J = 2, \quad M = \pm 1.
\end{aligned}
\tag{8}
$$

Below it will be shown that the excitations with $M = 0$ are independent of those with $M = \pm 1$. This fact can be easily interpreted if we take into account the symmetry of the system. All the excitations can be classified in terms of the two-particle states with the following quantum numbers: $0 \leqslant J \leqslant 2$, $|M| \leqslant J$, $L = 1$, $S = 1$. Since in the system the direction $\hat{k}$ is distinguished, the total angular momentum J is no longer a good quantum number (M is still a good quantum number). For this reason the states with the same number M mutually mix. All longitudinal effects are related to the excitations with $M = 0$, whereas the transverse or circular ones are related to those with $M = \pm 1$.

Let us introduce the 7-vector $\underline{S}$ such that

$$\tau^{in}_j = S_\alpha \hat{T}_\alpha \tag{9}$$

and the operator $\hat{Q}_l (l = 0, \dots, 11)$ such that

$$\hat{Q}_l \hat{T}_\alpha = \hat{T}^{(l)}_\alpha \quad \text{and} \quad \hat{T}^{(l)}_\alpha = \{\hat{Q}_l\}_{\alpha\beta}\, \hat{T}_\beta \ . \tag{10}$$

The matrices $\{\hat{Q}_l\}_{\alpha\beta}$ $(\alpha, \beta = 1, \dots, 7)$ have the following forms:

$$\{\hat{Q}_l\} = \left[\begin{array}{c|c} \{\hat{Q}^0_l\} & 0 \\ \hline 0 & \{\hat{Q}^1_l\} \end{array}\right] \tag{11}$$

$$\{\hat{Q}^0_l\}_{\alpha\beta}\,, \quad \alpha,\beta = 1,2,3; \qquad \{\hat{Q}^1_l\}_{\alpha\beta}\,, \quad \alpha,\beta = 4,5,6,7$$

where

$$
\begin{aligned}
\{\hat{Q}_0\}_{\alpha\beta} = {} & (15/4)\,\delta_{\alpha1}\delta_{\beta1} + \tfrac{5}{6}\,\delta_{\alpha2}\delta_{\beta2} + (29/12)\delta_{\alpha3}\delta_{\beta3} \\
& \pm \tfrac{3}{4}\sqrt{5}(\delta_{\alpha1}\delta_{\beta3} + \delta_{\alpha3}\delta_{\beta1}) - \tfrac{5}{6}\sqrt{2}(\delta_{\alpha2}\delta_{\beta3} + \delta_{\alpha3}\delta_{\beta2}) \\
& + \tfrac{5}{4}(\delta_{\alpha4}\delta_{\beta4} + \delta_{\alpha6}\delta_{\beta6}) + \tfrac{3}{4}(\delta_{\alpha5}\delta_{\beta5} + \delta_{\alpha7}\delta_{\beta7}) \\
& \pm \tfrac{1}{4}\sqrt{15}(\delta_{\alpha4}\delta_{\beta5} + \delta_{\alpha5}\delta_{\beta4} - \delta_{\alpha6}\delta_{\beta7} - \delta_{\alpha7}\delta_{\beta6}),
\end{aligned}
$$

$$\{\hat{Q}_1\}_{\alpha\beta} = \delta_{\alpha\beta} ,$$

$$\{\hat{Q}_2\}_{\alpha\beta} = \delta_{\alpha\beta} - 2(\delta_{\alpha 1}\delta_{\beta 1} + \delta_{\alpha 4}\delta_{\beta 4} + \delta_{\alpha 6}\delta_{\beta 6}),$$

$$\{\hat{Q}_3\}_{\alpha\beta} = 3\,\delta_{\alpha 2}\delta_{\beta 2},$$

$$\{\hat{Q}^0_4\}_{\alpha\beta} = \{\hat{Q}^0_5\}_{\alpha\beta} = \{\hat{Q}^0_6\}_{\alpha\beta} = \{\hat{Q}^0_7\}_{\alpha\beta} = \{\hat{Q}^0_{11}\}_{\alpha\beta}$$

$$= \frac{1}{3}\delta_{\alpha 2}\delta_{\beta 2} + \frac{2}{3}\delta_{\alpha 3}\delta_{\beta 3} - \frac{1}{3}\sqrt{2}(\delta_{\alpha 2}\delta_{\beta 3} + \delta_{\alpha 3}\delta_{\beta 2}),$$

$$\{\hat{Q}^1_4\}_{\alpha\beta} = \frac{1}{2}(\delta_{\alpha\beta} + \delta_{\alpha 4}\delta_{\beta 7} + \delta_{\alpha 5}\delta_{\beta 6} + \delta_{\alpha 6}\delta_{\beta 5} + \delta_{\alpha 7}\delta_{\beta 4}),$$

$$\{\hat{Q}^1_5\}_{\alpha\beta} = \{\hat{Q}^1_7\}_{\beta\alpha} = \frac{1}{2}(\{\hat{Q}^1_2\}_{\alpha\beta} - \delta_{\alpha 4}\delta_{\beta 7}$$

$$+ \delta_{\alpha 5}\delta_{\beta 6} - \delta_{\alpha 6}\delta_{\beta 5} + \delta_{\alpha 7}\delta_{\beta 4}),$$

$$\{\hat{Q}^1_6\}_{\alpha\beta} = - \{\hat{Q}^1_4\}_{\alpha\beta} + 2\delta_{\alpha\beta} ,$$

$$\{\hat{Q}_8\}_{\alpha\beta} = \{\hat{Q}_9\}_{\beta\alpha} = \delta_{\alpha 2}\delta_{\beta 2} - \sqrt{2}\,\delta_{\alpha 2}\delta_{\beta 3},$$

$$\{\hat{Q}_{10}\}_{\alpha\beta} = -2\,\delta_{\alpha 1}\delta_{\beta 1},$$

$$\{\hat{Q}^1_{11}\}_{\alpha\beta} = 0. \tag{12}$$

The action of the operator $\hat{Q}_l$ on τ^{in}_j should be understood in the followiing way

$$\hat{Q}_l\tau^{in}_j = \hat{Q}_l S_\alpha \hat{T}_\alpha = S_\alpha\{\hat{Q}_l\}_{\alpha\beta}\hat{T}_\beta . \tag{13}$$

The 7-vector C is defined by (cf. Section 33)

$$C = -\left[\frac{2m_2}{1-b_0I_1},0,0,\frac{2m_2+m_3}{1-b_0I_2},0,0,\frac{m_3}{1-b_0I_2}\right] \tag{14}$$

where $I_2 = l_0 - o_0 + 2o_2$, $I_1 = I_2 + 2o_3$, and the 3-vector X^{ij} of the second rank tensors is defined by

$$\hat{X}^{ij} = \left[\hat{k}_i\hat{k}_j,\ \delta_{ij} - \hat{k}_i\hat{k}_j,\ \pm\varepsilon_{ijm}\hat{k}_m\right]. \tag{15}$$

The plus sign should be used if we take the gap in the form defined by (19.9a) and $\theta = +\cos^{-1}(-1/4)$ or if we take the gap in the form (19.9b) and $\theta = -\cos^{-1}(1/4)$. The minus sign should be taken in the opposite case.

Let the operator $\hat{E}_t$ be defined by the formulas

$$\hat{E}_t\hat{T}_\alpha = \hat{\Xi}_\alpha^{(t)} \quad \text{and} \quad \hat{\Xi}_\alpha^{(t)} = \{\hat{E}_t\}_{\alpha\beta} \ \hat{X}_\beta^{ij} \tag{16}$$

where $\alpha = 1,\ldots,7$, $\beta = 1,2,3$; $t = 1,2,3$; and

$$\begin{aligned}
\{\hat{E}_1\}_{\alpha\beta} &= -2(\delta_{\alpha 1}\delta_{\beta 2} + \delta_{\alpha 4}\delta_{\beta 2} - \delta_{\alpha 6}\delta_{\beta 3}),\\
\{\hat{E}_2\}_{\alpha\beta} &= -\delta_{\alpha 4}\delta_{\beta 2} + \delta_{\alpha 5}\delta_{\beta 3} + \delta_{\alpha 6}\delta_{\beta 3} - \delta_{\alpha 7}\delta_{\beta 2},\\
\{\hat{E}_3\}_{\alpha\beta} &= -2\delta_{\alpha 1}\delta_{\beta 1}
\end{aligned} \tag{17}$$

and let two 3-vector $\underline{Y}$ and $\underline{Z}$ be defined by

$$Y = -\frac{2}{1-b_0I_2}\left[m_2,\ m_3,\ 2b_0m_2o_3(1-b_0I_1)^{-1}\right], \tag{18}$$

$$Z = -\left[I_1(1-b_0I_1)^{-1}, I_2(1-b_0I_2)^{-1}, 0\right]. \tag{19}$$

We also introduce the factors $D_l (l = 0,\ldots,11)$, which in the transformed equation (2) appeared near the tensors $\hat{T}_\alpha$. They will not be defined now because they are necessary only for the calculation of the $P_{\alpha\beta}$ (see below). We only emphasize that D_0 and D_4 depend on g. Using all the symbols introduced above, from Eqs. (1)-(3) we obtain

$$S_\alpha D_l\{\hat{Q}_l\}_{\alpha\beta}\hat{T}_\beta = C_\beta\hat{T}_\beta, \tag{20}$$

$$\chi_{ij} = \mu_B^2\nu_0 \left[Z_\beta - S_\alpha Y_t\{\hat{E}_t\}_{\alpha\beta}\right] \hat{X}_\beta^{ij}. \tag{21}$$

Since the $\hat{T}_\alpha$, are mutually orthogonal, we can rewrite Eq. (20) in the form

$$S_\alpha P_{\alpha\beta} = C_\beta \tag{22}$$

where

$$\underline{P} = D_l\{\hat{Q}_l\}.$$

The matrix $\underline{P}$ has an analogous block structure to that of the matrices $\{\hat{Q}_l\}$.

Let us introduce the inverse matrix $\underline{R} = \underline{P}^{-1}$ where

$$R_{\alpha\beta} = \| M_{\beta\alpha} \| / \| P \| \tag{23}$$

and

$$R_{\alpha\beta} = \| M_{\beta\alpha}^e \| / \| {}^eP \|, \quad e = 0 \text{ or } 1 \tag{24}$$

where $\| \ldots \|$ denotes a determinant.

It is evident that the matrix $\underline{R}$ has the same block structure as the matrices $\underline{P}$ and $\{\hat{Q}_l\}$.

From equation (22) we determine S; inserting it to Eq. (21), we find the following spin susceptibility tensor:

$$\chi_{ij} = \mu_B^2 \nu \left[Z_\alpha - C_\beta \{P^{-1}\}_{\beta\gamma} Y_t \{\hat{E}_t\}_{\gamma\alpha} \right] \hat{X}_\alpha^{ij} \ . \tag{25}$$

The poles of this expression are determined by the condition

$$\| \underline{P} \| = 0 \tag{26}$$

which is equivalent to

$$\| {}^e\underline{P} \| = 0 \quad \text{where} \quad e = 0 \text{ or } 1. \tag{27}$$

If equations (27) are fulfilled, the extra conditions are obtained

$$\underline{P}_{\alpha\beta} \, \| M^e_{\beta\gamma} \| = 0 \tag{28}$$

for all α and γ. These conditions can be very useful to calculate $\| M^e_{\alpha\beta} \|$ in the vicinity of poles.

The second part of expression (25) can be written as follows:

$$C_\alpha \, \{P^{-1}\}_{\alpha\beta} \, Y_t \{\hat{E}_t\}_{\beta\gamma} = W_\gamma \tag{29}$$

where

$$W_1 = \frac{-8m_2^2}{(1 - b_0 I_1)^2} R_{11},$$

$$W_2 = \frac{-2}{(1-b_0 I_2)^2} \left[(2m_2 + m_3)^2 R_{44} + (2m_2 + m_3) m_3 (R_{47} + R_{74}) + m_3^2 R_{77} \right],$$

$$W_3 = \frac{2}{(1-b_0 I_2)^2} \left[(2m_2 + m_3)^2 R_{46} + (2m_2 + m_3) m_3 (R_{45} + R_{76}) + m_3^2 R_{75} \right].$$

Now, let us write down the matrices ${}^e\underline{P} = D_1\{\hat{Q}_1^e\}$ in their explicit form:

$${}^0P = \begin{bmatrix} P_{11} & 0 & P_{13} \\ 0 & P_{22} & P_{23} \\ P_{13} & P_{23} & P_{33} \end{bmatrix} \tag{30}$$

where

$$P_{11} = \tfrac{9}{4} g + 8b_0 (m_2^2 + m_3^2)(1 - b_0 I_2)^{-1} - n_1 - o_1 - 4o_8,$$

$$P_{13} = \pm\, (9\sqrt{5}/20) g,$$

$$P_{22} = \tfrac{1}{3} I_3 + 10 o_6 + (50/3) o_7 - n_2 - o_2 - \tfrac{1}{3} n_3 - \tfrac{1}{3} o_3,$$

$$P_{23} = -(\sqrt{2}/3)(I_3 + 14o_7 - n_3 - o_3),$$

$$P_{33} = (9/20)g + \tfrac{2}{3}I_3 + 4o_6 + (28/3)o_7 - n_2 - o_2 - \tfrac{2}{3}o_3 - \tfrac{2}{3}n_3,$$

$$I_3 = 4b_0\left[m_3^2 - 2b_0 m_2^2 o_3(1 - b_0 I_1)^{-1}\right](1 - b_0 I_2)^{-1},$$

and

$$^1P = \begin{bmatrix} P_{44} & P_{45} & 0 & P_{47} \\ P_{45} & P_{55} & P_{56} & 0 \\ 0 & P_{47} & P_{44} & -P_{45} \\ P_{56} & 0 & -P_{45} & P_{55} \end{bmatrix} \qquad (31)$$

where

$$P_{44} = -n_2 - o_2 - \tfrac{1}{2}n_3 - \tfrac{1}{2}o_3 + 2b_0(2m_2 + m_3)m_3(1 - b_0 I_2)^{-1}$$

$$+ 8b_0 m_2^2(1 - b_0 I_2)^{-1},$$

$$P_{45} = \pm(3\sqrt{15}/20)g,$$

$$P_{47} = \tfrac{3}{4}g - \tfrac{1}{2}n_3 - \tfrac{1}{2}o_3 + 2b_0 m_3(4m_2 + m_3)(1 - b_0 I_2)^{-1},$$

$$P_{55} = -\tfrac{3}{10}g - n_2 - o_2 - \tfrac{1}{2}n_3 - \tfrac{1}{2}o_3 + 4o_6 + 8o_7 + 2b_0 m_3^2(1 - b_0 I_2)^{-1},$$

$$P_{56} = -\tfrac{3}{4}g - \tfrac{1}{2}n_3 - \tfrac{1}{2}o_3 + 2b_0 m_3(2m_2 + m_3)(1 - b_0 I_2)^{-1}.$$

Here the rule according to which the sign is to be chosen is the same as for (15). Matrices $M^e_{\alpha\beta}$ can be easily obtained from $^e\underline{P}$. The solution of the equation $\| ^e\underline{P} \| = 0$ is a good problem for numerical computation. That is why only some limiting cases of

$$\chi_{ij} = \mu_B^2 \nu \left[Z_\alpha - W_\alpha\right] \hat{X}^{ij}_\alpha \qquad (32)$$

will be considered (cf. Section 37). First, let us consider the acoustic limit at $T = 0$ assuming $g \ll 1$. In this case

$$Z_1 = Z_2 = \tfrac{2}{3}(1 + \tfrac{2}{3}b_0)^{-1},$$

$$W_1 = Z_1\omega^2\left[\omega^2 - 27\Delta^2 g(1 + \tfrac{2}{3}b_0) - \tfrac{1}{5}k^2 v^2(1 + \tfrac{2}{3}b_0)\right]^{-1},$$

$$W_2 = Z_1\omega^2\left[\omega^2 - \tfrac{2}{5}k^2 v^2(1 + \tfrac{2}{3}b_0)\right]^{-1},$$

$$W_3 = 0. \qquad (33)$$

In the homogeneous limit ($k = 0$) we obtain

$$Z_1 = Z_2 = \frac{2}{3} F\left(1 + \frac{2}{3} b_0 F\right)^{-1}, \tag{34}$$

$$\begin{aligned} W_1 = Z_1 &\left(\frac{\omega}{2\Delta}\right)^2 \left[\frac{2}{5} F - \left(\frac{\omega}{2\Delta}\right)^2 F + \frac{27}{20} g\right] \\ &\times \left(F\left(\frac{\omega}{2\Delta}\right)^2 \left[\frac{2}{5} - \left(\frac{\omega}{2\Delta}\right)^2\right] - \frac{9}{2} g \left\{\frac{3}{5}\left[1 - 3\left(\frac{\omega}{2\Delta}\right)^2\right]\right.\right. \\ &\left.\left. + b_0 F \left[\frac{2}{5} - \left(\frac{\omega}{2\Delta}\right)^2\right]\right\}\right)^{-1}, \end{aligned} \tag{35}$$

$$\begin{aligned} W_2 = Z_1 F &\left(\frac{\omega}{2\Delta}\right)^2 \left[\frac{2}{5} F - \left(\frac{\omega}{2\Delta}\right)^2 F - \frac{9}{10} g\right] \\ &\times \left\{F \left(\frac{\omega}{2\Delta}\right)^2 \left[\frac{2}{5} F - \left(\frac{\omega}{2\Delta}\right)^2 F - \frac{9}{10} g\right]\right. \\ &\left. - \frac{81}{10} g^2 \left(1 + \frac{2}{3} b_0 F\right)\right\}^{-1}, \end{aligned} \tag{36}$$

$$W_3 = 0. \tag{37}$$

In the static limit ($\omega = 0$)

$$Z_1 = - I_1(1 - b_0 I_1)^{-1}, \qquad Z_2 = - I_2(1 - b_0 I_2)^{-1} \tag{38}$$

where

$$I_1 = - 1 + \frac{1}{3} F_2, \qquad I_2 = - 1 + \frac{1}{2} F_0 - \frac{1}{6} F_2$$

and

$$W_\alpha = 0. \tag{39}$$

Let us analyse the formula (32) in two cases:

(i) For $g = 0$, i.e., when the dipole-dipole interaction is disregarded: in this case

$$\begin{aligned} W_1 &= 8m_2^2(1 - b_0 I_1)^{-1} \left[(n_2 + o_2 + 4o_8)(1 - b_0 I_1) - 8b_0(m_2^2 + m_3^2)\right]^{-1}, \\ W_2 &= -2(1 - b_0 I_2)^{-2} \left[(2m_2 + m_3)^2 P_{55} - (2m_2 + m_3) m_3 (P_{47} + P_{56})\right. \\ &\left.\quad + m_3^2 P_{44}\right] (P_{44} P_{45} - P_{47} P_{56})^{-1}, \end{aligned} \tag{40}$$

$W_3 = 0$, $\quad Z_\alpha$ as in Eq. (19).

(ii) For $b_0 = 0$, i.e., when the Fermi-liquid interaction is disregarded. In this case we obtain $P_{47} = P_{56}$ and hence

$$W_3 = 0. \tag{41}$$

In order our problem be analysed completely, let us discuss W_3 in detail. The expressions for $\|M_{\alpha\beta}\|$ can be found from (31). Using Equations (24) and (29), we obtain W_3 in the form

$$W_3 = \mp \sqrt{15} g(P_{47} - P_{56}) I_4 \left[5(1 - b_0 I_2)^2\right]^{-1} \|{}^1P\|^{-1} \tag{42}$$

where

$$I_4 = (2m_2 + m_3)^2 P_{55} - (2m_2 + m_3)m_3(P_{47} + P_{56}) + m_3^2 P_{44}.$$

Hence

$$W_3 = \mp (4\sqrt{15}/5)\ g\ b_0 m_2 m_3 I_4 (1 - b_0 I_2)^{-3} \|{}^1P\|^{-1}. \tag{43}$$

As we see, this effect is strongly coupled with the Fermi-liquid interaction. If we take now m_i in the form given in Section 37, we obtain

$$W_3 = \mp \frac{2\sqrt{15}}{225}\ g\ b_0 \left(\frac{\omega}{2\Delta}\right)^2 \left(\frac{kv}{2\Delta}\right)^2 F_h F_{(1)} I_4 (1 - b_0 I_2)^{-3} \|{}^1P\|^{-1}. \tag{44}$$

Comparing (15), (32) and (42)-(44) we state that the response of the system is independent of the choice of the equilibrium state (cf. Section 19). Assuming that all the terms g^n can be neglected for $n \geqslant 2$, we see that the inclusion of the dipole-dipole interaction modifies only the spectrum of the longitudinal spin waves in which there appears a gap, whereas the spectrum of the transverse spin waves does not change. With the above assumption we obtain that in the homogeneous limit $Z_2 = W_2$, due to which the transverse part of the spin susceptibility vanishes (cf. Section 26). The static limit is completely independent of the dipole-dipole interaction.

The circular component which appears in the spin susceptibility tensor is analogous to the components of the dielectric tensors responsible for the circular dichroism of optically active substances. The role of this component can be characterized in the following way. If a monochromatic and polarized electromagnetic wave is transmitted through a sample of superfluid ^{3}He-B, the polarization plane will be rotated. This is a result of the composition of two orthogonal vectors, which characterize the magnetic oscillations. This rotation is proportional to the value of g and its direction (right or left) depends on the sign of b_0. A similar situation will take place for the transverse collective excitations. In such a case the magnetic vector of the propagating mode will rotate tracing a helicoid, and for $b_0 < 0$ a suitable mode has positive helicity.

26. Dynamic properties of the system in the homogeneous limit [45]

Let us complete the above considerations with the results obtained in the homogeneous limit ($k = 0$), connected with the dynamic properties of the spin susceptibility and the properties of the spin collective excitation gaps. The new dynamic properties, i.e., non-zero response of the system in homogeneous fields, are due to the broken spin-orbit symmetry, which takes place if the weak dipole-dipole interaction is not neglected.

The inclusion of the dipole-dipole interaction leads to the quantitative and qualitative changes of the spin susceptibility tensor. Although now no additional restrictions imposed on the Fermi liquid interaction parameters are claimed, the final results hold only two Landau parameters b_0 and b_2. Let us discuss now the spin susceptibility tensor in two specific limits. In the static limit, when $k = 0$ and $\omega \to 0$, we have (cf. Section 34)

$$\chi_{ij} = \frac{\frac{2}{3}\mu_B^2\nu_0(1-X)\delta_{ij}}{1 + \frac{2}{3}b_0 - \frac{2}{3}b_0 X + \frac{1}{3}b_2 - \frac{1}{3}b_2 X} . \tag{1}$$

It is the function of temperature and its value changes from the static spin susceptibility value at $T = 0$ to zero, when the temperature increases to T_c.

In the limit $\omega \to 2\Delta$, when the function $F(\omega)$ tends to infinity, we get

$$\chi_{ij} = -9g\mu_B^2\nu_0\left[\frac{\hat{k}_i\hat{k}_j}{2-9g(b_0+b_2)} + \frac{2gb_2(\delta_{ij}-\hat{k}_i\hat{k}_j)}{2+3gb_2(1-6gb_0-3gb_2)}\right] . \tag{2}$$

The expression (2) is constant for all temperatures, but, as we remember, $\omega\,(= 2\Delta)$ essentially depends on temperature. This feature can be helpful in determining the temperature dependence of Δ. The longitudinal part of the spin susceptibility reveals the diamagnetic properties of the system, whereas the transverse response depends on the sign of the Landau parameter b_2.

If we neglect the parameter g^2 (cf. Section 42), the dynamic spin susceptibility is of the form

$$\chi_{ij} = \frac{-18\mu_B^2\nu_0 g\Delta^2\hat{k}_i\hat{k}_j}{\omega^2-27g\Delta^2\left[1+\frac{2}{3}b_0F(\omega)+\frac{1}{3}b_2F(\omega)\right]/F(\omega)} , \tag{3}$$

since the disregard of the elements proportional to g^2 causes the transverse part of the spin susceptibility to vanish. The magnetic resonance is connected with poles of the spin susceptibility (3) and is achieved if the following equation is fulfilled

$$\omega_r^2 = 27g\Delta^2\left[1+\frac{2}{3}b_0(1-X)+\frac{1}{3}b_2(1-X)\right](1-X)^{-1} \tag{4}$$

where we restrict ourselves to the first iterative step. Since the expression $\Delta^2(1 - X)^{-1}$ tends to zero alike Δ if temperature tends to T_c, the resonance frequency tends to zero if temperature tends to T_c. The expression (3) has no antiresonances (cf. [26, 27, 61, 68]) which are not detectable if g^2 is too small. In the static limit, the longitudinal part of the spin susceptibility is the same as previously in Eq. (1), whereas its transverse part is always equal to zero. In the limit $\omega\to 2\Delta$, we obtain again the longitudinal part alike in Eq. (2).

An investigation of the spin collective excitation gaps creates a separate problem which can be evaluated by means of extrapolation of the spin collective excitation spectrum. If we regard that the parameter g is small, then the gaps of the longitudinal and transverse spin collective excitation can be obtained from the following equations

$$\omega^2\left[1 + \frac{2}{5}b_2F(\omega)\right] = \frac{8}{5}\Delta^2\left[1 + b_2F(\omega)\right]\left[1 + 27g/8F(\omega)\right] , \tag{5}$$

$$\omega^2\left[1 + \frac{2}{5}b_2F(\omega)\right] = \frac{8}{5}\Delta^2\left[1 + b_2F(\omega)\right]\left[1 - 9g/4F(\omega)\right] . \tag{6}$$

Let us denote the solutions of these equations by $\omega_\parallel$ and $\omega_\perp$, ω_0 is the solution of Eqs. (5) and (6) in the limit $g = 0$. The accuracy of the applied formalism allows us to assume that $\omega_\parallel$ and $\omega_\perp$ if compared with ω_0 are modified by the quantities λ and μ of the order of g, and can be taken in the forms

$$\omega_\parallel = \omega_0(1 + \lambda) \quad \text{and} \quad \omega_\perp = \omega_0(1 + \mu). \tag{7}$$

Inserting Equations (7) into Eqs. (5) and (6) we find λ and μ (cf. Section 35)

$$\lambda = 27g/16E_{2/5}(\omega_0), \qquad \mu = -9g/8E_{2/5}(\omega_0) \tag{8}$$

where

$$\omega_0 = \frac{1}{5}(2\omega_\parallel + 3\omega_\perp). \tag{9}$$

Using (7) we get

$$\frac{\omega_\parallel - \omega_\perp}{\omega_0} = 45g/16E_{2/5}(\omega_0), \tag{10}$$

then the following inequalities are valid

$$\omega_{\perp} < \omega_{Q} < \omega_{\parallel}. \qquad (11)$$

The r.h.s. of Eq. (10) achieves its maximum if $\omega_0 = 2\sqrt{2/5}\Delta$, i.e., if $b_2 = 0$ (cf. Section 35). Since b_2 is probably small (cf. [6, 60]), split of the spectrum is not the effect to be neglected (cf. [128,150]).

An inclusion of the dipole-dipole interaction leads to changes in the spin-wave spectrum. Although the transverse spin wave remains the gapless mode, in the longitudinal spin wave spectrum there appears a gap which can be expressed by Eq. (4). This gap vanishes if the temperature tends to T_c. Moreover, the non-commutativity of two kinds of the static limit, i.e., $\omega = 0$, $k \longrightarrow 0$, and $k = 0$, $\omega \longrightarrow 0$ is a worth emphasizing effect. Both static susceptibilities split over T = 0 and the first one tends to the normal spin susceptibility value whereas the second one decreases to zero together with $T \longrightarrow T_c$. Hence, the static response of the system becomes completely univocal at T = 0, only.

VIII. Autocorrelation Functions and Spinless Oscillations in ^{3}He-B

27. The density-density autocorrelation function [53]

We developed the microscopic Green function formalism for a case when the two first harmonics of the spin symmetric Fermi liquid interaction, i.e., A_0 and A_1 and dipole-dipole interaction are included. The scantiness of the number of Landau parameters is caused by the fact that no additional restrictions were imposed on the wave vector and temperature. In such a situation the specified problem can be solved only when the number of Landau parameters is finite. However, the influence of the higher Landau parameters is set off by the temperature and inhomogeneity effects and it can be neglected if the mentioned parameters or effects are small. The presented calculations are carried out without any additional restrictions imposed on the frequency and wave vector. It causes the obtained results to be more precise in comparison with those given in [27, 65, 90, 91, 105, 145-147, 150] and they allow us e.g. to penetrate the deep (in the frequency and wave vector) region of spin-less collective excitations. Simultaneously we can eliminate some collective excitations which are not determined by poles of the autocorrelation functions.

Since the present methods differ in many details from the ones applied in the previous chapter we have developed them independently. In the specified case the proper vertex equations reduce to the forms

$$\overset{s}{\mathcal{T}} = 1 + A_0 \left\langle (L - O)\overset{s}{\mathcal{T}} + (L + O)\overset{a}{\mathcal{T}} - 2M\tau^k \hat{q}_k \right\rangle, \tag{1}$$

$$\overset{a}{\mathcal{T}} = A_1 \hat{p}_i \left\langle \hat{p}_i \left[(L - O)\overset{s}{\mathcal{T}} + (L + O)\overset{a}{\mathcal{T}} - 2M\tau^k \hat{q}_k \right] \right\rangle, \tag{2}$$

$$\tau^k = \left\langle \hat{V}_{k1} \left[(N - O - H)\tau^1 + 2O\tau^n \hat{q}_n \hat{q}_1 + 2M(\overset{s}{\mathcal{T}} + \overset{a}{\mathcal{T}})\hat{q}_1 \right] \right\rangle, \tag{3}$$

and the density-density autocorrelation function is of the form

$$S^{00} = \nu_0 \left\langle (L - O)\overset{s}{\mathcal{T}} + (L + O)\overset{a}{\mathcal{T}} - 2M\tau^k \hat{q}_k \right\rangle \tag{4}$$

where $\overset{s}{\mathcal{T}}$ and $\overset{a}{\mathcal{T}}$ are the symmetric and antisymmetric parts of the normal vertex function, respectively, τ^k is the anomalous vertex functions, and $\hat{q}_k = d_{kn}\hat{p}_n$. $\underline{\hat{V}}$ is full interaction in the particle-particle channel (Section 19) and the kernels L.M.N.O. are given in Section 37.

Introducing the additional symbols

$$\mathcal{T}_0 = \langle \overset{s}{\mathcal{T}} \rangle, \qquad \mathcal{T}_1 = 3\langle \overset{a}{\mathcal{T}}(\hat{k}\hat{p}) \rangle, \qquad \tau^{kn} = 3\left\langle \tau^k \hat{q}_n \right\rangle, \tag{5}$$

and taking into account (Section 19)

$$\hat{p}\hat{p}' = \hat{q}\hat{q}', \qquad \hat{k}_n = d_{np}\hat{k}_p \qquad \text{and} \qquad \hat{k}\hat{p} = \hat{k}\hat{q} \tag{6}$$

we can transform Eqs. (4), (2) and (4) to the forms

$$\mathcal{T}_0 = 1 + A_0(l_0 - o_0)\mathcal{T}_0 + A_0 l_1 \mathcal{T}_1 - 2A_0 m_2 \tau^{ii} - 2A_0 m_3 \tau^{ij}\hat{k}_i \hat{k}_j, \tag{7}$$

$$\begin{aligned} \mathcal{T}_1 \hat{k}_i &= A_1 l_1 \mathcal{T}_0 \hat{k}_i + A_1(l_2 + o_2 + l_3 + o_3)\mathcal{T}_1 \hat{k}_i \\ &\quad -2A_1\left[m_4(\tau^{jj}\hat{k}_i + \tau^{ij}\hat{k}_j + \tau^{ji}\hat{k}_j) + m_5 \tau^{jn} k_j \hat{k}_n \hat{k}_i\right], \end{aligned} \tag{8}$$

$$S^{00} = \nu_0\left[(l_0 + o_0)\mathcal{T}_0 + l_1\mathcal{T}_1 - 2m_2\tau^{ii} - 2m_3\tau^{ij}\hat{k}_i\hat{k}_j\right], \tag{9}$$

hence multiplying Eq. (8) by $\hat{k}_i$ we obtain

$$\begin{aligned} \mathcal{T}_1 &= A_1 l_1 \mathcal{T}_0 + A_1(l_2 + o_2 + l_3 + o_3)\mathcal{T}_1 \\ &\quad - 2A_1\left[m_4\tau^{ii} + (2m_4 + m_5)\tau^{ij}\hat{k}_i\hat{k}_j\right]. \end{aligned} \tag{10}$$

By multiplying both sides of Eq. (3) by $\underline{\hat{V}}^{-1}$ and after some algebra, we obtain

$$\frac{3}{2}\, g\tau^{in}\hat{k}_n\hat{k}_j - \frac{1}{5}\, gU^{jn}_{im}\tau^{mn} = (n_2 - o_2 + 2o_6)\tau^{ij} + 2o_6(\tau^{ji} + \tau^{nn}\delta_{ij})$$

$$+ (n_3 - o_3 + 2o_7)\tau^{in}\hat{k}_n\hat{k}_j + 2o_7(\tau^{ni}\hat{k}_n\hat{k}_j + \tau^{jn}\hat{k}_n\hat{k}_i + \tau^{nj}\hat{k}_n\hat{k}_i + \tau^{nm}\hat{k}_n\hat{k}_m\delta_{ij} + \tau^{nn}\hat{k}_i\hat{k}_j)$$

$$+ 2o_8\tau^{nm}\hat{k}_n\hat{k}_m\hat{k}_i\hat{k}_j + 2\mathcal{T}_0(m_2\delta_{ij} + m_3\hat{k}_i\hat{k}_j) + 2\mathcal{T}_1 m_4\delta_{ij} + 2\mathcal{T}_1(2m_4 + m_5)\hat{k}_i\hat{k}_j \tag{11}$$

where the parameter g is an invariant of the theory (Section 42). In order to solve so prepared problem we proceed in the following way:

From equations (7) and (10) we express the normal vertex functions $\mathcal{T}_0$ and $\mathcal{T}_1$ by means of the anomalous vertex function τ^{ij} combined with the Kronecker delta and the unit vector $\hat{k}$. Substituting the derived expressions into Eq. (11) we obtain the tensor equation which is analogous to Eq. (11) where the sole unknown quantity is the anomalous vertex function. In order to make our notation consistent we present this equation in the following operator form

$$B_l\hat{Q}_l\tau^{ij} = C_\beta T^{ij}_\beta \tag{12}$$

where $\hat{Q}_l\,(l = 0,\ldots,10)$ are the operators transforming the structure of the anomalous vertex function according to the relations occurring in Eq. (11). B_l denote the appropriate coefficients which are the respective functions of the frequency, wave vector, Landau parameters A_0 and A_1 and the dipole interaction parameter g. The terms T^{ij}_β ($\beta = 0,1,2$) denote the possible linearly independent tensors of the second rank that possess the rotational symmetry in relation to the distinguished direction $\hat{k}$. These tensors generate the three-dimension space; we choose them in the forms

$$T^{ij}_0 = \frac{1}{\sqrt{3}}\delta_{ij}, \quad T^{ij}_1 = \frac{1}{\sqrt{2}}\varepsilon_{inj}\hat{k}_n, \quad T^{ij}_2 = \frac{1}{\sqrt{6}}(\delta_{ij} - 3\hat{k}_i\hat{k}_j), \tag{13}$$

which ensure their orthonormality. In such a case they fulfil the relation

$$T^{ij}_\alpha T^{ij}_\beta = \delta_{\alpha\beta}\ . \tag{14}$$

Moreover, such a choice of the tensors T^{ij}_β allows us to classify the adequate collective excitations in terms of the two-particle states, and we have (cf. [42, 43, 90, 134])

$$T^{ij}_0:\ J = 0, \quad M = 0,$$

$$T^{ij}_1:\ J = 1, \quad M = 0,$$

$$T^{ij}_2:\ J = 2 \quad M = 0. \tag{15}$$

However, in the inhomogeneous systems distinguishing one external direction $\hat{k}$, J ceases to be a good quantum number which causes the collective excitations to mix mutually, hence they can be separated solely in the quasi-homogeneous limit [42].

The coefficients C_β are of the forms

$$C_0 = \frac{2\sqrt{3}}{3}\left\{(3m_2 + m_3)\left[1 - A_1(l_2 + o_2 + l_3 + o_3)\right] + A_1 l_1(5m_4 + m_5)\right\} W^{-1},$$

$$C_1 = 0,$$

$$C_2 = -2\sqrt{\frac{2}{3}}\left\{m_3\left[1 - A_1(l_2 + o_2 + l_3 + o_3)\right] + A_1 l_1(2m_4 + m_5)\right\} W^{-1} \tag{16}$$

where

$$W = \left[1-A_0(l_0-o_0)\right]\left[1-A_1(l_2+o_2+l_3+o_3)\right] - A_0A_1l_1^2.$$

According to equations (13) the anomalous vertex function τ^{ij} can be expressed in the form

$$\tau^{ij} = D_\beta T^{ij}_\beta, \tag{17}$$

then Eq. (12) reduces to the form

$$D_\alpha B_l\{\hat{Q}_l\}_{\alpha\beta}\, T^{ij}_\beta = C_\beta T^{ij}_\beta \tag{18}$$

where the matrices $\{\hat{Q}_l\}$ constituting the representation of the operators $\hat{Q}_l$ in the three-dimension space generated by the tensor T^{ij} are of the forms

$$\{\hat{Q}_0\}_{\alpha\beta} = \frac{5}{6}\delta_{\alpha1}\delta_{\beta1} + \frac{15}{4}\delta_{\alpha2}\delta_{\beta2} + \frac{29}{12}\delta_{\alpha3}\delta_{\beta3} - \frac{5\sqrt{2}}{6}(\delta_{\alpha1}\delta_{\beta3} + \delta_{\alpha3}\delta_{\beta1}) \pm \frac{3\sqrt{5}}{4}(\delta_{\alpha2}\delta_{\beta3} + \delta_{\alpha3}\delta_{\beta2}),$$

$$\{\hat{Q}_1\}_{\alpha\beta} = \delta_{\alpha\beta},$$

$$\{\hat{Q}_2\}_{\alpha\beta} = \delta_{\alpha\beta} - 2\delta_{\alpha2}\delta_{\beta2},$$

$$\{\hat{Q}_3\}_{\alpha\beta} = 3\delta_{\alpha1}\delta_{\beta1},$$

$$\{\hat{Q}_4\}_{\alpha\beta} = \{\hat{Q}_5\}_{\alpha\beta} = \{\hat{Q}_6\}_{\alpha\beta} = \{\hat{Q}_7\}_{\alpha\beta} = \{\hat{Q}_{10}\}_{\alpha\beta} = \frac{1}{3}\delta_{\alpha1}\delta_{\beta1} + \frac{2}{3}\delta_{\alpha3}\delta_{\beta3} - \frac{\sqrt{2}}{3}(\delta_{\alpha1}\delta_{\beta3} + \delta_{\alpha3}\delta_{\beta1}),$$

$$\{\hat{Q}_8\}_{\alpha\beta} = \{\hat{Q}_9\}_{\beta\alpha} = \delta_{\alpha 1}\delta_{\beta 1} - \sqrt{2}\,\delta_{\alpha 1}\delta_{\beta 3}. \tag{19}$$

In order to simplify Eq. (9), we introduce the following symbols: $\hat{R}_n$ and Π, where the operators of contraction $\hat{R}_n (n = 1,2)$ are such that

$$\hat{R}_n \tau^{ij} = (\delta_{n1}\delta_{ij} + \delta_{n2}\hat{k}_i\hat{k}_j)\tau^{ij}. \tag{20}$$

Then

$$\hat{R}_n T_{\alpha}^{ij} = \{\hat{R}_n\}_{\alpha} \tag{21}$$

where

$$\{\hat{R}_1\}_{\alpha} = \sqrt{3}\,\delta_{\alpha 1}, \quad \{\hat{R}_2\}_{\alpha} = \frac{\sqrt{3}}{3}\delta_{\alpha 1} - \sqrt{\frac{2}{3}}\,\delta_{\alpha 3},$$

and the matrix Π is defined by the relation (cf. [44])

$$\Pi = B_1\{\hat{Q}_1\}, \tag{22}$$

the elements of which are of the forms

$$\begin{aligned}
\Pi_{11} = &-(n_2 - o_2) - \tfrac{1}{3}(n_3 - o_3) - 10o_6 - \frac{20}{3}o_7 - \frac{2}{3}o_8 \\
&+ \tfrac{4}{3}\{A_0(3m_2 + m_3)^2[1 - A_1(l_2 + o_2 + l_3 + o_3)] \\
&+ A_1(5m_4 + m_5)^2[1 - A_0(l_0 - o_0)] \\
&+ 2A_0A_1l_1(3m_2 + m_3)(5m_4 + m_5)\}\,W^{-1},
\end{aligned}$$

$$\Pi_{12} = \Pi_{21} = 0,$$

$$\begin{aligned}
\Pi_{13} = \Pi_{31} = &\frac{\sqrt{2}}{3}\left[(n_3 - o_3) + 20o_7 + 2o_8\right] \\
&- \frac{4\sqrt{2}}{3}\{A_0m_3(3m_2 + m_3)\left[1 - A_1(l_2 + o_2 + l_3 + o_3)\right] \\
&+ A_1(2m_4 + m_5)(5m_4 + m_5)\left[1 - A_0(l_0 - o_0)\right] \\
&+ A_0A_1l_1\left[m_3(5m_4 + m_5) + (3m_2 + m_3)(2m_4 + m_5)\right]\}\,W^{-1},
\end{aligned}$$

$$\Pi_{22} = \frac{9}{4}g - (n_2 - o_2),$$

$$\Pi_{23} = \Pi_{32} = \pm\frac{9\sqrt{5}}{20}g,$$

$$\Pi_{33} = \frac{9}{20}g - (n_2 - o_2) - \tfrac{2}{3}(n_3 - o_3) - 4o_6 - \frac{16}{3}o_7 - \frac{4}{3}o_8$$

$$+ \frac{8}{3} \left\{ A_0 m_3^2 \left[1 - A_1(l_2 + o_2 + l_3 + o_3)\right] \right.$$

$$+ A_1(2m_4 + m_5)^2 \left[1 - A_0(l_0 - o_0)\right]$$

$$\left. + 2A_0 A_1 l_1 m_3 (2m_4 + m_5) \right\} W^{-1}.$$

Now, inserting the expressions $\mathcal{T}_0$ and $\mathcal{T}_1$ derived from Eqs. (7) and (10) into Eq. (9) and applying the above symbols, we obtain

$$S^{00} = \nu_0 (E - I_n \{\hat{R}_n\}_\alpha D_\alpha) \tag{23}$$

where

$$E = \left\{(l_0 - o_0) \left[1 - A_1(l_2 + o_2 + l_3 + o_3)\right] - A_1 l_1^2 \right\} W^{-1},$$

and

$$I_1 = 2\left\{m_2 \left[1 - A_1(l_2 + o_2 + l_3 + o_3)\right] + A_1 l_1 m_4 \right\} W^{-1},$$

$$I_2 = 2\left\{m_3 \left[1 - A_1(l_2 + o_2 + l_3 + o_3)\right] + A_1 l_1 (2m_4 + m_5)\right\} W^{-1}.$$

Exploiting equation (18) we find the terms D_α in the form

$$D_\alpha = C_\beta \Pi^{-1}_{\beta\alpha}, \tag{24}$$

where Π^{-1} is the inverse matrix fulfilling the relation

$$\Pi_{\alpha\beta} \Pi^{-1}_{\beta\gamma} = \delta_{\alpha\gamma}, \tag{25}$$

and by substituting the obtained expressions into Eq. (23), the autocorrelation function S^{00} reduces to the form

$$S^{00} = \nu_0 (E - K), \tag{26}$$

where

$$K = \sqrt{3}(I_1 + \tfrac{1}{3} I_2)(C_0 \Pi^{-1}_{11} + C_2 \Pi^{-1}_{31}) - \sqrt{\tfrac{2}{3}} I_2 (C_0 \Pi^{-1}_{13} + C_2 \Pi^{-1}_{33}),$$

and the suitable elements of the inverse matrix Π^{-1} can be expressed in the following way

$$\Pi^{-1}_{11} = (\Pi_{22} \Pi_{33} - \Pi^2_{23}) / \|\Pi\|,$$

$$\Pi^{-1}_{13} = \Pi^{-1}_{31} = -\Pi_{13} \Pi_{22} / \|\Pi\|,$$

$$\Pi^{-1}_{33} = \Pi_{11} \Pi_{22} / \|\Pi\|, \tag{27}$$

where $\|\Pi\|$ denotes the determinant of the matrix Π and

$$\|\Pi\| = \Pi_{11}\Pi_{22}\Pi_{33} - \Pi_{13}^2\Pi_{22} - \Pi_{11}\Pi_{23}^2. \tag{28}$$

Using the above relations the autocorrelation function can be transformed to the form

$$S^{00} = v_0\left\{E - \left[C_0^2(\Pi_{22}\Pi_{33} - \Pi_{23}^2)\right.\right. \tag{29}$$

$$\left.\left. - 2C_0C_2\Pi_{13}\Pi_{22} + C_2^2\Pi_{11}\Pi_{22}\right]/\|\Pi\|\right\},$$

which becomes more readable after applying Section 33. Finally we obtain

$$S^{00} = \frac{v_0}{W}\left(V - \frac{X}{Z}\right), \tag{30}$$

where

$$V = (l_0 - o_0)\left[1 - A_1(l_p + o_p)\right] + A_1 l_1^2,$$

$$W = 1 - A_1(l_p + o_p) - A_0 V,$$

$$X = (WC_0)^2\left[\Pi_{22}(W\Pi_{33}) - \Pi_{23}^2 W\right]$$

$$- 2(WC_0)(WC_2)(W\Pi_{13})\Pi_{22} + (WC_2)^2(W\Pi_{11})\Pi_{22},$$

$$Z = (W\Pi_{11})\Pi_{22}(W\Pi_{33}) - (W\Pi_{13})^2\Pi_{22} - W(W\Pi_{11})\Pi_{23}^2,$$

$$WC_0 = \frac{2\sqrt{3}}{3}\left\{m_0\left[1 - A_1(l_p + o_p)\right] + A_1 l_1 m_1\right\},$$

$$WC_2 = \sqrt{\frac{2}{3}}\left\{(m_0 - m_p)\left[1 - A_1(l_p + o_p)\right] + A_1 l_1(m_1 - 3m_r)\right\},$$

and

$$W\Pi_{11} = -\frac{1}{3}(n_0 + o_0)W + \frac{4}{3}A_0 m_0^2\left[1 - A_1(l_p + o_p)\right]$$

$$+ \frac{4}{3}A_1 m_1^2\left[1 - A_0(l_0 - o_0)\right] + \frac{8}{3}A_0 A_1 l_1 m_0 m_1,$$

$$W\Pi_{13} = \frac{\sqrt{2}}{12}\left\{\left[2(n_p - n_0) - 5o_0 + 8o_p - 3o_q\right]W\right.$$

$$- 8A_0 m_0(m_p - m_0)\left[1 - A_1(l_p + o_p)\right]$$

$$- 8A_1 m_1(3m_r - m_1)\left[1 - A_0(l_0 - o_0)\right]$$

$$\left. - 8A_0 A_1 l_1(m_p m_1 + 3m_0 m_r - 2m_0 m_1)\right\}.$$

$$\Pi_{22} = \frac{9}{4}g - \frac{1}{2}(n_0 - o_0) + \frac{1}{6}(n_p - o_p),$$

$$\Pi_{23} = \pm \frac{9\sqrt{5}}{20} g,$$

$$W\Pi_{33} = \left[\frac{9}{20} g - \frac{1}{6}(n_0 + o_0) - \frac{1}{2} n_p + \frac{5}{6} o_p - \frac{3}{5} o_q\right] W$$

$$+ \frac{2}{3}\left\{A_0(m_p - m_0)^2 \left[1 - A_1(l_p + o_p)\right]\right.$$

$$+ A_1(3m_r - m_1)^2 \left[1 - A_0(l_0 - o_0)\right]$$

$$\left. + 2A_0A_1l_1(m_p - m_0)(3m_r - m_1)\right\}.$$

The assumed structure of the autocorrelation function ensures all the introduced expressions V, W, X, Z to be of the polynomial-type and they have no poles. Hence, the collective excitations which are always determined by the poles of the correlation functions can be derived from the following equations

$$W = 0 \quad \text{and} \quad Z = 0. \tag{31}$$

The first equation defines the collective excitations of the normal component whereas the other one refers to the collective excitations appearing in the superfluid BW systems only. The obtained form of the autocorrelation function S^{00} is very extensive and its involved structure is due to the presence of the dipole-dipole interaction. We can reduce the number of the introduced symbols replacing the factors l_i, m_i, n_i, o_i by the frequency, wave vector and the appropriate averages of the function F (Sections 33, 34, 36, 37).

Let us consider now the possible spinless collective excitations determined by the poles of the derived density-density autocorrelation function (30) when the dipole-dipole interaction is omitted.

Putting $g = 0$ we have stated that the autocorrelation function S^{00} keeps its structure (30) and the terms V, W, X, Z reduce to the forms

$$V = -(\Gamma_0 - 1)(1 + \tfrac{1}{3}A_1 - \tfrac{1}{3}A_1F_2) - A_1 s^2 F_0 \Gamma_2, \tag{32}$$

$$W = (1 + A_0 - A_0\Gamma_0)(1 + \tfrac{1}{3}A_1 - \tfrac{1}{3}A_1F_2) - A_1 s^2 (1 + A_0F_0)\Gamma_2, \tag{33}$$

$$X = \left(\frac{\omega}{2\Delta}\right)^2 \left[(1 + A_0 - A_0\Gamma_0)\Sigma - A_0A_1s^2\Gamma_2^2\right]$$

$$\times\left\{\left[-3F_2 + \frac{9}{5}F_4 + \left(\frac{\omega}{2\Delta}\right)^2 F_2 - \frac{3}{5}\left(\frac{kv}{2\Delta}\right)^2 F_4\right](F_0\Sigma + \tfrac{1}{3}A_1F_2\Gamma_2)^2\right.$$

$$+ \left[\left(\frac{\omega}{2\Delta}\right)^2 F_0 - \frac{1}{3}\left(\frac{kv}{2\Delta}\right)^2\right]F_2\,(F_2\Sigma + \tfrac{3}{5}A_1F_4\Gamma_2)^2$$

$$+ \frac{3}{2}(F_0\Sigma + \frac{1}{3}A_1F_2\Gamma_2)\left[(F_0 - F_2)\Sigma + \frac{1}{3}A_1(F_2 - \frac{9}{5}F_4)\Gamma_2\right]$$

$$\times \left[\frac{4}{3}\left(\frac{\omega}{2\Delta}\right)^2 F_2 - \frac{4}{5}\left(\frac{kv}{2\Delta}\right)^2 F_4 - 2F_0 + 2F_2 - F_4\right]$$

$$- 9A_1\left(\frac{kv}{2\Delta}\right)^2 (\frac{1}{5}F_0F_4 - \frac{1}{9}F_2^2)^2\Sigma\Big\}, \tag{34}$$

$$Z = \Big\{\left(\frac{\omega}{2\Delta}\right)^2 F_0(1 - A_0\Gamma_2)(1 + \frac{1}{3}A_1 - \frac{1}{3}A_1F_2)$$

$$- \frac{1}{3}\left(\frac{kv}{2\Delta}\right)^2 F_2(1 + A_0 - A_0\Gamma_0)(1 + \frac{1}{3}A_1)$$

$$- A_1s^2\Gamma_2\left[\left(\frac{\omega}{2\Delta}\right)^2 F_0 - \frac{1}{3}\left(\frac{kv}{2\Delta}\right)^2 (1 - A_0F_0)\right]\Big\}$$

$$\times \Big\{\left(\frac{\omega}{2\Delta}\right)^2(1+ \frac{1}{3}A_1 - \frac{1}{3}A_1F_2)\left[(F_0+F_2)(1+A_0-A_0\Gamma_0) - A_0(F_0 - F_2)^2\right]$$

$$- \left(\frac{kv}{2\Delta}\right)^2 (1+A_0-A_0\Gamma_0)\left[(\frac{1}{3}F_2 + \frac{3}{5}F_4)(1+ \frac{1}{3}A_1 - \frac{1}{3}A_1F_2) + A_1(\frac{1}{3}F_2 - \frac{3}{5}F_4)^2\right]$$

$$- 3(F_2 - \frac{3}{5}F_4)(1 + A_0 - A_0\Gamma_0)(1 + \frac{1}{3}A_1 - \frac{1}{3}A_1F_2)$$

$$- A_1s^2\Gamma_2\left[\left(\frac{\omega}{2\Delta}\right)^2 (F_0 + F_2 + 3A_0F_0F_2 - A_0F_2^2)\right.$$

$$- \frac{1}{15}\left(\frac{kv}{2\Delta}\right)^2 (5F_2 + 9F_4 - 5A_0F_0F_2 + 27A_0F_0F_4 + 10A_0F_2^2 - 18A_0F_2F_4)$$

$$\left.- 3(F_2 - \frac{3}{5}F_4)(1 + A_0F_0)\right]\Big\}$$

$$- \Big\{\left(\frac{\omega}{2\Delta}\right)^2 (F_0-F_2)(1-A_0\Gamma_2)(1 + \frac{1}{3}A_1 - \frac{1}{3}A_1F_2)$$

$$- \left(\frac{kv}{2\Delta}\right)^2 (\frac{1}{3}F_2 - \frac{3}{5}F_4)(1+A_0-A_0\Gamma_0)(1 + \frac{1}{3}A_1)$$

$$+ \frac{3}{4}(F_0 - 2F_2 + F_4)(1 + A_0 - A_0\Gamma_0)(1 + \frac{1}{3}A_1 - \frac{1}{3}A_1F_2)$$

$$- A_0s^2\Gamma_2\left[\left(\frac{\omega}{2\Delta}\right)^2 (F_0-F_2) - \left(\frac{kv}{2\Delta}\right)^2 (\frac{1}{3}F_2 - \frac{3}{5}F_4)\right.$$

$$\left.- \frac{1}{3}\left(\frac{kv}{2\Delta}\right)^2 A_0F_2(F_2 - 2F_0) + \frac{3}{4}(1 + A_0F_0)(F_0 - 2F_2 + F_4)\right]\Big\}^2 \tag{35}$$

where

$$s = \frac{\omega}{kv}\,, \qquad \Gamma_2 = F_0 + \Gamma_0 - 1,$$

and

$$\Sigma = 1 + \frac{1}{3}A_1 - \frac{1}{3}A_1F_2 - A_1 s^2 \Gamma_2. \tag{36}$$

The functions F_{2i} and Γ_0 are defined and discussed in Section 36. It is easy to prove that the obtained form of the autocorrelation function S^{00} reduces to the well-known forms in some specific limits [22, 27, 44]. In order to derive the spectra of the longitudinal spinless excitations appearing in the B-phase of superfluid ^{3}He we have to solve the latter equation (31). That equation can be rewritten in the form

$$Z_1Z_2 - Z_3^2 = 0 \tag{37}$$

where $Z_i(i = 1,2,3)$ are the factors defined in Eq. (35). Let us note that the factor Z_3 is proportional to $(kv)^2$, thus it must be omitted in the acoustic and quasi-homogeneous limits. Hence we can state that the collective excitations of the types J = 0, M = 0 (zero sound) and J = 2, M = 0 connected with the factors Z_1 and Z_2, respectively, separate themselves within the mentioned limits. Applying the formulae of Section 36 we can refine the developed problem extremely. However, at least at non-zero temperatures, the problem cannot be solved to the end in an analytic manner. Therefore, it is necessary to investigate Eq. (37) in some specific limits.

1° The acoustic limit (ω, $kv \ll \Delta$)

After neglecting all terms of the higher order in ω and kv than the square ones we have stated that Eq. (37) is satisfied if $Z_1 = 0$. Hence, the zero sound dispersion can be derived from the following equation

$$\begin{aligned} & s^2\left[1 - \hat{\underline{P}}\Psi_0 + \hat{\underline{P}}\xi^2(\Psi_0 - 1)\right]\left[1 - A_0\hat{\underline{P}}\xi^2(\Psi_0 - 1)\right] \\ & \times\left[1 + A_1 s^2\hat{\underline{P}}\xi^2(1-\xi^2)(\Psi_0 - 1) + \frac{1}{3}A_1\hat{\underline{P}}\xi^2\right] \\ & - \frac{1}{3}\left[1 - 3s^2\hat{\underline{P}}\xi^2(1-\xi^2)(\Psi_0 - 1) - \hat{\underline{P}}\xi^2\right](1 + A_0 - A_0\hat{\underline{P}}\Psi_0)(1 + \frac{1}{3}A_1) \\ & - A_1 s^2\left\{s^2\left[1 - \hat{\underline{P}}\Psi_0 + \hat{\underline{P}}\xi^2(\Psi_0 - 1)\right]\right. \\ & \left. - \frac{1}{3}\left[1 - A_0 + A_0\hat{\underline{P}}\Psi_0 - A_0\hat{\underline{P}}\xi^2(\Psi_0 - 1)\right]\right\}\hat{\underline{P}}\xi^2(\Psi_0 - 1) \end{aligned} \tag{38}$$

where $\xi = E/\varepsilon$, the integral operator $\hat{\underline{P}}$ and the function Ψ_0 are given in Section 36. Although in the general case the velocity of the zero sound $v_0 = sv$ (v is the Fermi velocity) can be derived in numerical way only, this disadvantage can be omitted by assuming that $s > 1$ (out of the Landau damping).

As far as the superfluid ^{3}He-B is concerned the assumption is certainly proper for sufficiently low temperatures (cf. [22, 27, 91, 145, 146] and Section 30). It allows us to expand the function Ψ_0 in uniformly convergent series (Section 36).

Hence, neglecting all terms proportional to s in the negative powers, after some calculation, we get

$$s^2 = \frac{1}{3}(1 + A_0)(1 + \frac{1}{3} A_1) + \frac{1}{9} A_0 A_1 Y_2^2 + \frac{1}{5} A_1 Y_2$$
$$- \frac{1}{9} \frac{A_1 Y_2}{1-Y_2} - \frac{1}{3} \frac{Y_2 - Y_4}{1-Y_2} . \tag{39}$$

We remind that according to the assumed conditions the above result remains valid for sufficiently low temperatures when the functions Y_i ($i = 2, 4$) are small. In the limit $T = 0$, Eq. (39) reduces to the well-known formula obtained in [22, 90, 91, 94, 105, 145-147, 150].

2^o The quasi-homogeneous limit, i.e., $kv \ll \omega \sim \Delta$

A similar problem was considered in detail in [27, 90, 91, 147, 150]. Hence, we present only the most essential results. Since the term Z_3^2 is still small and it must be omitted and $Z_1 \sim \omega^2$, then Eq. (38) is fulfilled if $Z_2 = 0$. Hence the dispersion law of the collective excitation with a gap is given by the relation

$$\omega^2 = \frac{12}{5} \Delta^2 + \frac{7}{15} k^2 v^2 (1 + \frac{8}{105} A_1 + \frac{1}{7} h) \tag{40}$$

where we put $\omega = \sqrt{12/5}\Delta$ in all expressions on the RHS of Eq. (40) in order not to exceed the assumed accuracy. The obtained result describes the spectrum of the collective excitations to the state $J = 2$, $M = 0$, therefore it is independent of the zero Landau parameter. Moreover, the last factor in Eq. (40) is positive for all permitted values of the Landau parameter A_1.

Let us consider now the dispersion law of the collective excitations if all the quantities ω, kv and Δ are of the same order. Such a problem seems to be of a particular interest since the frequency of the collective excitations should increase if the wave vector tends to infinity. Then we can expect that the curve of dispersion will have a local minimum for sufficiently large value of the wave vector k_0 when the term Z_3^2 cannot be neglected any more. In order to specify the adequate problem in the analytic manner we differentiate Eq. (37) with respect to kv (ω is a function of kv) and put $d\omega/dkv = 0$. Then we obtain the relation

$$\frac{\partial}{\partial kv}(Z_1 Z_2) = 2Z_3 \frac{\partial Z_3}{\partial kv}. \tag{41}$$

A simultaneous realization of Eqs. (37) and (41) will allow us to derive the position of the local minimum, i.e., the positions of ω_0 and k_0 on the curve of dispersion. Due to the existence of the minimum in the collective excitation spectrum the suitable collective excitations are preferred. As an analogy to the effects interpreted in superfluid ^{4}He we can identify the distinguished excitation as rotons where ω_0 is the gap of the roton excitations. The both characteristic quantities k_0 and ω_0 depend on the Landau parameters and it is probable that the appearance of the local minimum is possible only for the particularly chosen values of the Landau parameters. Such additional restrictions imposed on the Landau parameters will create the criterion of the roton stability. However, the very complicated forms of Eqs. (37) and (41) cause that a more precise analysis can be performed by means of the numerical computations only.

28. Transverse collective excitations [53]

The formalism developed in the previous section can be easily repeated in order to compute the autocorrelation function of the transverse current. Because of the existence of only one type of the collective excitations ($J = 2$, $M = \pm 1$) the whole problem is considerably simplified and the suitable basic equations become independent of the Landau parameter A_0 [22, 26]. Since the question of the modification of the energy gap by the dipole-dipole interaction is considered independently, we put $g = 0$. Then the transverse current-transverse current autocorrelation function is of the form

$$S^{\perp\perp} = \frac{\nu_0 \left(\frac{p_0}{m}\right)^2}{W_\perp}\left(V_\perp - \frac{X_\perp}{Z_\perp}\right) \tag{1}$$

where

$$V_\perp = \frac{1}{2}\,\Gamma_2(1 - s^2) - \frac{1}{6}(F_2 - 1), \tag{2}$$

$$W_\perp = 1 - A_1 V_\perp, \tag{3}$$

$$X_\perp = 2\left(\frac{kv}{2\Delta}\right)^2\left(\frac{1}{3}F_2 - \frac{1}{5}F_4\right)^2\left[F_0 + \frac{1}{3}F_2 - \left(\frac{\omega}{2\Delta}\right)^2\left(F_0 + \frac{1}{3}F_2\right) + \frac{1}{3}\left(\frac{kv}{2\Delta}\right)^2\left(F_2 + \frac{3}{5}F_4\right)\right], \tag{4}$$

$$Z_{\perp} = \left\{\left[\left(\frac{\omega}{2\Delta}\right)^2 (F_0 + \tfrac{1}{3}F_2) - \tfrac{1}{3}\left(\frac{kv}{2\Delta}\right)^2 (F_2 + \tfrac{3}{5}F_4)\right.\right.$$

$$\left.\left. - (F_0 - F_2 + \tfrac{4}{5}F_4)\right] W_{\perp} - 2A_1\left(\frac{kv}{2\Delta}\right)^2 (\tfrac{1}{3}F_2 - \tfrac{1}{5}F_4)^2\right\}$$

$$\times \left[\left(\frac{\omega}{2\Delta}\right)^2 (F_0 + \tfrac{1}{3}F_2) - \tfrac{1}{3}\left(\frac{kv}{2\Delta}\right)^2 (F_2 + \tfrac{3}{5}F_4) - (F_0 + \tfrac{1}{3}F_2)\right]$$

$$- \left[\left(\frac{\omega}{2\Delta}\right)^2 (F_0 - F_2) - \frac{kv}{2\Delta}^2 (\tfrac{1}{3}F_2 - \tfrac{3}{5}F_4) - (F_0 - F_2)\right]^2 W_{\perp} \,. \qquad (5)$$

Although the accurate form of the collective excitation spectrum can be, of course, derived by numerical computations only, nevertheless, some estimations can be performed in the quasi-homogeneous limit. According to Eqs. (1) and (5) the denominator of the function $S^{\perp\perp}$ can be written down in the form

$$Z_{\perp} = Z_4 Z_5 - Z_6^2 W_{\perp} \qquad (6)$$

where the factor Z_6 is proportional to $(kv)^2$. While examining Eq. (4) we state that the numerator $X_{\perp}$ is proportional to Z_5. Hence the collective excitations defined by equation $Z_4 = 0$ are the only type of excitations available in the quasi-homogeneous limit. Their dispersion law expresses itself in the form

$$\omega^2 = \frac{12}{5}\Delta^2 + \frac{2}{5}k^2v^2\left[1 + \frac{1}{21}h + \frac{1}{15}\,\frac{A_1F_h}{1 + \frac{1}{6}A_1(1 - F_h)}\right] \,. \qquad (7)$$

Let us consider also the poles of the autocorrelation function $S^{\perp\perp}$ connected with the normal component. Putting $T = T_c(\Delta = 0)$ the system approaches the normal state and the function $S^{\perp\perp}$ reduces to the form

$$S^{\perp\perp} = \nu_0\left(\frac{p_0}{m}\right)^2 \frac{V_{\perp}}{W_{\perp}} \qquad (8)$$

where now

$$V_{\perp} = -\tfrac{1}{3} + \tfrac{1}{2}(\Psi_0 - \Psi_0 s^2 + s^2),$$

$$W_{\perp} = 1 - A_1 V_{\perp} \quad \text{and} \quad \eta = s.$$

Hence, after solving the equation $W_{\perp} = 0$, we may find the dispersion law of the transverse sound in the normal Fermi liquid. However, for all temperatures below the phase transition the function $S^{\perp\perp}$ attains the finite value if the denominator $W_\perp$ tends to zero. This limit value is a function of the Landau parameter A_1 and can be expressed in the form

$$S^{\perp\perp}_{W_\perp = 0} = \nu_0 \left(\frac{p_0}{m}\right)^2 \left(\frac{Z_5 Z_7}{A_1} - Z_6^2 - X_\perp\right) \tag{9}$$

where

$$Z_7 = \left(\frac{\omega}{2\Delta}\right)^2 \left(F_0 + \frac{1}{3} F_2\right) - \frac{1}{3} \left(\frac{kv}{2\omega}\right)^2 \left(F_2 + \frac{3}{5} F_4\right)$$

$$- \left(F_0 - F_2 + \frac{4}{5} F_4\right)$$

and

$$V_\perp \Big|_{W_\perp = 0} = \frac{1}{A_1} \, . \tag{10}$$

In this way we have proved that no extra collective excitations connected with a normal component of superfluid ^{3}He can appear in the phase B. Therefore the transverse sound appears only in the normal phase and it is suppressed rapidly after the phase transition.

29. Gaps of spinless collective excitations [44]

Since the parameter A_2 is strongly connected with the gaps in spectra of the density and transverse current collective excitations, this problem will be discussed additionally taking account of the dipole -dipole interaction. The values of the gaps can be derived from the following equations:

$$\omega^2\left[1 + \frac{3}{25} A_2 F(\omega)\right] = \frac{12}{5} \Delta^2 \left[1 + \frac{1}{5} A_2 F(\omega)\right] \left[1 + 9g/4F(\omega)\right] , \tag{1}$$

$$\omega^2\left[1 + \frac{3}{25} A_2 F(\omega)\right] = \frac{12}{5} \Delta^2 \left[1 + \frac{1}{5} A_2 F(\omega)\right] \left[1 - 3g/2F(\omega)\right] . \tag{2}$$

Let the solutions of these equations be denoted by $\omega_\parallel$ and $\omega_\perp$, and by ω_0 if $g = 0$.

The accuracy of our calculations allows us to assume that the values of $\omega_\parallel$ and $\omega_\perp$ if compared with that of ω_0 are modified respectively by quantities λ and μ (of order of g) and can be written in the form:

$$\omega_\parallel = \omega_0 (1 + \lambda) \quad \text{and} \quad \omega_\perp = \omega_0 (1 + \mu) . \tag{3}$$

Substituting the first formula of (3) into (1) and the latter into (2), confining ourselves to the terms up to the order of g and eliminating A_2 from the first order terms we obtain

$$\lambda = 9g/8E_{3/5}(\omega_0), \qquad \mu = -3g/4E_{3/5}(\omega_0), \tag{4}$$

hence,

$$\frac{\omega_\parallel - \omega_\perp}{\omega_0} = 15g/8E_{3/5}(\omega_0), \tag{5}$$

and

$$\omega_0 = \frac{1}{5}(2\omega_\parallel + 3\omega_\perp). \tag{6}$$

Since the function $E_s(\omega)$ (Section 35) is positive then from Eqs. (3) -(6) we obtain the following inequalities:

$$\omega_\perp < \omega_0 < \omega_\parallel . \tag{7}$$

The function $E_s(\omega)$ attains its minimum value for $\omega_0 = 2\sqrt{s}\,\Delta$, i.e., for $\Delta_2 \to 0$.

According to equation (5), the difference $\omega_\parallel - \omega_\perp$ is proportional to the function $\omega_0/E_{3/5}(\omega_0)$ which has a sharp maximum in the vicinity of $\omega_0 = 2\sqrt{3/5}\,\Delta$. For this reason the dipole interaction is the most important if Δ_2 is close to zero.

The developed formalism allows us to state that the density autocorrelation function vanishes in the homogeneous limit ($k = 0$) even without the extra assumption about g^n. We also state that the inclusion of the dipole interaction does not change the zero sound spectrum.

© The strong coupling effects which are connected with the mutual interaction of the two-particle states lead to the effects analogous to that of the dipole-dipole interaction in the collective excitation spectra with the gap [136]. On the other hand, we can generalize the total interaction in the particle-particle channel (19.2) assuming it in the form

$$V_{ij} = -3f_1\Big[\tfrac{1}{3}(1+\alpha_1)\delta_{ik}\delta_{jn} + \tfrac{1}{2}(1+\alpha_2)(\delta_{ij}\delta_{kn} - \delta_{in}\delta_{jk})$$
$$+ \tfrac{1}{2}(1+\alpha_3)(\delta_{ij}\delta_{kn} + \delta_{in}\delta_{jk} - \tfrac{2}{3}\delta_{ik}\delta_{jn})\Big]\,\hat{p}_k\hat{p}'_n$$

where $|\alpha_i| \ll 1$ which can be positive as well as negative. Their values should be fitted independently, e.g, from experiments. Let us note that this interaction is the rotational invariant in the total spin-momentum space (cf. Section 40)●

IX. Remarks on the Obtained Results

30. The values of the Landau parameter in ^{3}He

In order to illustrate the research of the obtained results we insert some information on the Landau parameters. All Landau parameters have to satisfy the Pomeranchuk inequalities [117]

$$- 1 < \frac{F^s}{2l + 1} < \infty \qquad \text{and} \qquad - 1 < \frac{F^a}{2l + 1} < \infty \tag{1}$$

and the Leggett condition [83]

$$F_1^a < F_1^s. \tag{2}$$

Moreover, they have to tend to zero if l tends to infinity. The Landau parameters depend on the pressure but do not depend on temperature,even after the phase transition. According to the experimental data [8, 9, 21, 59, 109, 110, 125, 139] their values for $l = 0$ and $l = 1$ range within the following intervals

$$\begin{aligned} 10.07 &< F_0^s < 94.13, \\ 6.04 &< F_1^s < 14.35, \\ - 0.67 &> F_0^a > -0.76, \\ - 0.21 &> F_1^a > -1.2 \end{aligned} \tag{3}$$

which are specified according to the growing pressure. The respective values of F_2^a estimated for high pressure in [118] and [60] are

$$\begin{aligned} F_2^a &= 0.3 \pm 0.5, \\ F_2^a &= -0.32 \pm 0.25. \end{aligned} \tag{4}$$

Thus the absolute value of F_2^a is rather small although its sign is uncertain. As to F_2^s we can only state that it is unlikely that F_2^s exceeds 5 [20].

31. Quntative and qualitative effects generated by the quasiparticle interactions

That is well-known that the Fermi liquid interaction modifies quantitatively the characteristic quantities which describe the Fermi liquid systems [22, 24, 26-28, 39, 40, 42, 64]. On the other hand, the Landau parameters must satisfy solely the Pomeranchuk inequalities [117] and the Leggett condition [83] and no additional restrictions arising from the stability conditions also for superconducting and superfluid systems have been found [25, 27, 28, 45, 46, 48, 141-143]. The only exceptions are the ferromagnetic Fermi liquids [23, 69, 70], but there the form of their Fermi-liquid interaction is defined in another way.

In the presented considerations we have shown that besides the obvious quantitative results there exist some qualitative effects in superconducting and superfluid systems, which are due to the Fermi liquid interaction. These effects can be classified ammong three groups:

1° The types of superconductivity and the behaviour of superfluid ^{3}He and mixtures ^{3}He-^{4}He in a strong magnetic field are determined by the Landau parameters. Hence the Landau parameters determine the order of a phase transition from the superconducting (superfluid) to the normal state (cf. [18, 19]). If the Fermi liquid interaction is omitted, the first order phase transition is realized only in charged and neutral BCS systems.

2° The Landau parameters determine the existence of additional branches of the collective excitations when the pairing channel contains additional harmonics. Then for sufficiently large Landau parameters no extra collective excitations can appear.

3° Considering ^{3}He-B we state that the circular dichroism, collective excitations of the helical type, and the transverse response of the system on the threshold of the quasiparticle destruction are not available if the Fermi liquid interaction is excluded. Hence, the Landau parameters (F_0^a, F_2^a) decide on these effects and their orientations. The fact that the spectrum of the spinless collective excitations possesses the roton minimum dependent on the Landau parameters F_1^s is plausible.

Let us note that the rest of the quasiparticle interactions, i.e., the dipole-dipole and the pure-pairing interactions are the cause of qualitative effects such as the splitting of the collective excitation gaps, the appearance of the gap in the longitudinal spin wave spectrum, the non-zero response of a system at homogeneous magnetic fields, and the suppression of the transverse sound in the superfluid phase.

High precision of the performed calculations allows us to eliminate a few types of collective excitations which are generally discussed.

Part Four

MATHEMATICAL METHODS AND APPENDICES

X. New mathematical methods

Theoretical investigations of the superconducting and superfluid systems required the development of new computing techniques which would permit us to derive some composed integral expressions. The principal advantage of the developed methods was connected with the qualification and utilization of symmetry properties of expressions being considered. Due to the high degree of complexity of the theory there appear some non-standard integrals connected with temperature series expansion and some new functions the properties of which should be defined independently. All these and other purely mathematical elements of the theory are collected and elaborated in this chapter.

32. Integral techniques of the Green function formalism [51]

In the presented considerations we use the temperature Green function formalism. In such a case the summation over possible quantum-mechanics states (εn, $\underline{p}$) concerns the Matsubara Green functions. In order to derive suitable sums we always apply one of the two following methods which can be reported as follows:

1^{o} The Eliashberg method [33]

$$T\sum_{\varepsilon_n}\sum_{\underline{p}}\mathcal{G}(\tilde{p}_+)\,\mathcal{G}'(\tilde{p}_-)\longrightarrow T\sum_{\varepsilon_n}\int\frac{d^3\underline{p}}{(2\pi)^3}\,\mathcal{G}(\tilde{p}_+)\,\mathcal{G}'(\tilde{p}_-)$$

$$\longrightarrow \frac{1}{2}T\sum_{\varepsilon_n}\int\frac{d\Omega}{4\pi}\int d\xi\,\nu(\xi)\,\mathcal{G}(\tilde{p}_+)\,\mathcal{G}'(\tilde{p}_-). \tag{1}$$

Using a residue theorem we can replace the summation over $i\varepsilon_n$ by integration over ε and transform expression (1) to the form

$$\longrightarrow \frac{1}{8\pi i}\cdot\int\frac{d\Omega}{4\pi}\int d\xi\,\nu(\xi)\int d\varepsilon\sum_{l=1}^{3} g_l\lambda_l \tag{2}$$

where the expressions g_l and λ_l have the forms:

$$\mathcal{E}_1 = \underline{G}_R(\tilde{\tilde{p}}_+)\underline{G}'_R(\tilde{\tilde{p}}_-), \qquad \mathcal{E}_2 = \underline{G}_R(\tilde{\tilde{p}}_+)\underline{G}'_A(\tilde{\tilde{p}}_-),$$

$$\mathcal{E}_3 = \underline{G}_A(\tilde{\tilde{p}}_+)\underline{G}'_A(\tilde{\tilde{p}}_-), \qquad \lambda_1 = \text{th}\,\frac{\varepsilon - \frac{\omega}{2}}{2T},$$

$$\lambda_2 = \text{th}\,\frac{\varepsilon + \frac{\omega}{2}}{2T} - \text{th}\,\frac{\varepsilon - \frac{\omega}{2}}{2T}, \qquad \lambda_3 = -\,\text{th}\,\frac{\varepsilon + \frac{\omega}{2}}{2T}. \tag{3}$$

2° The residuum method [35]

$$\lim_{\delta \to 0^+} T \sum_{\varepsilon_n} \sum_{\underline{p}} e^{\pm i\varepsilon_n\delta}\mathcal{G}(\tilde{p})$$

$$\longrightarrow \lim_{\delta \to 0^+} \frac{1}{2}\, T \int \frac{d\Omega}{4\pi} \int d\,\xi\nu\,(\xi) \sum_{\varepsilon_n} e^{\pm i\varepsilon_n\delta}\,\mathcal{G}(\tilde{p})$$

$$\longrightarrow \frac{1}{2}\int \frac{d\Omega}{4\pi}\int d\,\xi\nu(\xi)\sum_{\mathcal{G}} \text{Res}\left[\mathcal{G}(\tilde{\tilde{p}})\left(1 \mp \text{th}\,\frac{\varepsilon}{2T}\right)\right] \tag{4}$$

where all the residua are taken for the poles of the Matsubara Green function $\mathcal{G}$ only, and

$$\tilde{p}_{\pm} = (\underline{p} \pm \tfrac{1}{2}\,\underline{k},\ \ i\varepsilon_n \pm \tfrac{1}{2}\,i\omega_m, \underline{h}),$$

$$\tilde{\tilde{p}}_{\pm} = (\underline{p} \pm \tfrac{1}{2}\,\underline{k},\ \ \varepsilon \pm \tfrac{1}{2}\,\omega, \underline{h}),$$

$$\varepsilon_n = (2n+1)\pi T, \qquad \omega_m = 2m\pi T.$$

All the Green functions are the matrices and $\underline{G}_R$ and $\underline{G}_A$ are the retarded and advanced Green functions, respectively. Moreover, we assume that each of the Green functions $\underline{G}_n$, $\underline{G}_q$, $\underline{G}_s$, $\underline{F}$ (Section 3) can be substituted for the functions $\mathcal{G}$ or $\mathcal{G}'$.

33. Averaging and recurrent formulae [40,42,43,53]

In order to simplify the notation the average over spherical angles are denoted by

$$\langle \ldots \rangle = \frac{1}{4\pi}\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta\,\sin\theta\,\ldots \tag{1}$$

and in the case when only one direction is distinguished in the system, the integral over spherical angles can be reduced to the form

$$\langle \ldots \rangle = \frac{1}{2} \int_{-1}^{1} dx \ldots \tag{2}$$

where $\cos\theta$ is substituted by x.

The integration of all the appearing expressions constitutes an essential difficulty of calculations. However, applying the symmetry properties of the integrated expressions we obtain the following very helpful formulae:

$$\langle \hat{p}_{\alpha_1} \ldots \hat{p}_{\alpha_{2n}} \rangle = \frac{1}{(2n+1)!!} \sum^{(2n-1)!!} \delta_{\alpha_{k_1} \alpha_{k_2}} \ldots \delta_{\alpha_{k_{2n-1}} \alpha_{k_{2n}}} \tag{3}$$

where the product of deltas contains n factors and summation is extended over all possible sequences of deltas ([40, 43]).

1° The Legendre polynomials are of the form

$$P_l(x) = \frac{1}{2^l\, l!} \frac{d^l}{dx^l} \left[(x^2 - 1)^l\right] \tag{4}$$

and fulfil the relations

$$(l + 1)P_{l+1}(w) = (2l + 1)wP_l(w) - l\,P_{l-1}(w), \tag{5}$$

and

$$\langle P_i(\hat{p}\hat{p}') \, P_j(\hat{p}'\hat{k}) \rangle = \frac{\delta_{ij}}{2j + 1} P_j(\hat{p}\hat{k}). \tag{6}$$

If the function $c \equiv c(\hat{p}\hat{p}')$ can be presented in the form

$$c = \sum_{l=0}^{\infty} (2l + 1)c_l \, P_l(\hat{p}\hat{p}'), \tag{7}$$

then we have [42]

$$\langle c \rangle = c_0,$$

$$\langle c\, \hat{p}'_i \rangle = c_1 \hat{p}_i ,$$

$$\langle c\, \hat{p}'_i\hat{p}'_j \rangle = \frac{1}{3} (c_0 - c_2)\, \delta_{ij} + c_2\, \hat{p}_i\hat{p}_j,$$

$$\langle c\, \hat{p}'_i\hat{p}'_j\hat{p}'_k \rangle = \frac{1}{5} (c_1 - c_3)(\hat{p}_i\delta_{jk} + \hat{p}_j\delta_{ik} + \hat{p}_k\delta_{ij}) + c_3\, \hat{p}_i\hat{p}_j\hat{p}_k,$$

$$\langle c\, \hat{p}'_i \hat{p}'_j \hat{p}'_k \hat{p}'_n \rangle = (\frac{1}{15} c_0 - \frac{2}{21} c_2 + \frac{1}{35} c_4)(\delta_{ij}\delta_{kn} + \delta_{ik}\delta_{jn}$$

$$+ \delta_{in}\delta_{jk}) + \frac{1}{7}(c_2 - c_4)(\hat{p}_i\hat{p}_j\delta_{kn} + \hat{p}_i\hat{p}_k\delta_{jn}$$

$$+ \hat{p}_i\hat{p}_n\delta_{jk} + \hat{p}_j\hat{p}_k\delta_{in} + \hat{p}_j\hat{p}_n\delta_{ik} + \hat{p}_k\hat{p}_n\delta_{ij})$$

$$+ c_4\, \hat{p}_i\hat{p}_j\hat{p}_k\hat{p}_n. \tag{8}$$

2° The function expressing the dipole-dipole interaction is of the form [43]

$$D_{ij}\,(\hat{p},\hat{p}') = \delta_{ij} - 3(\hat{p}_i - \hat{p}'_i)(\hat{p}_j - \hat{p}'_j)\,/|\hat{p} - \hat{p}'|^2 \tag{9}$$

and it fulfils the relations

$$D_{ij}\,(\hat{p},\hat{p}') = D_{ji}\,(\hat{p},\hat{p}') \quad \text{and} \quad D_{ii}\,(\hat{p},\hat{p}') = 0. \tag{10}$$

By series expansion we get

$$D_{ij}\,(\hat{p},\hat{p}') = \sum_{n=0} [(2n+1)]!!^{\,2} << D_{ij}\,(\hat{p},\hat{p}')\,\hat{p}'_{k_1}\ldots\hat{p}'_{k_n} > \hat{p}_{l_1}\ldots\hat{p}_{l_n} >$$

$$\times\, \hat{p}'_{k_1} \ldots \hat{p}'_{k_n}\, \hat{p}_{l_1} \ldots \hat{p}_{l_n} \tag{11}$$

and employing the relations

$$\langle D_{ij}\,(\hat{p},\hat{p}')\rangle = \frac{1}{4}\,\delta_{ij} - \frac{3}{4}\,\hat{p}_i\hat{p}_j, \tag{12}$$

$$\langle D_{ij}\,(\hat{p},\hat{p}')\,p'_k\rangle = -\frac{1}{4}\,\delta_{ij}\hat{p}_k + \frac{1}{2}\,(\delta_{jk}\hat{p}_i + \delta_{ik}\hat{p}_j)$$

$$-\frac{1}{4}\,\hat{p}_i\hat{p}_j\hat{p}_k \tag{13}$$

we find two first fators of the series expansion (11) in the forms

$$\langle\langle D_{ij}\,(\hat{p},\hat{p}')\rangle\rangle = 0, \tag{14}$$

$$\langle\langle D_{ij}\,(\hat{p},\hat{p}')\,\hat{p}'_k\rangle\hat{p}_n\rangle = -\frac{1}{10}\,\delta_{ij}\delta_{kn} + \frac{3}{20}\,(\delta_{jk}\delta_{in} + \delta_{ik}\delta_{jn}). \tag{15}$$

The other tensor terms of the series expansion can be found in a similar manner after applying symmetry rules.

It is worth noticing that the contraction $i = j$ of each series expansion tensor must be always equal to zero, since $D_{ii}\ (\hat{p},\hat{p}') = 0$. This statement implies, e.g., Eq. (14).

3° If $\Lambda = \Lambda(\hat{k}\hat{p})$ is an arbitrary function of the scalar product $\hat{k}\hat{p}$, the suitable averages have the forms [53]

$$\langle \Lambda \rangle = \lambda_0,$$

$$\langle \Lambda \hat{p}_i \rangle = \lambda_1 \hat{k}_i,$$

$$\langle \Lambda \hat{p}_i \hat{p}_j \rangle = \lambda_2 \delta_{ij} + \lambda_3 \hat{k}_i \hat{k}_j,$$

$$\langle \Lambda \hat{p}_i \hat{p}_j \hat{p}_k \rangle = \lambda_4 (\delta_{ij}\hat{k}_k + \delta_{ik}\hat{k}_j + \delta_{jk}\hat{k}_i) + \lambda_5 \hat{k}_i \hat{k}_j \hat{k}_j,$$

$$\begin{aligned} \langle \Lambda \hat{p}_i \hat{p}_j \hat{p}_k \hat{p}_n \rangle = {} & \lambda_6 (\delta_{ij}\delta_{kn} + \delta_{ik}\delta_{jn} + \delta_{in}\delta_{jk}) \\ & + \lambda_7 (\delta_{ij}\hat{k}_k\hat{k}_n + \delta_{ik}\hat{k}_j\hat{k}_n + \delta_{in}\hat{k}_j\hat{k}_k + \delta_{jk}\hat{k}_i\hat{k}_n \\ & + \delta_{jn}\hat{k}_i\hat{k}_k + \delta_{kn}\hat{k}_i\hat{k}_j) + \lambda_8 \hat{k}_i\hat{k}_j\hat{k}_k\hat{k}_n. \end{aligned} \tag{16}$$

The terms $\lambda_0, \ldots, \lambda_8$ are not independent linearly, however, by introducing the following denotations

$$\lambda_p = 3\langle \Lambda (\hat{k}\hat{p})^2 \rangle,$$

$$\lambda_q = 5\langle \Lambda (kp)^4 \rangle,$$

$$\lambda_r = \langle \Lambda (kp)^3 \rangle \tag{17}$$

they can be expressed in the forms

$$\begin{aligned} \lambda_2 &= \frac{1}{2}\lambda_0 - \frac{1}{6}\lambda_p, \\ \lambda_3 &= -\frac{1}{2}\lambda_0 + \frac{1}{2}\lambda_p, \\ \lambda_4 &= \frac{1}{2}\lambda_1 - \frac{1}{2}\lambda_r, \\ \lambda_5 &= -\frac{2}{3}\lambda_1 + \frac{5}{2}\lambda_r, \\ \lambda_6 &= \frac{1}{8}\lambda_0 - \frac{1}{12}\lambda_p + \frac{1}{40}\lambda_q, \\ \lambda_7 &= -\frac{1}{8}\lambda_0 + \frac{1}{4}\lambda_p - \frac{1}{8}\lambda_q, \\ \lambda_8 &= \frac{3}{8}\lambda_0 - \frac{3}{4}\lambda_p + \frac{7}{8}\lambda_q \end{aligned} \tag{18}$$

where

$$\lambda_1 = \langle \Lambda(\hat{k}\hat{p}) \rangle \tag{19}$$

and then the functional factors λ_0, λ_1, λ_p, λ_q, λ_r are linearly independent.

4^0 If Q is an arbitrary scalar or vector function of vectorial variable $\underline{q}$ then we can write

$$\left\langle \left(|Q|^n - x|Q|^{n-2j} \langle |Q|^2 \rangle_{\underline{q}}^{\,j} \right)^2 \right\rangle_{\underline{q}} \geqslant 0 \tag{20}$$

where $\langle\ldots\rangle_q$ denotes averaging over $\underline{q}$ vector space. Equation (20) implies the relations

$$\beta^2_{n-j} \ll \beta_n \ \beta_{n-2j} \tag{21}$$

where

$$0 \leqslant j \leqslant \frac{n}{2}$$

and

$$\beta_n = \frac{\langle |Q|^{2n} \rangle_{\underline{q}}}{\langle |Q|^2 \rangle^n_{\underline{q}}}, \qquad \beta_0 = \beta_1 = 1.$$

Employing (21) we state that

$$\beta_n \geqslant \frac{\beta^{k+1}_{n-k}}{\beta^{k}_{n-k-1}} \tag{22}$$

where

$$0 \leqslant k \leqslant n-1$$

thus for $k = n-2$ we get

$$\beta_n \geqslant \beta_2^{n-1}. \tag{23}$$

Note that the equalities are attained only if the function $|Q(\underline{q})|$ is constant.

34. The Maki and Ebisawa function [43, 47]

The function F, so-called the Maki and Ebisawa function, is defined as follows:

$$F = \frac{4\pi T}{\nu(0)} \sum_{\varepsilon_n} \sum_{|\underline{p}|} \mathcal{F}(p_+)\,\mathcal{F}(p_-) \tag{1}$$

where $\mathcal{F}(p\underline{+})$ is the anomalous Matsubara-Green function taken in the limit H = 0. In order to calculate the function F we apply the method formulated by Eliashberg [33]. Then after some algebra we obtain

$$F = \Delta^2 \int_{-\infty}^{+\infty} d\xi \, \frac{\operatorname{tgh}\frac{E_-}{2T}}{E_-} \cdot \frac{\omega^2 - 2\xi \underline{kv}}{\left[\omega^2 - (\underline{kv})^2\right](4E^2 - \omega^2) + 4\Delta^2(\underline{kv})^2}$$

$$+ \Delta^2 \int_{-\infty}^{+\infty} d\xi \, \frac{\operatorname{tgh}\frac{E_+}{2T}}{E_+} \, \frac{\omega^2 + 2\xi \underline{kv}}{\left[\omega^2 - (\underline{kv})^2\right](4E^2 - \omega^2) + 4\Delta^2(\underline{kv})^2} \tag{2}$$

where

$$E_{\pm} = \left[\left(\xi \pm \frac{\underline{kv}}{2}\right)^2 + |\Delta|^2\right]^{1/2} . \tag{3}$$

The obtained integrals (2) exist, in general, (i.e., for arbitrary values of ω and kv) in the sense of principal values only, therefore the change of variable is forbidden. Let us consider now some other representations of the function F which turn out to be convenient for analytical evaluations and numerical calculations. After expanding the expressions

$$\frac{\operatorname{tgh}\frac{E_+}{2T}}{E_+} \quad \text{and} \quad \frac{\operatorname{tgh}\frac{E_-}{2T}}{E_-} \tag{4}$$

in the Taylor series we can rewrite Eq. (2) in the form

$$F = 4\Delta^2 \sum_{n=0}^{\infty} \frac{1}{(2n)!} \left(\frac{\underline{kv}}{2}\right)^{2n} \int_0^{\infty} \frac{d\xi}{\left[\omega^2 - (\underline{kv})^2\right](4E^2 - \omega^2) + 4\Delta^2(\underline{kv})^2}$$

$$\times \left[\omega^2 \frac{d^{2n}}{d\xi^{2n}} \left(\frac{\operatorname{tgh}\frac{E}{2T}}{E}\right) + (\underline{kv})^2 \frac{\xi}{2n+1} \frac{d^{2n+1}}{d\xi^{2n+1}} \left(\frac{\operatorname{tgh}\frac{E}{2T}}{E}\right)\right] . \tag{5}$$

Putting $\omega = \underline{kv}$ in Eq. (5) we can easily obtain the following property of the function F

$$F = 1 \quad \text{if} \quad \omega = \underline{kv} . \tag{6}$$

This property is very important because of the term $(F-1)/(\omega - \underline{kv})$ appearing in the linear response theory. Employing Equation (6) we can transform Eq. (33) to the form

$$F = 1 - \left[\omega^2 - (\underline{k}\underline{v})^2\right] \sum_{n=0}^{\infty} \frac{1}{(2n)!} \left(\frac{\underline{k}\underline{v}}{2}\right)^{2n} \int_0^{\infty} \frac{d\xi}{\omega^2(4E^2-\omega^2)-(\underline{k}\underline{v})^2(4\xi^2-\omega^2)}$$

$$\times \left[(4\xi^2-\omega^2)\frac{d^{2n}}{d\xi^{2n}}\left(\frac{\mathrm{tgh}\,\frac{E}{2T}}{E}\right) + \frac{\xi}{2n+1}(4E^2-\omega^2)\frac{d^{2n+1}}{d\xi^{2n+1}}\left(\frac{\mathrm{tgh}\,\frac{E}{2T}}{E}\right)\right]. \tag{7}$$

The obtained Eqs. (5)-(7) allow us to determine the forms of the function F in some specific limits. In the homogeneous limit ($k = 0$) Eq. 5 reduces to the form

$$F_h = \int_0^{\infty} d\xi \, \frac{\mathrm{tgh}\,\frac{E}{2T}}{E} \cdot \frac{4\Delta^2}{4E^2 - \omega^2} \,. \tag{8}$$

The function F_h, which becomes a function of variables ω and T, is an increasing function of the argument ω. Its value increases from

$$F_h(0) = 1 - X, \tag{9}$$

where

$$X \equiv Y_2 \quad \text{and} \quad Y_{2i} = \int_0^{\infty} \frac{d\varepsilon}{2T}\, \mathrm{ch}^{-2}\left(\frac{E}{2T}\right) \cdot \left(\frac{\varepsilon}{E}\right)^{2i}, \tag{10}$$

up to infinity if ω tends to 2Δ.

The function $F_h(\omega)$ decreases to zero for the fixed ω if temperature tends to T_c. Consequently, if temperature fluctuates from zero to T_c, the functions Y_{2i} increase from zero to unity and fulfil the relation

$$Y_{2i} < Y_{2j} \quad \text{if} \quad i > j \quad \text{and} \quad 0 < T < T_c. \tag{11}$$

In the static limit ($\omega = 0$) the function F reduces to the form

$$F_s = 1 - Y_0 - \Delta^2 \sum_{n=1}^{\infty} \frac{(kv/2)^{2n}}{(2n+1)!} \int_0^{\infty} \frac{d\xi}{\xi} \frac{d^{2n+1}}{d\xi^{2n+1}} \left(\frac{\mathrm{th}\,\frac{E}{2T}}{E}\right) \tag{12}$$

where Y_0 is the Yosida function.

The function F_s is a decreasing function of $\underline{kv}$ and its value tends to zero if $\underline{kv}$ tends to infinity (the Pippard limit). It is also a decreasing function of temperature, so it never exceeds the unity.

The function F in Eqs. (5) and (7) is given in form of a series. This series is rapidly convergent and in some problems we can limit ourselves to a few of its terms. The most often applied reduction is the so-called acoustic limit (i.e., small ω and small kv limit). In that case all terms but one are neglected and the function F reduces to the form [81]

$$F = 1 - \int_0^\infty \frac{d\xi}{2T}\left(\mathrm{ch}\,\frac{E}{2T}\right)^{-2}\frac{\xi^2}{E^2}\cdot\frac{\omega^2 - (\underline{kv})^2}{\omega^2 - (\underline{kv})^2\xi^2/E^2}\,. \tag{13}$$

Putting ω or $\underline{kv}$ equal to zero, Eq. (13) reduces to the form given by Eqs. (9) or (12), taken, in the same limits. The small $\underline{kv}$ and great ω case is an often used specific limit. In this case the function F becomes an analytic function of $(\underline{kv})^2$ and we can expand it according to powers of $(\underline{kv})^2$, and limit ourselves to a few terms of the series. Then we obtain

$$F = F_{(0)} + \left(\frac{\underline{kv}}{2\Delta}\right)^2 F_{(1)} + \left(\frac{\underline{kv}}{2\Delta}\right)^4 F_{(2)} + \left(\frac{\underline{kv}}{2\Delta}\right)^6 F_{(3)} + \dots \tag{14}$$

where $F_{(0)}$ is given by Eq. (8)

$$F_{(1)} = 16\Delta^4\int_0^\infty d\xi\,\frac{\mathrm{tgh}\,\frac{E}{2T}}{E}\cdot\frac{16E^2(2\xi^2 - \Delta^2) + \omega^2(4\xi^2 + \omega^2)}{\omega^2(4E^2 - \omega^2)^3} \tag{15}$$

and

$$F_{(n)} = \frac{(2\Delta)^{2n}}{n!}\cdot\left.\frac{\partial^n F}{\partial\left[(\underline{kv})^2\right]^n}\right|_{\underline{kv}=0}. \tag{16}$$

In the quasi-homogeneous limit ($\underline{kv} \ll \omega$) the reduced function F can be written down in the form

$$F = F_h\left(1 - \frac{(\underline{kv})^2}{\omega^2}\cdot h\right) \tag{17}$$

where

$$h = -\frac{\omega^2 F_{(1)}}{4\Delta^2 F_h}\,. \tag{18}$$

The function F is a non-analytic function of the variables ω and kv for finite temperatures. Its limits do not commutate if ω and kv tend to zero in various ways. This fact introduces large complications to the existing formalism. However, in the limit $T = 0$ the function F becomes an analytic function and has the form

$$F = \frac{\arcsin\beta}{\beta\sqrt{1-\beta^2}} \tag{19}$$

where

$$\beta = \sqrt{\left(\frac{\omega}{2\Delta}\right)^2 - \left(\frac{\underline{k}v}{2\Delta}\right)^2} . \tag{20}$$

First it was calculated by Vaks et al. [133]. The form of the function F given by Eq. (19) allows us to expand it according to the powers of $(\omega/2\Delta)^2$ and $(\underline{k}v/2\Delta)^2$, and to cut off the expression if the terms become small. All the presented forms of the non-zero temperature function F correspond to some analogous ones obtained from Eq. (19), which was not achieved by Maki and Ebisawa in [93].

The forms of F-function given by Eqs. (5) and (7) allow us to prove the existence of many interesting properties of the investigated function, they, however, are inconvenient in the numerical calculations, since the function F is defined by means of the series. Therefore, we propose to use in numerical computations the following forms of the function F:

$$F = \frac{1}{4}\int_{-\infty}^{+\infty} \frac{dx}{(t^2-u^2)(1+x^2-t^2)+u^2} \times \left\{ t^2\left[\frac{\operatorname{tgh}\alpha X_+}{X_+} + \frac{\operatorname{tgh}\alpha X_-}{X_-}\right] + xu\left[\frac{\operatorname{tgh}\alpha X_+}{X_+} - \frac{\operatorname{tgh}\alpha X_-}{X_-}\right]\right\} , \tag{21}$$

$$F = 1 - \frac{1}{4}(t^2-u^2)\int_{-\infty}^{+\infty} \frac{dx}{t^2(1+x^2-t^2) - u^2(x^2-t^2)} \times \left\{(x^2-t^2)\left[\frac{\operatorname{tgh}\alpha X_+}{X_+} + \frac{\operatorname{tgh}\alpha X_-}{X_-}\right] + \frac{x}{u}(1+x^2-t^2)\left[\frac{\operatorname{tgh}\alpha X_+}{X_+} - \frac{\operatorname{tgh}\alpha X_-}{X_-}\right]\right\} , \tag{22}$$

and

$$F = \int_0^{\infty} \frac{dx}{\sqrt{1+x^2}} \cdot \frac{\operatorname{tgh}\alpha\sqrt{1+x^2}}{1+x^2-t^2} \tag{23}$$

in the homogeneous limit where the quantities

$$t = \frac{\omega}{2\Delta}, \quad u = \frac{kv}{2\Delta}, \quad \alpha = \frac{\Delta}{2T},$$

$$x = \frac{\xi}{\Delta}, \quad X_{\pm} = \sqrt{1 + (x \pm u)^2} \tag{24}$$

form the set of dimensionless variables.

By introducing the above notations the whole temperature dependence in the function F can be reduced to one parameter α. Such a renormalization allows us to use the function F even when we do not know how Δ depends on temperature. Assuming that α is a number chosen from the interval $[0,\infty)$ we can find the value of function F for given t and u.

To give the full image of the function F we performed numerical investigations using Eqs. (21)-(24). The research was carried out for $t = \omega/2\Delta$ and $u = kv/2\Delta$ taken only from the unitary intervals, and for three chosen values of the parameter α, i.e., for α equal 500, 1 or 0.1. The first case ($\alpha = 500$) corresponds to very low temperatures and with great accuracy can be identified with the case $T = 0$. The second case ($\alpha = 1$) is connected with middle-range temperatures, and the last one ($\alpha = 0.1$) defines temperatures very near to the phase transition. The dependence of function F on t and u for fixed α can be illustrated by some particular surfaces shown in the Figures 7-9. These figures allow us to give some extra properties of F-function, namely, with the increasing temperature function F decreases in the large part of the considered square, and it increases only in those parts of space where t is near to u, hence, there appears a hummock on the appropriate surface. The more temperature increases, the steeper, narrower and higher the hummock becomes. The maximum of hummock is constant for fixed α and always lies in that part of space where t is smaller than u, although it comes near to the line $t = u$ if the temperature tends to T_c.

Moreover, from Figs. 7-9 we can also see that in the discussed region of quantities t and u there exist some local minima of the function F. We do not consider them because they have probably no physical meaning.

Let us define now some other characteristic functions being the generalization of the Yosida function. They have the forms

$$Y_{0,2} = \int_0^1 dx \int_0^\infty \frac{d\xi}{2T} \, ch^{-2} \frac{\sqrt{\xi^2 + \frac{3}{2}\Delta^2(1 - x^2)}}{2T}, \tag{25}$$

$$Y_{0,1} = \int_0^1 dx \int_0^\infty \frac{d\xi}{2T}\, ch^{-2}\, \frac{\sqrt{\xi^2 + 3\Delta^2 x^2}}{2T}\,, \tag{26}$$

and

$$Y_{,i} = \int_0^\infty \frac{d\xi}{2T}\, ch^{-(2+i)}\, \frac{\sqrt{\xi^2 + \Delta^2}}{2T} \tag{27}$$

where

$$Y_{,0} \equiv Y_0 .$$

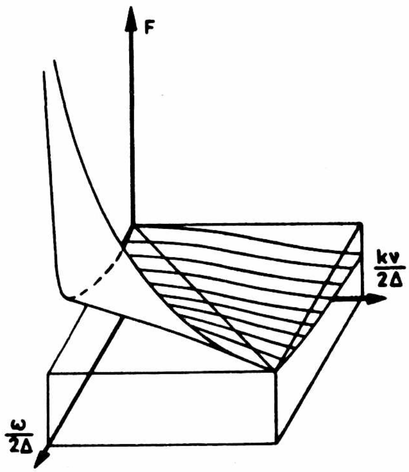

Fig. 7. F-function dependence on $t = \omega/2\Delta$ and $u = kv/2\Delta$ taken from the unitary square. Here $\alpha = 500$ (range of very low temperatures). The hummock is not perceptible yet

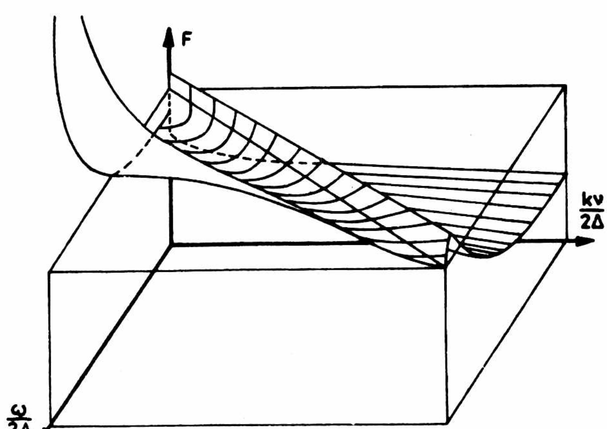

Fig. 8. F-function dependence on t and u. Here, $\alpha = 1$ (range of middle temperatures). The hummock is low and wide

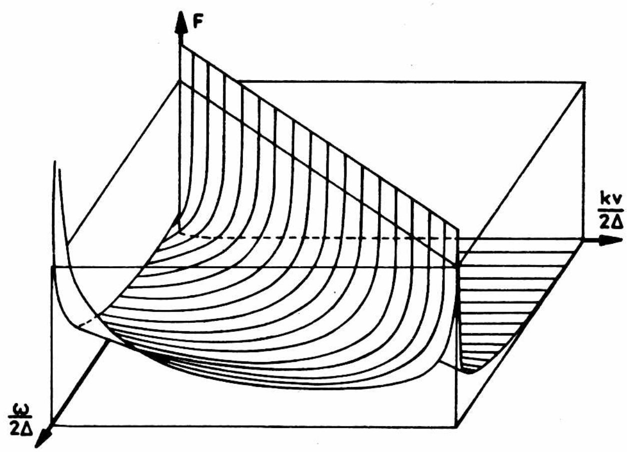

Fig. 9. F-function dependence on t and u. Here, $\alpha = 0.1$ (range of temperatures near to phase transition). The hummock becomes narrow and high

35. Some other characteristic functions and equations [44, 45, 49, 51]

1° Besides the function $F_h(\omega)$ (Eq. 34.8) the two following functions are strongly connected with the superfluidity studied in the homogeneous limit, namely

$$G(\omega) = 4\Delta^2 \frac{dF(\omega)}{d(\omega^2)} = \int_0^\infty d\xi \, \frac{\operatorname{th}\frac{E}{2T}}{E} \, \frac{16\Delta^4}{(4E^2 - \omega^2)^2} \tag{1}$$

which is always positive and increases from

$$G(0) = \frac{2}{3} F_h(0) - \frac{1}{3}\Delta^2 \int_0^\infty \frac{d\xi}{2T} \cdot \frac{\varepsilon^2}{E^4} \operatorname{ch}^{-2} \frac{E}{2T} \tag{2}$$

(the latter term disappears at $T = 0$ and $T = T_c$) to infinity if ω tends from zero to 2Δ and $T < T_c$, and

$$E_s(\omega) = F_h(\omega) + (1 - s)^{-1} G(\omega) \left[s - \left(\frac{\omega}{2\Delta}\right)^2 \right] \left[1 - \left(\frac{\omega}{2\Delta}\right)^2 \right] \tag{3}$$

where

$0 \leqslant s < 1.$

After applying the integral representations of $F_h(\omega)$ and $G(\omega)$ the function $E_s(\omega)$ can be rewritten in the form

$$E_s(\omega) = 16\Delta^2 \int_0^\infty \frac{d\varepsilon}{E} \frac{\operatorname{th}\frac{E}{2T}}{(4E^2 - \omega^2)^2} \left\{ \varepsilon^2 + \frac{\Delta^2}{1-s} \left[1 - \left(\frac{\omega}{2\Delta}\right)^2\right]^2 \right\} \tag{4}$$

While analysing the function $E_s(\omega)$ we find that its minimum value is reached at

$$\omega_m = 2\sqrt{s}\,\Delta. \tag{5}$$

Since $F_h(\omega)$ is an increasing function, the function $G(\omega)$ is always positive, hence the values of $E_s(\omega)$-function are greater than those of $F_h(\omega)$-function if $\omega < \omega_m$, and smaller if $\omega > \omega_m$. The function $E_s(\omega)$ is also always positive and for $\omega = \omega_m$ its value is equal to $F_h(\omega_m)$.

At the zero-temperature limit the above functions reduce to the forms

$$G(\omega) = \frac{1 - \left[1 - 2\left(\frac{\omega}{2\Delta}\right)^2\right] F_h(\omega)}{2\left(\frac{\omega}{2\Delta}\right)^2 \left[1 - \left(\frac{\omega}{2\Delta}\right)^2\right]}, \tag{6}$$

$$G(0) = \frac{2}{3} \tag{7}$$

and

$$E_s(\omega) = \frac{F_h(\omega)}{1-s}\left[1 - \left(\frac{\omega}{2\Delta}\right)^2\right] + \frac{s - \left(\frac{\omega}{2\Delta}\right)^2}{2\left(\frac{\omega}{2\Delta}\right)^2 (1-s)} \left[1 - F_h(\omega)\right], \tag{8}$$

$$E_s(0) = \frac{1 - \frac{1}{3}s}{1-s}. \tag{9}$$

2° Let us consider now the following integral equation

$$\omega^2 \left[1 + s\lambda F_h(\omega)\right] = 4s\Delta^2 \left[1 + \lambda F_h(\omega)\right]. \tag{10}$$

Since the parameter s in Eq. (10) is a fixed number, its solutions are examined in dependence on the value of the parameter λ. The frequency ω_0 is a solution of this equation provided that the following equation is fulfilled

$$\lambda = \Lambda(\omega) \tag{11}$$

where

$$\Lambda(\omega) = \left[1 - \frac{1}{3}\,\frac{1}{s}\,\left(\frac{\omega}{2\Delta}\right)^2\right] \left\{\left[\left(\frac{\omega}{2\Delta}\right)^2 - 1\right] F(\omega)\right\}^{-1} .$$

Differentiating the function $\Lambda(\omega)$, we obtain

$$\frac{d\,\Lambda(\omega)}{d(\omega^2)} = \frac{(1 - s)E_s(\omega)}{s\left[1 - \left(\frac{\omega}{2\Delta}\right)^2\right]^2 F^2(\omega)} . \tag{12}$$

Since the derivative of $\Lambda(\omega)$-function is always positive, $\Lambda(\omega)$ is the monotonously increasing function of ω. Its value increases from $(\bar{X} - 1)^{-1} \leqslant -1$ for $\omega = 0$ to infinity when ω tends to 2Δ. If the parameter λ is required to satisfy the condition

$$\lambda > -1 \tag{13}$$

(the typical stability condition of the Fermi systems) then the values of solutions of Eq. (11) begin from certain frequency $\omega_p (0 < \omega_p < \omega_m)$, if $T > 0$, or $\omega = 0^+$, if $T = 0$ on the border of stability of a system, and tend to infinity if λ tends to infinity, too. If $\lambda = 0$ or $T \to T_c$, the solution is given by Eq. (5).

3^o Let us specify now: the main relations employed in the zero -temperature limit

$$\lim_{T=0} \operatorname{th} \frac{x}{2T} = \operatorname{sgn} x, \tag{14}$$

$$\lim_{T=0} 2T \ln \operatorname{ch} \frac{x}{2T} = |x|, \tag{15}$$

$$\frac{1}{2} \lim_{T=0} \left(\operatorname{th} \frac{|x| + |y|}{2T} \pm \operatorname{th} \frac{|x| - |y|}{2T}\right)$$

$$= \Theta\left[\pm (|x| - |y|)\right] \tag{16}$$

and the integrals

$$\int_{-\infty}^{+\infty} d\xi \left(\operatorname{th} \frac{\xi + x}{2T} - \operatorname{th} \frac{\xi - x}{2T}\right) = 4x, \tag{17}$$

$$\int_{-\infty}^{+\infty} d\xi \left[\operatorname{sgn}(\xi + x) - \operatorname{sgn}(\xi - x)\right] = 4x. \tag{18}$$

36. The averages containing the Maki and Ebisawa function [43, 53]

All appearing averages can be expressed by the following terms

$$F_{2i} = (2i + 1)\left\langle F(\hat{k}\hat{p})^{2i}\right\rangle , \tag{1}$$

and

$$\Gamma_0 = \left\langle \frac{\omega^2}{\omega^2 - k^2v^2(\hat{k}\hat{p})^2}(1 - F)\right\rangle \tag{2}$$

which can be computed by the methods developed in [40]. We introduce the following notations

$$\Psi_0 = \eta^2 \left\langle [\eta^2 - (\hat{k}\hat{p})^2]^{-1}\right\rangle , \tag{3}$$

$$\Psi_0 = \begin{cases} \frac{1}{2}\eta \ln\left|\frac{\eta+1}{\eta-1}\right| - \frac{1}{2} i\pi\eta\theta(1 - \eta^2) & \text{if} \quad \eta^2 > 0, \\ |\eta| \operatorname{arctg}\frac{1}{|\eta|} & \text{if} \quad \eta^2 < 0, \end{cases} \tag{4}$$

$$\Psi_{2l} = \eta^{2l}\Psi_0 - \sum_{j=0}^{l-1} \frac{\eta^{2(l-j)}}{2j + 1} \tag{5}$$

where

$$\eta^2 = s^2 \frac{4E^2 - \omega^2}{4\xi^2 - \omega^2} , \tag{6}$$

and

$$E^2 = \xi^2 + \Delta^2, \qquad s = \frac{\omega}{kv} .$$

The imaginary part of the function Ψ_0 is responsible for the Landau damping where its threshold is modified according to Eq. (6) then it never exceeds its normal-phase value ($s = 1$).

If $|\eta| > 1$, then function Ψ_{2l} can be presented as

$$\Psi_{2l} = \sum_{j=0}^{\infty} \frac{\eta^{-2j}}{2(l + j) + 1} . \tag{7}$$

Applying equations (3)-(5) the expressions F_{2i} and Γ_0 reduce to the forms

$$F_{2i} = 4(2i + 1)\Delta^2 \sum_{n=0}^{\infty} \frac{1}{(2n)!}\left(\frac{kv}{2}\right)^{2n} \int_0^{\infty} \frac{d\xi}{4E^2 - \omega^2}$$

$$\times \left[\left(\Psi_{2(n+i)} - s^{-2}\Psi_{2(n+i+1)} \right) \frac{\partial^{2n}}{\partial \xi^{2n}} \left(\frac{\mathrm{th}\,\frac{E}{2T}}{E} \right) + \frac{s^{-2}}{2n+1} \Psi_{2(n+i+1)} \frac{\partial^{2n+1}}{\partial \xi^{2n+1}} \left(\frac{\varepsilon}{E}\,\mathrm{th}\,\frac{E}{2T} \right) \right] \tag{8}$$

or

$$F_{2i} = 4(2i+1)\Delta^2 \sum_{n=0}^{\infty} \frac{1}{(2n)!} \left(\frac{kv}{2} \right)^{2n} \int_0^{\infty} \frac{d\xi}{4E^2 - \omega^2}$$

$$\times \left[\Psi_{2(n+i)} \frac{\partial^{2n}}{\partial \xi^{2n}} \left(\frac{\mathrm{th}\,\frac{E}{2T}}{E} \right) + \frac{s^{-2}\varepsilon}{2n+1} \Psi_{2(n+i+1)} \frac{\partial^{2n+1}}{\partial \xi^{2n+1}} \left(\frac{\mathrm{th}\,\frac{E}{2T}}{E} \right) \right] \tag{9}$$

and

$$\Gamma_0 = \sum_{n=0}^{\infty} \frac{1}{(2n)!} \left(\frac{kv}{2} \right)^{2n} \int_0^{\infty} d\xi \Psi_{2n}$$

$$\times \left[\frac{1}{2n+1} \cdot \frac{\partial^{2n+1}}{\partial \xi^{2n+1}} \left(\frac{\xi}{E}\,\mathrm{th}\,\frac{E}{2T} \right) - \frac{4\Delta^2}{4E^2 - \omega^2} \cdot \frac{\partial^{2n}}{\partial \xi^{2n}} \left(\frac{\mathrm{th}\,\frac{E}{2T}}{E} \right) \right]. \tag{10}$$

In the acoustic limit, i.e., when ω, $kv \ll \Delta$, the expressions(8) and (10) reduce to the forms which are obtained by putting $\omega = 0$ at the denominator $(4E^2 - \omega^2)$ and in the term η and with taking only one $n=0$ term of the series. Then we get

$$F_{2i} = 1 - (2i+1)(\hat{\underline{P}}\Psi_{2i} - s^{-2}\hat{\underline{P}}\Psi_{2(i+1)}), \tag{11}$$

$$\Gamma_0 = \hat{\underline{P}}\Psi_0$$

where $\hat{\underline{P}}$ denotes the integral operator defined as follows

$$\hat{\underline{P}}\Phi(\xi) = \int_0^{\infty} \frac{d\xi}{2T}\,\mathrm{ch}^{-2}\left(\frac{E}{2T} \right) \left(\frac{\xi}{E} \right)^2 \Phi(\xi), \tag{12}$$

and

$$\eta = \frac{\omega}{kv} \cdot \frac{E}{\varepsilon}.$$

According to equation (5) the function Ψ_{2i} for $i > 0$ can be expressed by means of the functions Ψ_0 and Y_{2i} (Eq. 34. 10). In the quasi-homogeneous limit the expressions F_{2i} and Γ_0 reduce to the forms

$$F_{2i} = F_h \left(1 - \frac{2i+1}{2i+3}\, s^{-2} h\right), \tag{13}$$

$$\Gamma_0 = (1 - F_h) + \frac{1}{3}\,(1 - F_h + F_h \cdot h)s^{-2}. \tag{14}$$

37. The kernels L, M, N, O [43]

1° The kernels L, M, N, O in the matrix rotation are defined as follows:

$$\begin{aligned}
L &\equiv \underline{G}_s(p_+)\underline{G}_s(p_-) - (\underline{G}^2(p))^{\omega},\\
M &\equiv \underline{G}_s(p_+)\underline{F}(p_-),\\
N &\equiv \underline{G}_s(p_+)\underline{G}_s^-(p_-) - \underline{G}_s(p)\underline{G}^-(p),\\
O &\equiv \underline{F}(p_+)\underline{F}(p_-)
\end{aligned} \tag{1'}$$

where the functions $\underline{G}$, $\underline{G}_s$ and $\underline{F}$ are taken in the limit $H = 0$. In order to compute these kernels we can utilize the Eliashberg method (Section 31). Then we obtain [22, 40, 43, 81]

$$\begin{aligned}
L &= \frac{\underline{kv}}{\omega - \underline{kv}} - \frac{1}{2} \cdot \frac{\omega + \underline{kv}}{\omega - \underline{kv}}\, F,\\
M &= -\frac{\omega + \underline{kv}}{4\Delta}\, F,\\
N &= \left[\frac{\omega^2 - (\underline{kv})^2}{4\Delta^2} - \frac{1}{2}\right] F,\\
O &= \frac{1}{2}\, F.
\end{aligned} \tag{2}$$

2° The type (33.3°) averages of the above kernels can be expressed by the terms F_{2i} and Γ_0 combined with the frequency ω and the wave vector $\underline{k}$. Some examples of such procedure used in some characteristic limits are given below. In the general case we obtain

$$l_0 - o_0 = -1 + \Gamma_0,$$

$$m_2 = -\omega\,(F_0 - \frac{1}{3}\, F_2)/8\Delta\,,$$

$$m_3 = \omega\,(F_0 - F_2)/8\Delta\,,$$

$$n_2 + o_2 = \left[\omega^2 F_0 - \tfrac{1}{3}(\omega^2 + k^2v^2)F_2 + \tfrac{1}{5}k^2v^2F_4\right]/8\Delta^2 ,$$

$$n_3 + o_3 = -\left[\omega^2 F_0 - (\omega^2 + \tfrac{1}{3}k^2v^2)F_2 + \tfrac{3}{5}k^2v^2F_4\right]/8\Delta^2,$$

$$o_0 = \frac{1}{2} F_0,$$

$$o_2 = \frac{1}{4} F_0 - \frac{1}{12} F_2,$$

$$o_3 = -\frac{1}{4} F_0 + \frac{1}{4} F_2,$$

$$o_6 = \frac{1}{16} F_0 - \frac{1}{24} F_2 + \frac{1}{80} F_4,$$

$$o_7 = -\frac{1}{16} F_0 + \frac{1}{8} F_2 - \frac{1}{16} F_4,$$

$$o_8 = \frac{3}{16} F_0 - \frac{15}{24} F_2 + \frac{7}{16} F_4,$$

$$o_1 = o_4 = o_5 = 0. \qquad (3)$$

In the acoustic limit (ω, $kv \ll 2\Delta$) and at $T = 0$ these expressions reduce to the form ($F \equiv 1$ and $F_{2i} = 1$, $\Gamma_0 = 0$)

$$l_0 - o_0 = -1, \quad m_2 = -\frac{1}{6}\,\frac{\omega}{2\Delta}, \quad n_2 + o_2 = \frac{1}{3}\left(\frac{\omega}{2\Delta}\right)^2 - \frac{1}{15}\left(\frac{kv}{2\Delta}\right)^2$$

$$n_3 + o_3 = -\frac{2}{15}\left(\frac{kv}{2\Delta}\right)^2, \quad o_0 = \frac{1}{2}, \quad o_2 = \frac{1}{6}, \quad o_6 = \frac{1}{30},$$

$$m_3 = o_3 = o_7 = o_8 = 0. \qquad (4)$$

In the homogeneous limit ($k = 0$) the expressions (3) have the forms

$$l_0 - o_0 = -F_h,$$

$$m_2 = -\frac{1}{6}\,\frac{\omega}{2\Delta} F_h,$$

$$n_2 + o_2 = \frac{1}{3}\left(\frac{\omega}{2\Delta}\right)^2 F_h,$$

$$o_0 = \frac{1}{2} F_h, \qquad o_2 = \frac{1}{6} F_h,$$

$$o_6 = \frac{1}{30} F_h,$$

$$m_3 = n_3 = o_3 = o_7 = o_8 = 0. \qquad (5)$$

In the static limit ($\omega = 0$) the expressions (3) have the forms

$$l_0 - o_0 = -1,$$

$$n_2 + o_2 = -\frac{1}{6}\left(\frac{kv}{2\Delta}\right)^2 (F_2 - \frac{3}{5}F_4),$$

$$n_3 + o_3 = \frac{1}{6}\left(\frac{kv}{2\Delta}\right)^2 (F_2 - \frac{9}{5}F_4),$$

$$m_2 = m_3 = 0, \quad o_i \text{ as in Eqs. (3).} \qquad (6)$$

In the quasi-homogeneous limit ($\omega \gg kv$) the expressions (3) reduce to the forms

$$l_0 - o_0 = \frac{1}{3}\left(\frac{kv}{\omega}\right)^2 - \left[1 + \frac{1}{3}\left(\frac{kv}{\omega}\right)^2\right] F_h + \frac{1}{3}\left(\frac{kv}{\omega}\right)^2 F_h \cdot h,$$

$$m_2 = \frac{1}{6}\,\frac{\omega}{2\Delta}\left[1 - \frac{1}{5}\left(\frac{kv}{\omega}\right)^2 h\right] F_h,$$

$$m_3 = \frac{1}{15}\,\frac{k^2v^2}{2\Delta\omega}\, F_h \cdot h,$$

$$n_2 + o_2 = \frac{1}{3}\left(\frac{\omega}{2\Delta}\right)^2 F_h - \frac{1}{15}\left(\frac{kv}{2\Delta}\right)^2 (1 + h)F_h,$$

$$n_3 + o_3 = -\frac{2}{15}\left(\frac{kv}{2\Delta}\right)^2 (1 + h)\, F_h,$$

$$o_0 = \frac{1}{2}\left[1 - \frac{1}{3}\left(\frac{kv}{\omega}\right)^2 h\right] F_h,$$

$$o_2 = \frac{1}{6}\left[1 - \frac{1}{5}\left(\frac{kv}{\omega}\right)^2 h\right] F_h,$$

$$o_3 = -\frac{1}{15}\left(\frac{kv}{\omega}\right)^2 F_h \cdot h,$$

$$o_6 = \frac{1}{30}\left[1 - \frac{1}{7}\left(\frac{kv}{\omega}\right)^2 h\right] F_h,$$

$$o_7 = -\frac{1}{105}\left(\frac{kv}{\omega}\right)^2 F_h \cdot h,$$

$$o_1 = o_4 = o_5 = o_8 = 0. \qquad (7)$$

All other terms can be obtained in the analogous manner.

38. Elliptic integrals [52]

The elliptic integrals introduced by Legendre are following:

The elliptic integral of the first type is of the form

$$F(\varphi_0 k) = \int_0^{\varphi_0} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2\varphi}} ,$$

and

$$F\left(\frac{\pi}{2}, k\right) \equiv F(k). \tag{1}$$

The elliptic integral of the second type is of the form

$$E(\varphi_0, k) = \int_0^{\varphi_0} d\varphi \sqrt{1 - k^2 \sin^2\varphi} ,$$

and

$$E\left(\frac{\pi}{2}, k\right) \equiv E(k). \tag{2}$$

The elliptic integral of the third type is of the form

$$\Pi(\varphi_0, n, k) = \int_0^{\varphi_0} \frac{d\varphi}{(1 + n \sin^2\varphi)\sqrt{1 - k^2 \sin^2\varphi}} . \tag{3}$$

The function $\Pi(\varphi_0, n, k)$ is divergent if n is negative and fulfils the condition

$$n < - 1/\sin\varphi_0 . \tag{4}$$

In order to omit the appearing disadvantages we have introduced an additional type of integral

$$G(\varphi_0, l, k) = \int_0^{\varphi_0} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2\varphi} + l \sin\varphi} ,$$

and

$$G\left(\frac{\pi}{2}, l, k\right) \equiv G(l, k) \tag{5}$$

which is convergent and can be expressed by means of elliptic integrals in the following way

$$G(\varphi, l, k) = - \frac{1}{n} \left[k^2 F(\varphi, k) - l^2 \Pi(\varphi, n, k)\right] + \frac{1}{2}\sqrt{\frac{k^2+n}{-n(n+1)}} \ln\left[\frac{\left|\cos\varphi - \sqrt{1+\frac{1}{n}}\right|}{\cos\varphi + \sqrt{1+\frac{1}{n}}} |n| \left(1+ \sqrt{1 + \frac{1}{n}}\right)^2\right] \tag{6}$$

where

$$n = -(k^2 + l^2).$$

39. Temperature expansion [54, 55]

Standard integrals which appear in the theory of the superconductivity and superfluidity when the temperature expansion nearby T_c is applied have the forms (cf. [66, 73])

$$\int_0^{\omega_D/2T} \frac{dx}{x}\,\mathrm{th}\,x = \ln\frac{\omega_D}{2T}\,\mathrm{th}\,\frac{\omega_D}{2T} + \ln\frac{4e^C}{\pi} = \ln\frac{2\omega_D}{\Delta(0)} + \frac{T_c}{T}\,, \tag{1}$$

$$\int_0^{\omega_D/2T} dx\,x\,\frac{d}{dx}\left(\frac{\mathrm{th}\,x}{x}\right) = \mathrm{th}\,\frac{\omega_D}{2T} - \int_0^{\omega_D/2T}\frac{dx}{x}\,\mathrm{th}\,x$$

$$= 1 - \ln\frac{2\omega_D}{\Delta(0)} + \ln\frac{T}{T_c} \tag{2}$$

where

$$\Delta(0) = T_c\pi e^{-C} \tag{3}$$

and $C = 0.577\,215$ is Euler's constant. Moreover

$$\int_0^\infty \frac{dx}{x}\,\frac{d}{dx}\left(\frac{\mathrm{th}\,x}{x}\right) = -16T^3\sum_{\varepsilon_n}\int_0^\infty\frac{d\xi}{(\varepsilon_n^2+\xi^2)^2} = -\frac{7\zeta(3)}{\pi^2}, \tag{4}$$

$$\int_0^\infty \frac{dx}{x}\,\frac{d^2\,\mathrm{th}\,x}{dx^2} = 16T^3\left[\sum_{\varepsilon_n}\int_0^\infty\frac{d\xi}{(\varepsilon_n^2+\xi^2)^2}\right.$$

$$\left. - 4\sum_{\varepsilon_n}\varepsilon_n^2\int_0^\infty\frac{d\xi}{(\varepsilon_n^2+\xi^2)^3}\right] = -\frac{14\zeta(3)}{\pi^2}\,, \tag{5}$$

$$\int_0^\infty \frac{dx}{x}\,\frac{d}{dx}\left[\frac{1}{x}\,\frac{d}{dx}\left(\frac{\mathrm{th}\,x}{x}\right)\right] = 256T^5\sum_{\varepsilon_n}\int_0^\infty\frac{d\xi}{(\varepsilon_n^2+\xi^2)^3} = \frac{93\zeta(5)}{\pi^4}\,, \tag{6}$$

$$\int_0^\infty \frac{dx}{x}\frac{d}{dx}\left(\frac{1}{x}\frac{d^2 \operatorname{th} x}{d\,x^2}\right) = 256T^5\left[-\sum_{\varepsilon_n}\int_0^\infty \frac{d\xi}{(\varepsilon_n^2+\xi^2)^3}\right.$$

$$\left.+\,6\sum_{\varepsilon_n}\varepsilon_n^{\,2}\int_0^\infty \frac{d\xi}{(\varepsilon_n^{\,2}+\xi^2)^4}\right] = \frac{372\,\zeta(5)}{\pi^4}\,, \qquad (7)$$

$$\int_0^\infty \frac{dx}{x}\frac{d^4 \operatorname{th} x}{d\,x^4} = 768T^5\left[\sum_{\varepsilon_n}\int_0^\infty \frac{d\xi}{(\varepsilon_n^{\,2}+\xi^2)^3} - 12\sum_{\varepsilon_n}\varepsilon_n^{\,2}\int_0^\infty \frac{d\xi}{(\varepsilon_n^{\,2}+\xi^2)^4}\right.$$

$$\left.+\,16\sum_{\varepsilon_n}\varepsilon_n^{\,4}\int_0^\infty \frac{d\xi}{(\varepsilon_n^{\,2}+\xi^2)^5}\right] = \frac{744\,\zeta(5)}{\pi^4}, \qquad (8)$$

$$\int_0^\infty \frac{dx}{x}\frac{d}{dx}\left\{\frac{1}{x}\frac{d}{dx}\left[\frac{1}{x}\frac{d}{dx}\left(\frac{\operatorname{th} x}{x}\right)\right]\right\}$$

$$= -\,6144T^7\sum_{\varepsilon_n}\int_0^\infty \frac{d\xi}{(\varepsilon_n^{\,2}+\xi^2)^4} = -\,\frac{1095\,\zeta(7)}{\pi^6} \qquad (9)$$

where $\varepsilon_n = T\pi\,(2n+1)$, n is an integer of any sign or zero, $\zeta(s)$ is Riemann's ξ-function. We also use the following formulae

$$2T\sum_{\varepsilon_n}\varepsilon_n^{-s} = \frac{4T}{\pi^s T^s}\,(1-2^{-s})\,\zeta(s), \qquad (10)$$

$$\int_0^\infty \frac{dx}{(1+x^2)^{n+1}} = \frac{\pi}{2}\,\frac{(2n-1)!!}{2^n\,n!}\,. \qquad (11)$$

Moreover we state that

$$\int_0^\infty dx\,\frac{d^2}{dx^2}\left(\frac{\operatorname{th} x}{x}\right) = 0, \qquad (12)$$

$$\int_0^\infty dx\,\frac{d}{dx^2}\,(\operatorname{ch}^{-2} x) = 0. \qquad (13)$$

Since the terms $\omega_D/2T \gg 1$ or $\xi_p/2T \gg 1$, if $T \leqslant T_c$ (where ω_D is the

Debye characteristic frequency and ξ_p is the cut-off parameter), they can be put equal to infinity in all convergent integrals and functions. The integrals (4)-(9) have been derived by comparing suitable terms of the series expansion in the residuum method. The applied values of Riemann's ζ-function are the following

$$\zeta(3) = 1.202\,057\ldots,$$
$$\zeta(5) = 1.036\,928\ldots,$$
$$\zeta(7) = 1.008\,349\ldots\,. \qquad (14)$$

XI. Appendices

40. Two-particle states

a) The concept of two-particle states

Quasiparticles which are the fermions possessing half-integral spins couple in Cooper's pairs if their momenta are opposite. All Cooper's pairs create a coherent two-particle state. Such sta..s are eigen-states of the operators $\hat{J}^2$ and $\hat{J}_3$ where $\hat{J}$ is the total angular momentum operator, composed of operators of angular momentum and total spin of Cooper's pairs. Since the total spin operator is composed of two operators of half-integral spins ($s_1 = s_2 = 1/2$), the suitable eigen-spin-states in the irreducible base have the following forms:

1^{o} the antisymmetric spin-state $S = 0$, $\sigma = 0$

$$|0\ 0\rangle , \tag{1}$$

2^{o} three symmetric spin-states $S = 1$, $\sigma = -1,\ 0,\ +1$

$$|1\ -1\rangle , \qquad |1\ 0\rangle , \qquad |1\ 1\rangle . \tag{2}$$

The orbital angular momentum of Cooper's pair is defined by the quantum numbers L and m. In general, L can be an arbitrary positive integer or zero and $|m| \leqslant L$. Hence, the orbital angular momentum states are of the form

$$|L\ m\rangle \tag{3}$$

and the states with even and odd values of L are space-symmetric and space-antisymmetric, respectively.

If we now form the sum of the operators

$$\hat{J} = \hat{L} + \hat{S}, \tag{4}$$

then it possesses all the properties of the angular-momentum operator if the components $\underline{L}$ and $\underline{S}$ transform identically under rotation of the coordinate system. The eigen-states of $\hat{S}^2$ and $\hat{S}_3$ generate the one- and three-dimension spaces, respectively, and the eigen-states of $\hat{L}^2$ and $\hat{L}_3$ generate the $(2L + 1)$ - dimension spaces.

The tensor-product of the above spaces contains $1\cdot(2L_e + 1)$ and $3\cdot(2L_0 + 1)$ independent states $|L\ m\ \ S\,\sigma\rangle$, respectively. The latter are the eigen-states of the operators $\hat{L}^2$, $\hat{L}_3$, $\hat{S}^2$, $\hat{S}_3$ where in order to satisfy the Pauli exclusion principle the numbers L_e and L_o are even and odd, respectively. In the space under consideration we can introduce the irreducible bases of states which are the eigen-states of the operators $\hat{J}^2$, $\hat{J}_3$, $\hat{L}$ and $\hat{S}$. We identify these eigen-states with the two-particle states of Cooper's pairs and denote them with

$$|J\ M\ L\ S\rangle \tag{5}$$

where J (the total angular-momentum number) and M (the total angular-momentum projection number) are good quantum numbers defining the two-particle states in systems with full rotational symmetry. In the systems where solely one direction is distinguished, M is still a good quantum number. The quantum numbers L and S are connected with the type of pairing interaction. The relation between the states $|L\ m\ S\ \sigma\rangle$ and $|J\ M\ L\ S\rangle$ can be expressed by means of the Clebsh-Gordon coefficients [16, 32]. According to the Pauli exclusion principle we can distinguish two types of the two-particle states, namely:

1° the orbital-angular momentum symmetric and spin-antisymmetric states which are of the form

$$|J\ M\quad L = 2l\quad S = 0\rangle\ , \tag{6}$$

then $J = 2l$, $|M| \leqslant J$ and

2° the orbital-angular momentum antisymmetric and spin-symmetric states which are of the form

$$|J\ M\ L{=}2l{+}1\ S{=}1\rangle\ , \tag{7}$$

then

$$2l \leqslant J \leqslant 2(l+1) \qquad |M| \leqslant J$$

and $l = 0, 1, 2 \ldots$ Let us note that in the latter case the different-structure states which are of the forms:

$$|J{=}2(l{+}1)\ M\ L{=}2l{+}1\ S{=}1\rangle \tag{8}$$

and

$$|J{=}2(l{+}1)\ M\ L{=}2l{+}3\ S{=}1\rangle \tag{9}$$

have the same quant numbers J and M.
Applying standard selection rules we can determine the allowed and forbidden transitions ammong the two-particle states in relation to forcing fields.

b) The representation and properties [42, 43, 130]

We are going now to give the representation of the two-particle states in the momentum-spin space in the most significant cases.

1° the case $L = 0$ and $S = 0$. The eigen-function is of the form

$$<\hat{\underline{p}} - \underline{\sigma}|0\ 0\ 0\ 0> = i\ \sigma^2. \tag{10}$$

2° the case $L = 1$ and $S = 1$. The eigen-functions are the spherical tensors which have the form

$$<\hat{\underline{p}} - \underline{\sigma}|J\ M\ \ 1\ 1> = B_{JM} = B_{JM}^{jn}\,\sigma^j p_n i\sigma^2 \tag{11}$$

where

$$0 \leqslant J \leqslant 2, \qquad |M| \leqslant J,$$

$$B_{00}^{jn} = (1/\sqrt{3})\delta_{jn},$$

$$B_{10}^{jn} = (1/\sqrt{2})(\delta_{j2}\delta_{n1} - \delta_{j1}\delta_{n2}),$$

$$B_{11}^{jn} = \tfrac{1}{2}(\delta_{j2}\delta_{n3} - \delta_{j3}\delta_{n2} + i\delta_{j3}\delta_{n1} - i\delta_{j1}\delta_{n3}),$$

$$B_{20}^{jn} = -(1/\sqrt{6})(\delta_{jn} - 3\delta_{j3}\delta_{n3}),$$

$$B_{21}^{jn} = -\tfrac{1}{2}(\delta_{j1}\delta_{n3} + \delta_{j3}\delta_{n1} + i\delta_{j2}\delta_{n3} + i\delta_{j3}\delta_{n2}),$$

$$B_{22}^{jn} = \tfrac{1}{2}(\delta_{j1}\delta_{n1} - \delta_{j2}\delta_{n2} + i\delta_{j1}\delta_{n2} + i\delta_{j2}\delta_{n1}),$$

and

$$B_{JM}^{*in} = (-1)^M B_{J-M}^{jn}. \tag{12}$$

The tensors B_{JM}^{jn} have the following properties:

$$B_{JM}^{jn}B_{J'M'}^{*jn} = \delta_{JJ'}\delta_{MM'}$$

$$B_{JM}^{ik}B_{JM}^{*jn} = \delta_{ij}\delta_{kn},$$

$$B_{00}^{jn} = B_{00}^{nj}, \qquad B_{00}^{nn} = \sqrt{3},$$

$$B_{1M}^{jn} = -B_{1M}^{nj}, \qquad B_{1M}^{nn} = 0,$$

$$B_{2M}^{jn} = B_{2M}^{nj}, \qquad B_{2M}^{nn} = 0. \tag{13}$$

Moreover, the tensors of rank four defined by the sums

$$B_{00}^{ik}\,B_{00}^{*jn} = \frac{1}{3}\,\delta_{ik}\delta_{jn},$$

$$B_{1M}^{ik}\,B_{1M}^{*jn} = \frac{1}{2}\,(\delta_{ij}\delta_{kn} - \delta_{in}\delta_{jk}),$$

$$B_{2M}^{ik}\,B_{2M}^{*jn} = \frac{1}{2}\,(\delta_{ij}\,\delta_{kn} + \delta_{in}\delta_{jk} - \frac{2}{3}\,\delta_{ik}\delta_{jn}) \tag{14}$$

are rotational invariants.

41. The normal neutral Fermi liquid in the presence of a constant magnetic field. The linear response theory [51]

The inclusion of a constant magnetic field (which is assumed as $h \ll \varepsilon_F$) into the linear response theory generates conspicuous difficulties. Let us demonstrate it for the normal Fermi liquid of neutral fermions with half-integral spins (^{3}He). Such considerations lead to the generalization of the basic equations of the linear response which were obtained by Luttinger and Noziérès [67], [106] for the case of h=0. While considering this problem we must realize that now the Green functions are matrices and that they do not commutate with vertex functions $\hat{\mathcal{T}}^i$ which are proportional to the appropriate Pauli matrices σ^0, σ^x, σ^y, σ^z. Nevertheless, assuming that $\underline{h} = h\hat{z}$, the Green-function part proportional to σ^z commutates with $\hat{\mathcal{T}}^0$ and $\hat{\mathcal{T}}^z$ and anticommutates with $\hat{\mathcal{T}}^x$ and $\hat{\mathcal{T}}^y$. Taking this into account we can transpose one Green function with respect to the vertex function from right to left and calculate the conventional function $\underline{L}$ defined as follows

$$\underline{L}_{(\pm)} = \frac{2\pi T}{\nu(0)} \sum_{\varepsilon_n} \sum_{|\underline{p}|} \mathcal{G}(\tilde{p}_+)\, \mathcal{G}^{(\pm)}(\tilde{p}_-) \tag{1}$$

in two independent cases mentioned above. The sign $(\pm)$ is connected with the manner of the Green function transposition and the sign $(+)$ refers to the full-commutation case. The second sum over $|\underline{p}|$ is taken only over this vector length. Using the Eliashberg technique (Section 31) after some calculations we obtain

$$\begin{aligned}\underline{L}_{(\pm)} = -\frac{1}{4}\int_{-\infty}^{+\infty} d\xi \Bigg[&\operatorname{sgn}\left(\xi - \frac{\underline{kv}}{2} - h\right) \frac{(\omega - \underline{kv} - h \mp h)(\sigma^0 \mp \sigma^z)}{(\omega - \underline{kv})(\omega - \underline{kv} - 2h)} \\ &+ \operatorname{sgn}\left(\xi - \frac{\underline{kv}}{2} + h\right) \frac{(\omega - \underline{kv} + h \pm h)(\sigma^0 \pm \sigma^z)}{(\omega - \underline{kv})(\omega - \underline{kv} + 2h)} \\ &+ \operatorname{sgn}\left(\xi + \frac{\underline{kv}}{2} - h\right) \frac{(-\omega + \underline{kv} - h \mp h)(\sigma^0 - \sigma^z)}{(-\omega + \underline{kv})(-\omega + \underline{kv} - 2h)} \\ &+ \operatorname{sgn}\left(\xi + \frac{\underline{kv}}{2} + h\right) \frac{(-\omega + \underline{kv} + h \pm h)(\sigma^0 + \sigma^z)}{(-\omega + \underline{kv})(-\omega + \underline{kv} + 2h)}\Bigg],\end{aligned} \tag{2}$$

and next, by separating the counts we finally obtain

$$\underline{L}_{(+)} = \frac{\underline{kv}}{\omega - \underline{kv}}\, \sigma^0 \tag{3}$$

and

$$\underline{L}_{(-)} = \frac{\underline{k}\underline{v}(\omega - \underline{k}\underline{v}) + \omega_L^2}{(\omega - \underline{k}\underline{v})^2 - \omega_L^2}\,\sigma^0 + \frac{\omega\,\omega_L}{(\omega - \underline{k}\underline{v})^2 - \omega_L^2}\,\sigma^z \tag{4}$$

where $\omega_L = 2h$ is the Larmor frequency and its value is determined by the total spin inversion. As one can see, the function $\underline{L}_{(+)}$ which is connected with the spinless and spin-longitudinal responses of a system, is independent of the magnetic field, just like in the non-magnetic case. On the other hand, the function $\underline{L}_{(-)}$ is connected with the spin-transverse response and has some quite understandable new properties, namely

(i) if $H_E = 0$, then $\underline{L}_{(-)} = \underline{L}_{(+)}$,

(ii) if $k = 0$, then $\underline{L}_{(-)} = \begin{pmatrix} \underline{L}^{\dagger}_{(-)}, & 0 \\ 0, & \underline{L}^{\dagger}_{(-)} \end{pmatrix}$,

and $L^{\dagger}_{(-)}(H_E) = L^{\dagger}_{(-)}(-H_E)$ where $L^{\dagger}_{(-)} = \dfrac{\omega_L}{\omega - \omega_L}$

(iii) if $\omega = 0$, then $\underline{L}_{(-)} = -1\sigma^0$. (5)

The obtained results do not depend on temperature, which is in agreement with Landau's ideas, and the mathematical equivalence of that statement arises from the formulae (35.17) and (35.18). Now we can formulate the system of the basic equations for the normal Fermi liquid in the presence of the strong magnetic field. The vertex equations are of the forms

$$\begin{aligned} \mathcal{T}^0 &= \mathcal{T}^0_\omega + \langle AL_0\,\mathcal{T}^0\rangle, \\ \mathcal{T}^z &= \mathcal{T}^z_\omega + \langle BL_0\,\mathcal{T}^z\rangle, \\ \mathcal{T}^{x(y)} &= \mathcal{T}^{x(y)}_\omega + \langle BL_1\,\mathcal{T}^{x(y)}\rangle \mp i\,\langle BL_2\,\mathcal{T}^{y(x)}\rangle, \end{aligned} \tag{6}$$

and hence adequate correlation functions can be performed in the forms

$$\begin{aligned} S^{00} &= \langle \mathcal{T}^0_\omega L_0\,\mathcal{T}^0\rangle, \\ S^{zz} &= \langle \mathcal{T}^z_\omega\,L_0\,\mathcal{T}^z\rangle, \\ S^{xx} &= \langle \mathcal{T}^x_\omega L_1\,\mathcal{T}^x\rangle, \\ S^{xy} &= -\,i\langle \mathcal{T}^x_\omega\,L_2\,\mathcal{T}^y\rangle, \end{aligned} \tag{7}$$

and

$$S^{xx} = S^{yy}, \quad S^{xy} = (S^{yx})^*$$

where

$$\underline{L}_{(+)} = L_0\sigma^0, \quad \underline{L}_{(-)} = L_1\sigma^0 + L_2\sigma^z,$$

and $\mathcal{T}^{\,i}_{\omega}$ denotes suitable vertex in the ω limit. The correlation functions (7) describe only the linear response of the system to the additional perturbating external field, while the static longitudinal magnetization effects, coming from the presence of constant magnetic field, have to be included independently. The set of the basic equations (6)-(7) is analogous to the known one obtained in the limit $k = 0$ which is useful to apply to the nuclear resonance, but now we can also investigate the collective excitations, i.e., the spin wave propagation in the presence of the external magnetic field.

In order to explain the sense of the Larmor frequency we must consider the problem of the static magnetization. For the field under discussion ($h \ll \varepsilon_F$) it is of the form (cf. [140])

$$M = \mu_B^2 \nu_0 H_T, \tag{8}$$

then after regard of the Fermi liquid interaction we derive the Larmor frequency in the form

$$\omega_L = \frac{2\mu_B}{1 + F_0^a} H. \tag{9}$$

Although this equation is well known, it completes the developed problem. While investigating the interacting fermion liquid we should however be aware that for very strong external magnetic fields the magnetization changes its form and all the omitted effects cannot be neglected.

42. The dipole contribution parameter as an invariant of the theory [44]

The Green function formalism contains an artificial parameter, called a cut-off parameter ξ_p, and related to it a pairing parameter, g_1 or a dimensionless parameter f_1 ($f_1 = \nu(0)g_1$, cf. Section 6). If we include the dipole-dipole interaction, there appears an additional parameter in energy scale $g_{\bar{D}}$ or a dimensionless parameter $\nu(0)g_{\bar{D}}$.

Since the final results contain contributions proportional to the dimensionless parameter

$$g = \frac{g_D}{\nu(0)g_1^2} = \frac{\nu(0)g_D}{f_1^2} \tag{1}$$

one can ask if the final results depend on ξ_p. We show that the parameter, g, is independent of ξ_p. Deriving the basic equations of the formalism, we apply renormalizing transformation given by the equation [22]:

$$\hat{\underline{V}} = (\hat{\underline{1}} + \hat{\underline{V}}\mathcal{A})\,\hat{\underline{\Gamma}}. \tag{2}$$

We use the matrix notation, where $\hat{\underline{V}}$ and $\hat{\underline{\Gamma}}$ are the renormalized and unrenormalized interactions in the particle-particle channel, respectively. $\mathcal{A}$ is the cut-off kernel and if $\Delta \ll \xi_p \ll \varepsilon_F$ it has the form

$$\mathcal{A} = G^{-}G\Theta(|\xi| - \xi_p). \tag{3}$$

We assume that the renormalization transformation does not change the interaction structure, i.e., $\hat{\underline{V}}$ and $\hat{\underline{\Gamma}}$ have the same structure. The interaction, $\hat{\underline{V}}$, in the presence of the dipole forces has the form: (Section 19)

$$\hat{\underline{V}} = -\,g_1(1+\alpha)\left[\hat{\underline{1}} + \frac{3}{5}\alpha\hat{\underline{U}}\right], \tag{4}$$

then $\hat{\underline{\Gamma}}$ must have the form:

$$\hat{\underline{\Gamma}} = -\,h(1+\beta)\,[\hat{\underline{1}} + \gamma\hat{\underline{U}}]. \tag{5}$$

The factors α (= g_1/g_D), β, and γ are small quantities in comparison with unity, and in all equations they can be kept in the first order only. If we insert Eqs. (4) and (5) into Eq. (2) and compare the suitable factors standing near the operators $\hat{\underline{1}}$ and $\hat{\underline{U}}$ on both sides of the equation, we obtain two independent equations

$$g_1(1+\alpha) = h(1+\beta)\,[1 - g_1(1+\alpha)\bar{\mathcal{A}}], \tag{6}$$

$$\frac{3}{5}\alpha g_1 = -\frac{3}{5}\alpha g_1\bar{\mathcal{A}}h + \gamma(1 - g_1\bar{\mathcal{A}})h. \tag{7}$$

Now, we can compare the quantities of the same order in the first equation, whence we obtain the two following equations:

$$g_1 = (1 - g_1\bar{\mathcal{A}})h, \tag{8}$$

$$\alpha g_1 = -\alpha g_1\bar{\mathcal{A}}h + \beta(1 - g_1\bar{\mathcal{A}})h. \tag{9}$$

Inserting equation (8) into Eq. (9) and transforming it we obtain

$$\frac{\alpha}{g_1} = \frac{\beta}{h} . \tag{10}$$

Both renormalized parametrs g_1 and g_D are the functions of the cut--off parameter ξ_p. However, according to Eq. (10), the choice of the cut-off parameter ξ_p is quite arbitrary, thus, the parameter g does not depend on ξ_p and is the invariant of the theory, in spite of that the parameters

$$g_1 = g_1(\varepsilon_p), \qquad g_D = g_D(\varepsilon_p), \tag{11}$$

and

$$g = \frac{g_D(\varepsilon_p)}{g_1^2(\varepsilon_p)\, \nu(0)} . \tag{12}$$

Comparing Equations (7) and (9) we get

$$\gamma = \frac{3}{5} \beta . \tag{13}$$

Since the dipole contribution parameter g is independent of the artificial, cut-off parameter ξ_p, it possesses a physical sense. However, we do not have enough information to estimate the parameter g correctly. That is why we will think of g as a phenomenological parameter. The parameters discussed above fulfil the inequalities

$$\nu(0) g_D \ll f_1 < 1 \tag{14}$$

and in the problems under consideration they appear in dimensionless relations

$$\alpha^n = \left(\frac{g_D}{g_1}\right)^n \tag{15}$$

and

$$g^n = \left(\frac{g_D}{g_1^2\, \nu(0)}\right)^n . \tag{16}$$

According to (14) the quantity α^n is negligibly small if $n \geqslant 2$. Since, however, $\alpha < g$, it can happen that the quantities α and g^2 are of the same order, hence the parameter g^n can be held in account up to $n = 2$.

43. Bifurcation points

According to the theory of non-linear integral equations, solutions of the integral equations with the parameter can split at a certain fixed point, so-called a bifurcation point. Let us consider the non-linear integral equation in the operational form

$$u = \underline{S}(u, \lambda) \tag{1}$$

where λ is a parameter of the equation.

Let us assume that we know a solution $u(\lambda)$ for all values of λ and that we look for the other solutions such that

$$u = u(\lambda) + v, \tag{2}$$

the function v is determined by the equation

$$v = \underline{T}(v, \lambda) \tag{3}$$

where

$$\underline{T}(v, \lambda) = \underline{S}[u(\lambda) + v; \lambda] - u(\lambda). \tag{4}$$

Equation (3) possesses zeroth (trivial) solution for all values of λ.

The fixed value, λ_0, is the bifurcation point of Eq. (3) if for each $\varepsilon > 0$ there exists such a value of the parameter λ taken from the interval $(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$ for which Eq. (3) possesses the non-trivial solution $v(\lambda)$ satisfying the condition $\|v(\lambda)\| < \varepsilon$.

The general theory of the bifurcations is more complicated and its other problems can be found in the books [14, 71, 72]. However, the ideas presented here allow us to understand the principal properties of the non-linear integral equations.

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EFEKTY JAKOŚCIOWE WYWOŁANE OBECNOŚCIĄ ODDZIAŁYWANIA FERMICIECZOWEGO W UKŁADACH NADPRZEWODZĄCYCH I NADCIEKŁYCH

Monografia poświęcona jest teoretycznym badaniom własności układów nadprzewodzących i nadciekłych. Szczególny nacisk został położony na określenie efektów jakościowych wywołanych oddziaływaniem fermicieczowym. Rozwinięto teorię funkcji Greena uwzględniając obecność silnego pola magnetycznego. Monografia zawiera najistotniejsze rezultaty określające własności nadprzewodników, nadciekłego ^{3}He oraz mieszanin ^{3}He-^{4}He w silnych polach magnetycznych dla T = 0 i T bliskich T_c. Ponadto przebadano układy typu BCS i BW metodami reakcji liniowej. Rozwinięta teoria pozwala uwzględnić dodatkowe harmoniki parujące, oddziaływanie dipol-dipol, wpływ temperatury, wysokie częstości oraz silną niejednorodność układu. Zastosowane podejścia pozwoliły uzyskać szereg jakościowych rezultatów. W ostatniej części zostały podane pewne metody matematyczne opracowane przez autora.

КАЧЕСТВЕННЫЕ ЭФФЕКТЫ ВЫЗВАННЫЕ ВОЗДЕЙСТВИЕМ ЖИДКОСТИ ФЕРМИ В СВЕРХПРОВОДЯЩИХ И СВЕРХТЕКУЧИХ СИСТЕМАХ

Монография посвящена теоретическим исследованиям свойств сверхпроводящих и сверхтекучих систем. Основной нажим положен на определение качественных эффектов вызванных воздействием жидкости Ферми. Развернуто теорию функции Грина учитывающую присутствие сильного магнитного поля. Монография содержит самые существенные результаты определяющие свойства сверхпроводников, сверхтекучего ^{3}He, а также смесей ^{3}He-^{4}He в сильных магнитных полях для T = 0 и T близких T_c. Кроме того исследовано системы типа BCS и BW с помощью методов линейной реакции. Развернута теория позволяет учитывать добавочные спаренные гармоники, воздействование диполь-диполь, влияние температуры, высокие частоты и сильную неоднородность системы. Применены подходы позволили получить ряд качественных результатов. В последней части даны некоторые математические методы разработанные автором.

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