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Identification of mass, damping and stiffness matrices of multi degree of freedom system subjected to kinematic excitations

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Abstract: This paper presents theoretical considerations relating to the possibility of fully identifying the parameters of a numerical model describing a building structure. An input–output method with system momentum change is proposed for this purpose, thanks to which the basic matrices describing the system were identified, that is, the mass matrix M, the damping matrix C and the stiffness matrix K. The proposed way of system identification is based on the knowledge of the vibration excitation (the input signal) and the structure's dynamic response (the output signal) to the applied excitation, and the analyses are performed in the time domain. The reverse problem defined in this way consists of determining the coefficients of matrices M, C and K at any discrete point of time. In the case when the vibrations of the system are excited by kinematic excitation (ground motion), in order for the inverse problem to be solvable, either knowledge of the mass matrix or a known modification of the system masses is required. This is due to the representation of excitation forces, which in the case of kinematic excitation contains a mass matrix in their full description. This paper presents a method based on an inertial modification, that is, adding known masses to the analysed system, which entails a change in system momentum. The addition of known masses to the system being identified results in the introduction of additional known forces into the system. In this way, a heterogenous linear algebraic system of equations is obtained in the reverse problem and the coefficients of the particular matrices M, C and K are calculated from this system of equations. Moreover, considering the fact that the input signal and the output signal are known in many time points, the proposed procedure leads to a set of systems of equations.

In order to verify the correctness and effectiveness of the proposed system identification method, numerical analyses for a shear building model were carried out as part of this study. Model vibrations were induced kinematically, and the functions describing the displacements of the subgrade were assumed in the form of a harmonic, the sum of three asynchronous harmonic functions, and a real earthquake. The results of numerical analyses confirmed the effectiveness of the proposed method in each of the three excitations.

Keywords: numerical analyses; input–output method; change of system momentum; identification of system parameters; kinematic excitation.

1 Introduction

According to Newton's second law, concerning the change of the momentum of a material particle, the cause of the motion of the particle is always the force acting on this particle. In order to be able to precisely indicate at any instant the position of a material particle in space, one needs a full description of the forces acting on this particle: the external (motion inducing) force and the force of inertia. The two forces counter each other at any instant, which is the basis for the d'Alembert's principle [10, 15]. Let us assume that the material particle is tied with a base by means of an elastic constraint and a damping constraint, and the considerations are limited to one possible direction of motion of the particle. Then, the material particle will be acted on by the following forces: external active force F(t); reaction force S(t) originating from the elastic constraint, as a restitution force; reaction force R(t) originating from the damping force, as a resistance-to-motion force; and inertial force B(t) arising as a result of the motion of the particle with an acceleration. According

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to the d'Alembert's principle at any instant, all the above-mentioned forces must form a balanced system, whereby a reverse problem can be formulated. Let us assume now that in a given time interval of the observation of material particle motion, the trajectory of the motion and external active force F(*t*) causing this motion are known. Thus, generally speaking, the system's input signal and output signal are known and on their basis one can determine the system's parameters (the particle's mass, the elastic constraint stiffness and the damping constraint parameter). The reverse problem presented in this way, consisting in identifying the system's parameters on the basis of the knowledge of the input signal and the output signal, expresses an experimental modal analysis (EMA). This is one of the basic experimental methods used in, i.a., building structure dynamics, the theoretical foundations of which are extensively described in [9]. In one of this handbook's chapters, several topics connected with EMA (e.g., system modal characteristics determination, kinds of measurements, error estimation) are discussed and procedural algorithms for this method as applied to a dynamical system with one or many degrees of freedom are exhaustively described.

In specialist literature, besides classical modal analysis, one can also find operational modal analysis (OMA). Its basic assumption is that the input signal (the excitation of the analysed building structure) is not known as it is of environmental nature. Using OMA, from a system's dynamic response (the input signal) one can derive such modal parameters as eigenfrequencies, damping coefficients and vibration eigenforms. The evolution of modal analysis from the input–output type method to the input-alone type method was described by Cunha and Caetano in [6]. The authors presented experimental results of many years of research for real building structures such as dams, buildings and bridges.

For several decades owing to their popularity, the two methods have been the subject of numerous scientific publications. The theoretical foundations for the methods of identifying system parameters were provided in, i.a., studies [10, 12, 16]. In professor Ljung's monograph [16], published in 1987, many topics, from linear and non-linear systems, through methods of estimating system parameters, to model validation, were discussed. The authors of study [12], published in 1989, analysed the existing system identification methods with regard to the dynamic behaviour of building structures under various environmental loads (earthquakes, wind, excitation by water waves). In order to verify the methods, they carried out numerical analyses on theoretical models of a suspension bridge, a marine tower and a shear building. Another valuable monograph dealing with modal analysis is [10] published in 2001. It reviews modal analysis methods, discusses in detail several theoretical problems and provides examples of modal analysis application in the investigation of real building structures.

In the dynamics of building structures, the importance of modal analysis increases when a building structure (being designed or existing) is located in a seismic activity zone. Then, engineers work out structural solutions protecting building structures against environmental hazards. Hence, several scientific papers on modal analysis are devoted to (existing or model) building structures subjected to seismic loads. The dynamic parameters of structures under seismic loads are presented in, i.a., studies [2, 4, 5, 8, 11, 20, 22]. The results of modal analysis investigations carried out on laboratory models are reported in, i.a., papers [3, 14, 21].

One of the basic computational building models frequently used in building dynamics is a shear building structure. This model is particularly popular in dynamic analyses of real slender buildings to which rigid floor assumption applies. The results of modal analyses carried out on a shear building model are reported in, i.a., papers [13, 19, 21]. Their authors showed the proposed algorithms to be highly effective in identifying the system's modal parameters when the structure's excitation and dynamic response are known [19, 21] or when only the input signal is known [13].

The knowledge of the input signal is one of the basic prerequisites in modal analyses of real building structures and determines the choice of an identification method. The artificial excitation of vibrations by means of force exciters can be impossible to accomplish in a real building structure, whereas its environmental excitation can be difficult to identify. The impossibility of getting the input signal rules out the use of classical modal analysis, but one can still use operational modal analysis. However, also the latter method has a certain drawback – having no knowledge of the excitation forces one cannot precisely determine the eigenforms. This means that the identified generalized displacements are proportional to and proper for a given form, but accurate to a constant multiplier. This problem had been addressed for over a decade or so, and a new method consisting in changing the mass of an investigated system was proposed [1, 7, 17, 18, 23]. The authors of the studies showed, on the basis of experimental investigations [1, 7, 18, 23] or numerical analyses [17], that thanks to a modification of an investigated structure, consisting in a controlled change of its mass, one can obtain precise eigenforms. By adding known masses to the system, when knowing its dynamic response (the input signal), one actually introduces additional known forces into the system, whereby it is possible to properly scale the generalized displacements.

The idea of modifying the system by changing its mass is also used in the present paper. According to Newton's second law, a change in a system's mass entails a change in its momentum. Considering the above, it is proper to call the method used as a 'input–output method with system momentum change'. The known input signal constitutes the system's kinematic excitation, while the output signal is the system's dynamic response (displacements, velocities, accelerations). The numerical analyses were carried out in the time domain, changing stepwise the momentum of the system which was a shear building model. The main aim of this study was to demonstrate the possibility of full identification of the mass M, damping C and stiffness K matrices, which completely describe the analysed object. In difference to the cited scientific works and the identification methods used in them, the approach presented in this work leads to obtaining linear, algebraic systems of equations, the unknowns of which are the coefficients of the **M**, **C** and **K** matrices. This way significantly simplifies the calculations that must be performed to identify the parameters of the system compared to the methods discussed earlier (EMA, OMA). However, the proposed method requires the system mass modification, consisting in adding known masses to the analysed model and conducting the experiment iteratively. Such a procedure is necessary due to the loading of the structure in the form of kinematic excitation. This is related to the specific description of the excitation vector (input signal) in this case, which requires knowledge of the mass matrix **M**. However, if this matrix is unknown, the only way to determine the excitation vector is the known modification of the masses of the analysed system, which in effect allows obtaining non-zero solutions from algebraic systems of equations.

2 Theoretical basis

The theoretical considerations presented in this section apply to linear discrete dynamical systems which can constitute numerical models of real building structures. In a general case, the vibration of such systems are described by the equation [9, 15]:

$$\mathbf{M}\,\ddot{\mathbf{q}}(t)+\mathbf{C}\,\dot{\mathbf{q}}(t)+\mathbf{K}\,\mathbf{q}(t)=\mathbf{F}(t) \tag{1}$$

where **M**, **C** and **K** are mass, damping and stiffness matrices, respectively. The system excitation is vector $\mathbf{F}(t)$ of the excitation forces, while the system vibrations are described by a set of generalized Lagrangian coordinates contained in vector $\mathbf{q}(t)$. In the case when the system excitation is a kinematic excitation instead of a strictly force excitation, the equivalent vector of the excitation forces consists of the inertial forces produced by the motion of the base. Then, equation of motion (1) assumes the form

$$\mathbf{M}\,\ddot{\mathbf{q}}(t)+\mathbf{C}\,\dot{\mathbf{q}}(t)+\mathbf{K}\,\mathbf{q}(t)=-\mathbf{M}\,\ddot{\mathbf{z}}(t) \tag{2}$$

where $\ddot{\mathbf{z}}(t)$ is the vector representing the accelerations of the base.

The full identification of the system consists in the precise determination of matrices **M**, **C** and **K** through a stepwise change of its momentum. This change is effected using the method of a set of successive realizations. In the first step, one gets the system's dynamic response as the reference. In each successive step, the momentum is changed by adding known masses to the system, in the places and along the directions of the structure's generalized displacements. Let us then denote the matrix of the known added masses as follows:

$$\Delta\mathbf{M}_i\ ,\ i=1,2,\ldots \tag{3}$$

where the subscript 'i' designates the number of the step. In the first step, the matrix of the added masses amounts to zero matrix, and equation (2) assumes the form

$$\mathbf{M}_1\,\ddot{\mathbf{q}}_1(t)+\mathbf{C}_1\,\dot{\mathbf{q}}_1(t)+\mathbf{K}_1\,\mathbf{q}_1(t)=-\mathbf{M}_1\,\ddot{\mathbf{z}}(t) \tag{4}$$

where $\mathbf{q}_1(t)$, $\dot{\mathbf{q}}_1(t)$, $\ddot{\mathbf{q}}_1(t)$ are the known displacement, velocity and acceleration vectors, respectively, describing the structure's motion caused by known acceleration $\ddot{\mathbf{z}}(t)$ of the subgrade. In the next step, equation (2) is written as follows:

$$\mathbf{M}_2\,\ddot{\mathbf{q}}_2(t)+\mathbf{C}_2\,\dot{\mathbf{q}}_2(t)+\mathbf{K}_2\,\mathbf{q}_2(t)=-\mathbf{M}_2\,\ddot{\mathbf{z}}(t) \tag{5}$$

Besides the obvious effect on the mass matrix, the system modification can also affect the damping matrix. Therefore, the Rayleigh damping model, which takes into account a possible effect of a change in mass on the structure's damping, was adopted for further transformations. Hence, the relations between the first (reference) step matrices and the ones in the next steps (denoted as 'i+1') are as follows:

$$\begin{aligned}
&\mathbf{M}_{i+1} = \mathbf{M}_1 + \Delta\mathbf{M}_{i+1} \\
&\mathbf{K}_{i+1} = \mathbf{K}_1 \\
&\mathbf{C}_1 = \mu\mathbf{M}_1 + \kappa\mathbf{K}_1 \\
&\mathbf{C}_{i+1} = \mu\mathbf{M}_{i+1} + \kappa\mathbf{K}_{i+1} = \mu\left(\mathbf{M}_1 + \Delta\mathbf{M}_{i+1}\right) + \kappa\mathbf{K}_1 = \mathbf{C}_1 + \mu\Delta\mathbf{M}_{i+1}
\end{aligned} \tag{6}$$

In relation (6), m and k are Rayleigh damping coefficients, which in a discrete dynamical system can be interrelated with the reliable dimensionless damping coefficient of the structure. Then, damping in the discrete system applies to the structure's two selected significant eigenfrequencies wj, wk (e.g., the two first base frequencies), and it is postulated that the following relation is true

$$\frac{\mu}{\omega_j} + \kappa\,\omega_j = \gamma_j = \gamma_k = \frac{\mu}{\omega_k} + \kappa\,\omega_k \tag{7}$$

whereas

$$\gamma_j = \gamma_k = \gamma = \text{const} \tag{8}$$

where g is then the structure's reliable dimensionless damping coefficient. By transforming relations (7) and (8), one gets

$$\mu = \frac{\omega_j\omega_k}{\omega_j + \omega_k}\gamma\,,\ \kappa = \frac{1}{\omega_j + \omega_k}\gamma \tag{9}$$

Considering that in each successive step of system momentum change, the input signal (kinematic excitation) and the output signal (the dynamic response of the system) are known, and equation (5) can be generalized and written for step 'i+1' as follows:

$$\mathbf{M}_{i+1}\,\ddot{\mathbf{q}}_{i+1}(t) + \mathbf{C}_{i+1}\,\dot{\mathbf{q}}_{i+1}(t) + \mathbf{K}_{i+1}\,\mathbf{q}_{i+1}(t) = -\mathbf{M}_{i+1}\,\ddot{\mathbf{z}}(t) \tag{10}$$

Taking into account relations (6), the equation in step 'i+1' can be written as follows:

$$\begin{aligned}
&\left(\mathbf{M}_1 + \Delta\mathbf{M}_{i+1}\right)\ddot{\mathbf{q}}_{i+1}(t) + \left(\mathbf{C}_1 + \mu\Delta\mathbf{M}_{i+1}\right)\dot{\mathbf{q}}_{i+1}(t) + \mathbf{K}_1\,\mathbf{q}_{i+1}(t) = -\mathbf{M}_1\,\ddot{\mathbf{z}}(t) - \Delta\mathbf{M}_{i+1}\,\ddot{\mathbf{z}}(t) \\
&\mathbf{M}_1\,\ddot{\mathbf{q}}_{i+1}(t) + \mathbf{C}_1\,\dot{\mathbf{q}}_{i+1}(t) + \mathbf{K}_1\,\mathbf{q}_{i+1}(t) = -\mathbf{M}_1\,\ddot{\mathbf{z}}(t) - \Delta\mathbf{M}_{i+1}\left(\ddot{\mathbf{z}}(t) + \ddot{\mathbf{q}}_{i+1}(t) + \mu\,\dot{\mathbf{q}}_{i+1}(t)\right)
\end{aligned} \tag{11}$$

According to equation (4), its left side, as an equivalent vector of the excitation forces in the first step, can be substituted into relation (11), whereby one gets

$$\begin{aligned}
&\mathbf{M}_1\,\ddot{\mathbf{q}}_{i+1}(t) + \mathbf{C}_1\,\dot{\mathbf{q}}_{i+1}(t) + \mathbf{K}_1\,\mathbf{q}_{i+1}(t) = \\
&= \mathbf{M}_1\,\ddot{\mathbf{q}}_1(t) + \mathbf{C}_1\,\dot{\mathbf{q}}_1(t) + \mathbf{K}_1\,\mathbf{q}_1(t) - \Delta\mathbf{M}_{i+1}\left(\ddot{\mathbf{z}}(t) + \ddot{\mathbf{q}}_{i+1}(t) + \mu\,\dot{\mathbf{q}}_{i+1}(t)\right)
\end{aligned} \tag{12}$$

and finally after ordering expression (12), one gets

$$\begin{aligned}
&\mathbf{M}_1\left(\ddot{\mathbf{q}}_{i+1}(t) - \ddot{\mathbf{q}}_1(t)\right) + \mathbf{C}_1\left(\dot{\mathbf{q}}_{i+1}(t) - \dot{\mathbf{q}}_1(t)\right) + \mathbf{K}_1\left(\mathbf{q}_{i+1}(t) - \mathbf{q}_1(t)\right) = \\
&= -\Delta\mathbf{M}_{i+1}\left(\ddot{\mathbf{z}}(t) + \ddot{\mathbf{q}}_{i+1}(t) + \mu\,\dot{\mathbf{q}}_{i+1}(t)\right)
\end{aligned} \tag{13}$$

To simplify the notation of relation (13), let us assume that

$$\begin{aligned}
\ddot{\mathbf{q}}_{i+1}(t) - \ddot{\mathbf{q}}_1(t) &= \Delta\ddot{\mathbf{q}}_{i+1}(t) \\
\dot{\mathbf{q}}_{i+1}(t) - \dot{\mathbf{q}}_1(t) &= \Delta\dot{\mathbf{q}}_{i+1}(t) \\
\mathbf{q}_{i+1}(t) - \mathbf{q}_1(t) &= \Delta\mathbf{q}_{i+1}(t)
\end{aligned} \tag{14}$$

Then, formula (13) transforms to this form

$$\begin{aligned}
&\mathbf{M}_1\,\Delta\ddot{\mathbf{q}}_{i+1}(t) + \mathbf{C}_1\,\Delta\dot{\mathbf{q}}_{i+1}(t) + \mathbf{K}_1\,\Delta\mathbf{q}_{i+1}(t) = \\
&-\Delta\mathbf{M}_{i+1}\left(\ddot{\mathbf{z}}(t) + \ddot{\mathbf{q}}_{i+1}(t) + \mu\,\dot{\mathbf{q}}_{i+1}(t)\right)
\end{aligned} \tag{15}$$

As already mentioned, the full identification of a discrete dynamical system here consists in determining matrices $\mathbf{M}_1$, $\mathbf{C}_1$ and $\mathbf{K}_1$, but because of the use of the Rayleigh damping model, besides damping matrix $\mathbf{C}_1$, an unknown coefficient m appears in equation (15). In linear discrete dynamical systems, matrices $\mathbf{M}_1$ and $\mathbf{K}_1$ are symmetric positive definite matrices whose dimension is equal to the number of the system's dynamic degrees of freedom. Thus, also damping matrix $\mathbf{C}_1$ will satisfy this condition. Let us assume that parameter d specifies the number of the system's dynamic degrees of freedom. Then, thanks to relation (15), one can formulate exactly d equations in step 'i+1', and the total number of unknowns in the identification of the system can be expressed by the formula

$$n_\mathrm{u} = \frac{3}{2}d^2 + \frac{3}{2}d + 1 \tag{16}$$

while the number of needed iterations is given by the formula

$$n_\mathrm{r} = \frac{n_\mathrm{u}}{d} \tag{17}$$

Table 1: Number of needed iterations – changes in momentum of system being identified.

Number of dynamic degrees of freedom *d*	Number of relations given by relation (15)	Number of unknowns n_u	Number of iterations n_r
2	2	10	5
3	3	19	7
4	4	31	8
5	5	46	10
6	6	64	11
7	7	85	13
8	8	109	14

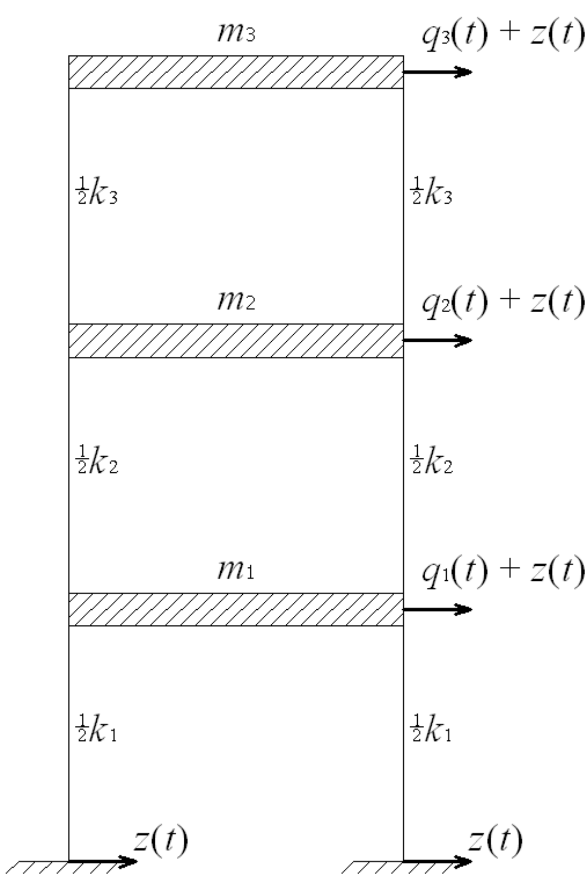

Figure 1: Shear building computational model.

In the case of fractional values, number n_r should be rounded upwards. Table 1 shows the number of needed system momentum changes for the number of dynamic degrees of freedom.

It should be added here that in the dynamics of building structures, many computational models are described using sparse matrices or simply diagonal matrices. This depends on the choice of generalized Lagrangian matrices and is connected with the occurrence of static couplings (the stiffness matrix) or inertial couplings (the mass matrix). If in the process of identifying the system's parameters one assumes generalized coordinates in such a way that no inertial couplings occur, the mass matrix will be diagonal and the number of unknowns will decrease in comparison with the one determined from relation (16). Also the number of needed iterations connected with a change of system momentum can be lower, which undoubtedly has an advantageous effect on the realization of the experiment.

3 Description of computational model

The calculation model in this paper is a discrete linear dynamical system having a shear building structure, shown in Figure 1. The model has three dynamic degrees of freedom and its vibrations are induced kinematically. An energy analysis carried out using the Lagrangian method [15] yields the following matrices describing the considered system:

– The mass matrix

$$E_k = \frac{1}{2}m_1\left(\dot{q}_1(t)+\dot{z}(t)\right)^2 + \frac{1}{2}m_2\left(\dot{q}_2(t)+\dot{z}(t)\right)^2 + \frac{1}{2}m_3\left(\dot{q}_3(t)+\dot{z}(t)\right)^2$$

$$\begin{aligned} \frac{d}{dt}\frac{\partial E_k}{\partial \dot{q}_1(t)} &= m_1\ddot{q}_1(t)+m_1\ddot{z}(t) \\ \frac{d}{dt}\frac{\partial E_k}{\partial \dot{q}_2(t)} &= m_2\ddot{q}_2(t)+m_2\ddot{z}(t) \\ \frac{d}{dt}\frac{\partial E_k}{\partial \dot{q}_3(t)} &= m_3\ddot{q}_3(t)+m_3\ddot{z}(t) \end{aligned} \tag{18}$$

It follows from the notation of the system's kinetic energy E_k and the calculations of the particular derivatives that the mass matrix is diagonal and has the form

$$\mathbf{M}_1 = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \tag{19}$$

Table 2: Computational model data in particular iteration steps.

Iteration step	1 ref.	2	3	4	5	6	7
m_1 [kg]	10.724						
m_2 [kg]	10.134						
m_3 [kg]	20.32						
Δm_1 [kg]	0	0.1	0.2	0.3	0.4	0.5	0.6
Δm_2 [kg]	0	0.1	0.2	0.3	0.4	0.5	0.6
Δm_3 [kg]	0	0.1	0.2	0.3	0.4	0.5	0.6
k_1 [N/m]	2100						
k_2 [N/m]	2100						
k_3 [N/m]	2100						
μ [1/s]	0.077466						
κ [s]	0.00094753						

– The stiffness matrix

$$
\begin{aligned}
E_{\mathrm{p}} &= \frac{1}{2}\left(\frac{1}{2}k_1+\frac{1}{2}k_1\right)\left(q_1(t)+z(t)-z(t)\right)^2+\\
&+\frac{1}{2}\left(\frac{1}{2}k_2+\frac{1}{2}k_2\right)\left(q_2(t)+z(t)-q_1(t)-z(t)\right)^2\\
&+\frac{1}{2}\left(\frac{1}{2}k_3+\frac{1}{2}k_3\right)\left(q_3(t)+z(t)-q_2(t)-z(t)\right)^2\\
\frac{\partial E_{\mathrm{p}}}{\partial q_1(t)} &= (k_1+k_2)q_1(t)-k_2\,q_2(t)\\
\frac{\partial E_{\mathrm{p}}}{\partial q_2(t)} &= -k_2\,q_1(t)+(k_2+k_3)q_2(t)-k_3\,q_3(t)\\
\frac{\partial E_{\mathrm{p}}}{\partial q_3(t)} &= -k_3\,q_2(t)+k_3\,q_3(t)
\end{aligned}
\tag{20}
$$

It follows from the notation of the system's potential energy E_{p} and the calculations of the particular derivatives that the stiffness matrix has the form

$$
\mathbf{K}_1=\begin{bmatrix} k_1+k_2 & -k_2 & 0\\ -k_2 & k_2+k_3 & -k_3\\ 0 & -k_3 & k_3\end{bmatrix}=\begin{bmatrix} k_{11} & k_{12} & k_{13}\\ k_{21} & k_{22} & k_{23}\\ k_{31} & k_{32} & k_{33}\end{bmatrix}=\begin{bmatrix} k_{11} & k_{12} & k_{13}\\ & k_{22} & k_{23}\\ \text{sym.} & & k_{33}\end{bmatrix}
\tag{21}
$$

– The equivalent vector of excitation forces

It follows from the system's kinetic energy written according to relation (18) that the structure's load constitutes the inertial forces resulting from the motion of the base

$$
\mathbf{F}(t)=\begin{bmatrix} -m_1\ddot{z}(t)\\ -m_2\ddot{z}(t)\\ -m_3\ddot{z}(t)\end{bmatrix}=-\begin{bmatrix} m_1 & 0 & 0\\ 0 & m_2 & 0\\ 0 & 0 & m_3\end{bmatrix}\begin{bmatrix}\ddot{z}(t)\\ \ddot{z}(t)\\ \ddot{z}(t)\end{bmatrix}=-\mathbf{M}_1\,\ddot{\mathbf{z}}(t)
\tag{22}
$$

– The damping matrix

Consistently with the assumptions presented in Section 2, for computer simulations, the Rayleigh damping model in which the damping matrix has the form (cf. formula 6)

$$
\begin{aligned}
\mathbf{C}_1=\mu\mathbf{M}_1+\kappa\mathbf{K}_1&=\mu\begin{bmatrix} m_1 & 0 & 0\\ 0 & m_2 & 0\\ 0 & 0 & m_3\end{bmatrix}+\kappa\begin{bmatrix} k_1+k_2 & -k_2 & 0\\ -k_2 & k_2+k_3 & -k_3\\ 0 & -k_3 & k_3\end{bmatrix}=\\
&=\begin{bmatrix} c_{11} & c_{12} & c_{13}\\ c_{21} & c_{22} & c_{23}\\ c_{31} & c_{32} & c_{33}\end{bmatrix}=\begin{bmatrix} c_{11} & c_{12} & c_{13}\\ & c_{22} & c_{23}\\ \text{sym.} & & c_{33}\end{bmatrix}
\end{aligned}
\tag{23}
$$

was assumed. Taking into account relations (19), (21), (22), (23), equation (15), constituting the basis for the identification of the analysed system, assumes the form

$$
\begin{aligned}
&\begin{bmatrix} m_{11} & 0 & 0\\ 0 & m_{22} & 0\\ 0 & 0 & m_{33}\end{bmatrix}\begin{bmatrix}\Delta\ddot{q}_1^{i+1}(t)\\ \Delta\ddot{q}_2^{i+1}(t)\\ \Delta\ddot{q}_3^{i+1}(t)\end{bmatrix}+\begin{bmatrix} c_{11} & c_{12} & c_{13}\\ & c_{22} & c_{23}\\ \text{sym.} & & c_{33}\end{bmatrix}\begin{bmatrix}\Delta\dot{q}_1^{i+1}(t)\\ \Delta\dot{q}_2^{i+1}(t)\\ \Delta\dot{q}_3^{i+1}(t)\end{bmatrix}+\\
&+\begin{bmatrix} k_{11} & k_{12} & k_{13}\\ & k_{22} & k_{23}\\ \text{sym.} & & k_{33}\end{bmatrix}\begin{bmatrix}\Delta q_1^{i+1}(t)\\ \Delta q_2^{i+1}(t)\\ \Delta q_3^{i+1}(t)\end{bmatrix}=\\
&=-\begin{bmatrix}\Delta m_1^{i+1} & 0 & 0\\ 0 & \Delta m_2^{i+1} & 0\\ 0 & 0 & \Delta m_3^{i+1}\end{bmatrix}\left(\begin{bmatrix}\ddot{z}(t)\\ \ddot{z}(t)\\ \ddot{z}(t)\end{bmatrix}+\begin{bmatrix}\ddot{q}_1^{i+1}(t)\\ \ddot{q}_2^{i+1}(t)\\ \ddot{q}_3^{i+1}(t)\end{bmatrix}+\mu\begin{bmatrix}\dot{q}_1^{i+1}(t)\\ \dot{q}_2^{i+1}(t)\\ \dot{q}_3^{i+1}(t)\end{bmatrix}\right)
\end{aligned}
\tag{24}
$$

where

$$
-\Delta\mathbf{M}_{i+1}=-\begin{bmatrix}\Delta m_1^{i+1} & 0 & 0\\ 0 & \Delta m_2^{i+1} & 0\\ 0 & 0 & \Delta m_3^{i+1}\end{bmatrix}
\tag{25}
$$

is the known matrix of the masses added to the system in step 'i+1'. In relations (24) and (25), the superscript of the expressions does not represent raising to a power, but only the number of the step.

In the analysed computational shear building structure model, the mass matrix is diagonal and in fact contains three unknowns (m_{11}, m_{22}, m_{33}). The stiffness

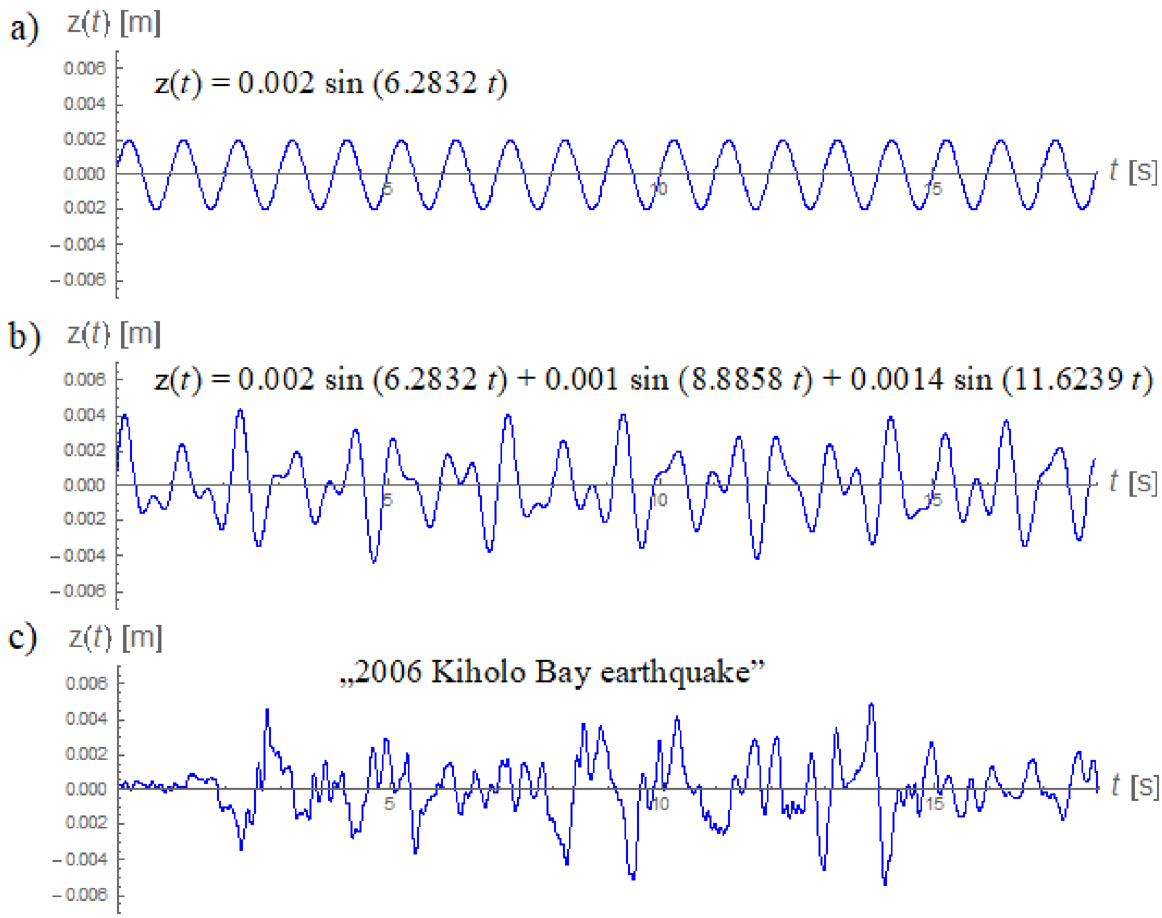

Figure 2: Diagrams of applied kinematic excitations: a) harmonics, b) sum of three harmonic functions and c) real earthquake.

matrix, although not diagonal, is symmetric, and so it contains six unknowns (k_{11}, k_{12}, k_{13}, k_{22}, k_{23}, k_{33}). The damping matrix for the Rayleigh damping model is symmetric, and so it also contains six unknowns (c_{11}, c_{12}, c_{13}, c_{22}, c_{23}, c_{33}). Taking additionally unknown parameter into account, one gets precisely 16 unknowns describing the investigated system. Relation (24) at any instant of the action of the kinematic excitation makes it possible to formulate three independent algebraic equations. Therefore, by changing the system's momentum six times, one can obtain a sufficient number of independent algebraic equations to determine the 16 unknowns. It should be emphasized that in a general case, the number of equations, the number of unknowns and the number of needed iterations (changes of system momentum) are consistent with Table 1 and formulas (16), (17). The peculiar structure of the computational model, its number of dynamic degrees of freedom and the way of choosing generalized coordinates describing the motion of the structure have a decisive bearing on the reduction of the number of unknowns. In the presented example, although the dynamic scheme has three degrees of freedom, from the overall number of 19 unknowns (see Table 1), the problem boils down to 16 unknown coefficients at any time instant.

4 Numerical analyses – description and results

4.1 General description

The identification of the parameters of the investigated system in this paper is based on the input–output method with system momentum change, and the numerical analyses are carried out in the time domain. As already known, this approach requires that the input signal and the output signal be known, which means that a numerical model for computer calculations must be precisely defined. The model data needed to perform the calculations are presented in Table 2. These are the assumed floor mass, column stiffness and Rayleigh damping model coefficient values and the values of the masses added in the particular iteration steps. In the first iteration step, the reference system response to the set kinematic excitation is determined, whereby vectors $\mathbf{q}_1(t), \dot{\mathbf{q}}_1(t), \ddot{\mathbf{q}}_1(t)$ are determined. In the second step, the known masses Δm_1, Δm_2, Δm_3 (see tab. 2) are introduced into the system and then the excitation of structure vibrations is repeated using the same input signal as in the first step. This time one gets vectors $\mathbf{q}_2(t), \dot{\mathbf{q}}_2(t), \ddot{\mathbf{q}}_2(t)$. In each subsequent iteration step, the procedure is carried out similarly as in the second step, that is, the system's masses are appropriately modified (see Table 2) and the dynamic response of the structure for the known excitation is modified.

Since the system being identified in this paper is a numerical model, all the parameters describing it are known by assumption. Thus, the model's equation of motion on the basis of which the reference response is determined has precisely this form

$$\begin{bmatrix} 10.724 & 0 & 0 \\ 0 & 10.134 & 0 \\ 0 & 0 & 20.32 \end{bmatrix} \begin{bmatrix} \ddot{q}_1^1(t) \\ \ddot{q}_2^1(t) \\ \ddot{q}_3^1(t) \end{bmatrix} + \begin{bmatrix} 4.8104 & -1.9898 & 0 \\ -1.9898 & 4.7647 & -1.9898 \\ 0 & -1.9898 & 3.5639 \end{bmatrix} \begin{bmatrix} \dot{q}_1^1(t) \\ \dot{q}_2^1(t) \\ \dot{q}_3^1(t) \end{bmatrix} + \begin{bmatrix} 4200 & -2100 & 0 \\ -2100 & 4200 & -2100 \\ 0 & -2100 & 2100 \end{bmatrix} \begin{bmatrix} q_1^1(t) \\ q_2^1(t) \\ q_3^1(t) \end{bmatrix} = -\begin{bmatrix} 10.724 & 0 & 0 \\ 0 & 10.134 & 0 \\ 0 & 0 & 20.32 \end{bmatrix} \begin{bmatrix} \ddot{z}(t) \\ \ddot{z}(t) \\ \ddot{z}(t) \end{bmatrix} \tag{26}$$

When a real building structure is the system being identified, the equation describing its vibration is not known at the start as it is the target of identification. Then, generalized Lagrangian coordinates consistent with the considered dynamic degrees of freedom are assumed. Consequently, one obtains the equation of motion in form (1) or (2), as for a discrete dynamical system. Whereas both the dynamic response and the excitation of a real system are obtained by registering, for example, acceleration by means of accelerometers, velocity by means of vibrometers and displacements by means of linear displacement transducers. In the numerical model being identified accelerations, velocities and displacements are not registered, but determined in each iteration step. Then, when one has got all the input and output signals, the system being identified is treated as a real building structure whose parameters are unknown.

In order to demonstrate that the identification method proposed in this paper is effective, the investigated system was subjected to three different kinematic excitations. In the first case, it was assumed that the motion of the base is harmonic – described by the sine function. In the second case, it was assumed that the base motion is described by a function being the sum of three asynchronous harmonic functions, which means that the resultant motion is not periodic. In the third case, the record of a real earthquake – the 2006 Kiholo Bay earthquake with a magnitude of 6.7 degrees on the Richter scale which took place in the archipelago of Hawaiian islands on 15 October 2006 – was assumed as the kinematic excitation. The earthquake data are available on the server of the United States Geological Survey which continuously monitors earthquakes in North America. Exemplary diagrams of the functions describing the displacements of the base are shown in Figure 2.

4.2 First kinematic excitation – harmonic function

In this case, the vibrations of the computational model were induced by subgrade displacement described by the following harmonic function:

$$z(t) = 0.002\sin(6.2832\,t) \tag{27}$$

In each step of iteration, connected with a change of system momentum, the dynamic response in the form of displacements, velocities and accelerations of the structure was determined. It should be noted here that the numerical integration of the equations of motion yields discrete time functions describing the motion of the structure. Then, the time step assumed for numerical analyses, connected with the integration of the equations, is a parameter inverse to the sampling frequency of the measuring apparatus used to register the vibration of real structures. Thus, in both experimental investigations and the computer calculations presented here, all the values of the input signal and the output signal are known in discrete time points. Thanks to this in each discrete time point within the experiment range, one gets a system of algebraic equations (see 28) from which the unknown parameters of the system are determined. Exemplary values of the structure's displacements, velocities and accelerations determined at instant $t = 5$s for each iteration step are presented in Table 3. In the first iteration step, equation (26) was solved, while in each subsequent step of system, momentum change equation (26) modified in accordance with relation (15) was solved.

Using the data in Table 3 and relation (24), the following system of equations was obtained for time $t = 5$s:

Table 3: Dynamic response of computational model at time instant t = 5s.

Iteration step	Displacements [mm]			Velocities [mm/s]			Accelerations [mm/s²]			
	q_1	q_2	q_3	$\dot{q}_1$	$\dot{q}_2$	$\dot{q}_3$	$\ddot{q}_1$	$\ddot{q}_2$	$\ddot{q}_3$	$\ddot{z}$
1	1.051	1.933	2.552	-4.227	-9.333	-13.72	-33.09	-53.37	-62.55	0
2	0.855	1.568	2.067	-3.574	-8.309	-12.53	-27.55	-43.05	-49.98	0
3	0.655	1.199	1.581	-3.028	-7.475	-11.56	-21.21	-32.47	-37.77	0
4	0.452	0.830	1.097	-2.602	-6.827	-10.80	-14.24	-21.83	-25.92	0
5	0.249	0.462	0.615	-2.305	-6.361	-10.23	-6.878	-11.30	-14.41	0
6	0.047	0.097	0.141	-2.140	-6.069	-9.842	0.541	-0.984	-3.222	0
7	-0.151	-0.261	-0.326	-2.102	-5.945	-9.627	7.686	9.010	7.682	0

$$
\begin{gathered}
6.530109231c_{11} + 10.22958289c_{12} + 11.86421095c_{13} - 1.960495038k_{11} + \\
-3.645966364k_{12} - 4.848365447k_{13} + 55.41966307m_{11} - 3.574125214\mu = 27.55106708 \\
6.530109231c_{12} + 10.22958289c_{22} + 11.86421095c_{23} - 1.960495038k_{12} + \\
-3.645966364k_{22} - 4.848365447k_{23} + 103.2177915m_{22} - 8.309547129\mu = 43.05295334 \\
6.530109231c_{13} + 10.22958289c_{23} + 11.86421095c_{33} - 1.960495038k_{13} + \\
-3.645966364k_{23} - 4.848365447k_{33} + 125.6921749m_{33} - 12.53276334\mu = 49.98419561 \\
11.99138447c_{11} + 18.57767897c_{12} + 21.54901416c_{13} - 3.963530244k_{11} + \\
-7.330953086k_{12} - 9.710841415k_{13} + 118.7993302m_{11} - 6.05599538\mu = 42.42620073 \\
11.99138447c_{12} + 18.57767897c_{22} + 21.54901416c_{23} - 3.963530244k_{12} + \\
-7.330953086k_{22} - 9.710841415k_{23} + 209.0141991m_{22} - 14.94947504\mu = 64.94662517 \\
11.99138447c_{13} + 18.57767897c_{23} + 21.54901416c_{33} - 3.963530244k_{13} + \\
-7.330953086k_{23} - 9.710841415k_{33} + 247.7987888m_{33} - 23.12856604\mu = 75.54706844 \\
16.25219689c_{11} + 25.05880963c_{12} + 29.1839893c_{13} - 5.991749269k_{11} + \\
-11.02771016k_{12} - 14.55974236k_{13} + 188.5633784m_{11} - 7.805749346\mu = 42.71008662 \\
16.25219689c_{12} + 25.05880963c_{22} + 29.1839893c_{23} - 5.991749269k_{12} + \\
-11.02771016k_{22} - 14.55974236k_{23} + 315.4086279m_{22} - 20.47987337\mu = 65.5016091 \\
16.25219689c_{13} + 25.05880963c_{23} + 29.1839893c_{33} - 5.991749269k_{13} + \\
-11.02771016k_{23} - 14.55974236k_{33} + 366.308967m_{33} - 32.40235651\mu = 77.76754921 \\
19.22349368c_{11} + 29.71998c_{12} + 34.88815366c_{13} - 8.025522019k_{11} + \\
-14.70992173k_{12} - 19.37005109k_{13} + 262.1456565m_{11} - 9.219147078\mu = 27.51387095 \\
19.22349368c_{12} + 29.71998c_{22} + 34.88815366c_{23} - 8.025522019k_{12} + \\
-14.70992173k_{22} - 19.37005109k_{23} + 420.7806996m_{22} - 25.44202967\mu = 45.18665013 \\
19.22349368c_{13} + 29.71998c_{23} + 34.88815366c_{33} - 8.025522019k_{13} + \\
-14.70992173k_{23} - 19.37005109k_{33} + 481.3925653m_{33} - 40.92147627\mu = 57.65662627 \\
20.87593448c_{11} + 32.63071047c_{12} + 38.76815579c_{13} - 10.04404035k_{11} + \\
-18.35269116k_{12} - 24.11906503k_{13} + 336.3435484m_{11} - 10.69771345\mu = -2.706607271 \\
20.87593448c_{12} + 32.63071047c_{22} + 38.76815579c_{23} - 10.04404035k_{12} + \\
-18.35269116k_{22} - 24.11906503k_{23} + 523.9056159m_{22} - 30.34717186\mu = 4.920854511 \\
20.87593448c_{13} + 32.63071047c_{23} + 38.76815579c_{33} - 10.04404035k_{13} + \\
-18.35269116k_{23} - 24.11906503k_{33} + 593.3106079m_{33} - 49.21184428\mu = 16.11176152 \\
21.2475239c_{11} + 33.87468607c_{12} + 40.91799336c_{13} - 12.02666078k_{11} + \\
-21.93279749k_{12} - 28.78604474k_{13} + 407.7915769m_{11} - 12.61430249\mu = -46.11674581
\end{gathered}
\tag{28}
$$

The solution of system of equations (28) yielded the following results:

$$
\begin{gathered}
m_{11} = 10.724\,\mathrm{kg}, \quad m_{22} = 10.134\,\mathrm{kg}, \quad m_{33} = 20.32\,\mathrm{kg}, \\
k_{11} = 4200\frac{\mathrm{N}}{\mathrm{m}}, \quad k_{12} = -2100\frac{\mathrm{N}}{\mathrm{m}}, \quad k_{13} = 0 \\
k_{22} = 4200\frac{\mathrm{N}}{\mathrm{m}}, \quad k_{23} = -2100\frac{\mathrm{N}}{\mathrm{m}}, \quad k_{33} = 2100\frac{\mathrm{N}}{\mathrm{m}} \\
c_{11} = 4.8104\frac{\mathrm{N\cdot s}}{\mathrm{m}}, \quad c_{12} = -1.9898\frac{\mathrm{N\cdot s}}{\mathrm{m}}, \quad c_{13} = 0 \\
c_{22} = 4.7647\frac{\mathrm{N\cdot s}}{\mathrm{m}}, \quad c_{23} = -1.9898\frac{\mathrm{N\cdot s}}{\mathrm{m}}, \quad c_{33} = 3.5639\frac{\mathrm{N\cdot s}}{\mathrm{m}} \\
\mu = 0.0774659\,\mathrm{s}^{-1}
\end{gathered}
\tag{29}
$$

Using results (27) and relation (23), parameter k of the assumed Rayleigh damping model was calculated:

$$
\begin{gathered}
\mu m_{11} + \kappa k_{11} = c_{11} \\
0.0774659 \cdot 10.724 + \kappa \cdot 4200 = 4.8104 \quad \Rightarrow \quad \kappa = 0.00094753\,\mathrm{s}
\end{gathered}
\tag{30}
$$

All the obtained results (see 29 and 30) are perfectly consistent with the data assumed for the computational model (see 26 and tab. 2.). This is evidence of the high effectiveness of the proposed input–output method with system momentum change. One should also bear in mind that analyses made using this method are performed in the time domain. Hence, in each discrete time point for which the input signal and the output signal are determined, one gets a system of equations analogous to (28). The solution of the whole set of systems of equations obviously yields sets of values of the particular model parameters being determined. In numerical analyses, these sets contain the same recurring elements, for example, m_{11} = {10.724, 10.724, ..., 10.724}. Whereas in the case of experiments conducted on real structures, the set of values determined for a particular parameter most probably will range around the expected value. This is owing to, among other things, the accuracy with which physical quantities are measured. Then, statistical calculus can prove helpful in determining the sought values of the parameters of the model being identified.

Table 4: Dynamic response of computational model at time instant $t = 9.3$s.

Iteration step	Displacements [mm]			Velocities [mm/s]			Accelerations [mm/s²]			
	q_1	q_2	q_3	$\dot{q}_1$	$\dot{q}_2$	$\dot{q}_3$	$\ddot{q}_1$	$\ddot{q}_2$	$\ddot{q}_3$	$\ddot{z}$
1	-3.578	-7.647	-11.07	-21.71	-53.28	-83.45	144.3	378.9	603.6	-240.5
2	-3.044	-6.558	-9.494	-21.76	-53.65	-84.01	149.0	363.4	552.0	-240.5
3	-2.502	-5.469	-7.946	-22.22	-53.78	-83.11	151.1	344.6	503.3	-240.5
4	-1.936	-4.385	-6.465	-22.85	-53.46	-80.94	143.3	319.3	461.3	-240.5
5	-1.344	-3.319	-5.085	-23.17	-52.44	-77.80	122.1	287.1	427.9	-240.5
6	-0.742	-2.297	-3.826	-22.70	-50.48	-74.00	89.30	250.2	402.7	-240.5
7	-0.162	-1.346	-2.697	-21.10	-47.46	-69.80	51.76	212.2	383.6	-240.5

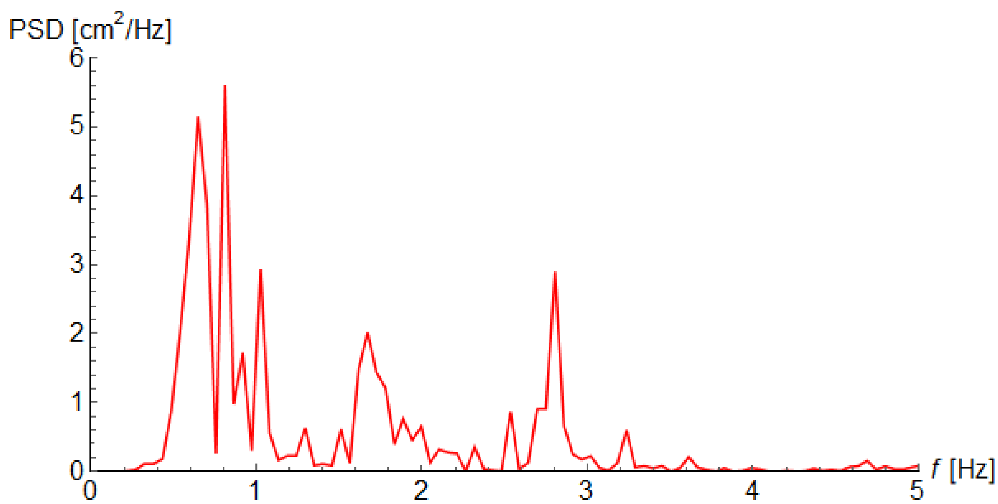

Figure 3: Spectral power density determined from displacements registered during 2006 Kiholo Bay earthquake.

4.3 Second kinematic excitation – sum of three harmonics

In this case, computational model vibrations were induced by subgrade displacement described by a function being a sum of three asynchronous harmonic functions:

$$z(t) = 0.002\sin(6.2832t) + 0.001\sin(8.8858t) + 0.0014\sin(1\ \ .6239t) \tag{31}$$

Owing to this, the obtained function $z(t)$ is neither harmonic nor periodic. The choice of such a function was dictated by a desire to see what effect the non-periodicity of the function representing system excitation can have on model identification.

The system parameter identification procedure was carried out similarly as in Section 4.2, that is, in each step of iteration connected with a change of system momentum, the dynamic response of the model was determined. Exemplary values of structure displacements, velocities and accelerations determined at instant $t = 9.3$s for each iteration step are presented in Table 4. Similarly as in the case of the first kinematic excitation, also this time equation (26) was solved in the first iteration step, whereas in each successive step of system, momentum change equation (26) modified consistently with relation (15) was solved.

Using the data contained in Table 4 and relation (24), the following system of equations was obtained at time instant $t = 9.3$s

Table 5: Dynamic response of computational model at time instant t = 7.71s.

Iteration step	Displacements [mm]			Velocities [mm/s]			Accelerations [mm/s²]			
	q_1	q_2	q_3	$\dot{q}_1$	$\dot{q}_2$	$\dot{q}_3$	$\ddot{q}_1$	$\ddot{q}_2$	$\ddot{q}_3$	$\ddot{z}$
1	5.039	7.062	9.491	17.36	23.66	13.55	-1274	-605.3	-933.9	648.8
2	4.324	6.151	8.897	25.15	22.09	14.97	-1172	-498.1	-965.6	648.8
3	3.508	5.281	8.293	31.56	19.94	16.40	-1025	-432.6	-991.9	648.8
4	2.614	4.444	7.685	35.78	17.88	17.78	-843.2	-398.6	-1014	648.8
5	1.681	3.623	7.079	37.28	16.25	19.11	-645.9	-379.1	-1034	648.8
6	0.754	2.798	6.479	36.02	14.96	20.44	-454.0	-356.7	-1056	648.8
7	-0.122	1.960	5.888	32.30	13.60	21.88	-285.5	-318.9	-1079	648.8

$$
\begin{aligned}
&0.4103211551c_{11} - 11.89047305c_{12} - 30.50430108c_{13} + 5.362554301k_{11} + \\
&+10.51327577k_{12} + 14.83812907k_{13} - 28.26435387m_{11} - 14.08876767\mu = 166.28415 \\
&0.4103211551c_{12} - 11.89047305c_{22} - 30.50430108c_{23} + 5.362554301k_{12} + \\
&+10.51327577k_{22} + 14.83812907k_{23} - 184.2992374m_{22} - 32.89418304\mu = -132.6862801 \\
&0.4103211551c_{13} - 11.89047305c_{23} - 30.50430108c_{33} + 5.362554301k_{13} + \\
&+10.51327577k_{23} + 14.83812907k_{33} - 463.941934m_{33} - 50.75469856\mu = -434.0948903 \\
&-2.075003428c_{11} - 21.34180718c_{12} - 46.56780678c_{13} + 10.59745391k_{11} + \\
&+21.12547402k_{12} + 30.15086742k_{13} + 13.80823768m_{11} - 28.67460025\mu = 324.1537817 \\
&-2.075003428c_{12} - 21.34180718c_{22} - 46.56780678c_{23} + 10.59745391k_{12} + \\
&+21.12547402k_{22} + 30.15086742k_{23} - 333.6180758m_{22} - 67.6786329\mu = -235.5087924 \\
&-2.075003428c_{13} - 21.34180718c_{23} - 46.56780678c_{33} + 10.59745391k_{13} + \\
&+21.12547402k_{23} + 30.15086742k_{33} - 964.0689238m_{33} - 104.7220983\mu = -768.1643827 \\
&-8.931268949c_{11} - 28.7009364c_{12} - 47.88619858c_{13} + 15.89947459k_{11} + \\
&+31.8413011k_{12} + 45.47860409k_{13} + 51.58352522m_{11} - 45.06878003\mu = 474.8980863 \\
&-8.931268949c_{12} - 28.7009364c_{22} - 47.88619858c_{23} + 15.89947459k_{12} + \\
&+31.8413011k_{22} + 45.47860409k_{23} - 504.2283499m_{22} - 103.7256881\mu = -302.0801065 \\
&-8.931268949c_{13} - 28.7009364c_{23} - 47.88619858c_{33} + 15.89947459k_{13} + \\
&+31.8413011k_{23} + 45.47860409k_{33} - 1452.669801m_{33} - 157.4786649\mu = -1005.666311 \\
&-18.54646449c_{11} - 32.36217053c_{12} - 35.84481265c_{13} + 21.463095689k_{11} + \\
&+42.63809288k_{12} + 60.38762045k_{13} + 7.06559793m_{11} - 63.93778492\mu = 651.0046194 \\
&-18.54646449c_{12} - 32.36217053c_{22} - 35.84481265c_{23} + 21.463095689k_{12} + \\
&+42.63809288k_{22} + 60.38762045k_{23} - 734.873516m_{22} - 139.7654111\mu = -310.5154088 \\
&-18.54646449c_{13} - 32.36217053c_{23} - 35.84481265c_{33} + 21.463095689k_{13} + \\
&+42.63809288k_{23} + 60.38762045k_{33} - 1887.872506m_{33} - 205.1549989\mu = -1166.807333 \\
&-26.73148372c_{11} - 29.76695652c_{12} - 13.03203309c_{13} + 27.36836296k_{11} + \\
&+53.4091166k_{12} + 74.52395036k_{13} - 168.8190666m_{11} - 84.01474077\mu = 901.6981065 \\
&-26.73148372c_{12} - 29.76695652c_{22} - 13.03203309c_{23} + 27.36836296k_{12} + \\
&+53.4091166k_{22} + 74.52395036k_{23} - 1036.662776m_{22} - 173.4091569\mu = -237.249631 \\
&-26.73148372c_{13} - 29.76695652c_{23} - 13.03203309c_{33} + 27.36836296k_{13} + \\
&+53.4091166k_{23} + 74.52395036k_{33} - 2244.831879m_{33} - 245.0373588\mu = -1280.029479 \\
&-28.3480409c_{11} - 18.42478678c_{12} + 17.51709969c_{13} + 33.51029929k_{11} + \\
&+63.94926615k_{12} + 87.64176381k_{13} - 473.4602865m_{11} - 101.7876232\mu = 1264.82246
\end{aligned}
\tag{32}
$$

The solution of system of equations (32) yielded the following results:

$$
\begin{aligned}
&m_{11} = 10.724\,\mathrm{kg}, \quad m_{22} = 10.134\,\mathrm{kg}, \quad m_{33} = 20.32\,\mathrm{kg}, \\
&k_{11} = 4200\frac{\mathrm{N}}{\mathrm{m}}, \quad k_{12} = -2100\frac{\mathrm{N}}{\mathrm{m}}, \quad k_{13} = 0 \\
&k_{22} = 4200\frac{\mathrm{N}}{\mathrm{m}}, \quad k_{23} = -2100\frac{\mathrm{N}}{\mathrm{m}}, \quad k_{33} = 2100\frac{\mathrm{N}}{\mathrm{m}} \\
&c_{11} = 4.8104\frac{\mathrm{N\cdot s}}{\mathrm{m}}, \quad c_{12} = -1.9898\frac{\mathrm{N\cdot s}}{\mathrm{m}}, \quad c_{13} = 0 \\
&c_{22} = 4.7647\frac{\mathrm{N\cdot s}}{\mathrm{m}}, \quad c_{23} = -1.9898\frac{\mathrm{N\cdot s}}{\mathrm{m}}, \quad c_{33} = 3.5639\frac{\mathrm{N\cdot s}}{\mathrm{m}} \\
&\mu = 0.0774659\,\mathrm{s}^{-1}
\end{aligned}
\tag{33}
$$

Using results (33) and relation (23), also parameter k of the assumed Rayleigh damping model was determined:

$$
\begin{aligned}
&\mu m_{11} + \kappa k_{11} = c_{11} \\
&0.0774659 \cdot 10.724 + \kappa \cdot 4200 = 4.8104 \quad \Rightarrow \quad \kappa = 0.00094753\,\mathrm{s}
\end{aligned}
\tag{34}
$$

Also this time all the obtained results (see 33 and 34) perfectly agree with the data assumed for the computational model (see 26 and Table 2). Based on obtained results, it is possible to formulate the fundamental conclusion that the identification method proposed in this paper is effective in the case of describing kinematic excitation using a non-periodic function that meets the conditions of oscillatory motion (limited amplitudes, passing through a neutral position).

4.4 Third kinematic excitation – 2006 Kiholo Bay earthquake

In this case, computational model vibrations were excited by subgrade displacement – the 2006 Kiholo Bay earthquake record made by the United States Geological Survey. Data, in the form of discrete functions of displacements, velocities and accelerations, relating to this earthquake are available from the USGS servers under the National Strong Motion Project. A diagram of the displacements registered during the earthquake is presented in Figure 2. On their basis, spectral power density was determined to show the dominant harmonic components of the signal (see fig.3).

The system parameters identification procedure was carried out similarly as in Section 4.2, that is, in each step of iteration connected with system momentum change, the model's dynamic response was determined. Exemplary values of the displacements, velocities and accelerations of the structure determined at instant t = 7.71s for each iteration step are presented in Table 5. Similarly as in the case of the first and second kinematic excitations, also this time equation (26) was solved in the first iteration step, whereas in each subsequent iteration step of system momentum change, equation (26) modified consistently with relation (15) was solved.

Using the data presented in Table 5 and relation (24), the following system of equations was obtained at time instant t = 7.71s:

$$
\begin{aligned}
&77.925656167c_{11} - 15.633080026c_{12} + 14.151297625c_{13} - 7.1488584815k_{11} + \\
&-9.1154927678k_{12} - 5.9366834385k_{13} + 1021.3051498m_{11} - 25.154704165\mu = 487.80454281 \\
&77.925656167c_{12} - 15.633080026c_{22} + 14.151297625c_{23} - 7.1488584815k_{12} + \\
&-9.1154927678k_{22} - 5.9366834385k_{23} + 1071.3275911m_{22} - 22.09443638\mu = -186.67215605 \\
&77.925656167c_{13} - 15.633080026c_{23} + 14.151297625c_{33} - 7.1488584815k_{13} + \\
&-9.1154927678k_{23} - 5.9366834385k_{33} - 317.76967925m_{33} - 14.967817358\mu = 280.84136637 \\
&142.02789503c_{11} - 37.158419541c_{12} + 28.500895065c_{13} - 15.310837955k_{11} + \\
&-17.812543652k_{12} - 11.977268115k_{13} + 2497.1919157m_{11} - 63.129856102\mu = 680.43173244 \\
&142.02789503c_{12} - 37.158419541c_{22} + 28.500895065c_{23} - 15.310837955k_{12} + \\
&-17.812543652k_{22} - 11.977268115k_{23} + 1726.6986834m_{22} - 39.883804858\mu = -504.41853055 \\
&142.02789503c_{13} - 37.158419541c_{23} + 28.500895065c_{33} - 15.310837955k_{13} + \\
&-17.812543652k_{23} - 11.977268115k_{33} - 579.90442042m_{33} - 32.805554204\mu = 614.10968097 \\
&184.13248993c_{11} - 57.731752723c_{12} + 42.318821939c_{13} - 24.253898714k_{11} + \\
&-26.178053538k_{12} - 18.060014519k_{13} + 4315.5560843m_{11} - 107.32616262\mu = 475.13834809 \\
&184.13248993c_{12} - 57.731752723c_{22} + 42.318821939c_{23} - 24.253898714k_{12} + \\
&-26.178053538k_{22} - 18.060014519k_{23} + 2066.9536927m_{22} - 53.653707332\mu = -858.70429861 \\
&184.13248993c_{13} - 57.731752723c_{23} + 42.318821939c_{33} - 24.253898714k_{13} + \\
&-26.178053538k_{23} - 18.060014519k_{33} - 801.49291292m_{33} - 53.353709368\mu = 987.6410692 \\
&199.22773148c_{11} - 74.046421114c_{12} + 55.551762671c_{13} - 33.58759407k_{11} + \\
&-34.395368234k_{12} - 24.121928108k_{13} + 6288.5876119m_{11} - 149.13964679\mu = -155.69481359 \\
&199.22773148c_{12} - 74.046421114c_{22} + 55.551762671c_{23} - 33.58759407k_{12} + \\
&-34.395368234k_{22} - 24.121928108k_{23} + 2261.4183226m_{22} - 65.012409086\mu = -1222.7249168 \\
&199.22773148c_{13} - 74.046421114c_{23} + 55.551762671c_{33} - 33.58759407k_{13} + \\
&-34.395368234k_{23} - 24.121928108k_{33} - 1007.6396101m_{33} - 76.43145545\mu = 1399.3134378 \\
&186.55882771c_{11} - 86.952949034c_{12} + 68.839933587c_{13} - 42.851892702k_{11} + \\
&-42.64556801k_{12} - 30.117723245k_{13} + 8207.0559541m_{11} - 180.0901066\mu = -1153.8526881 \\
&186.55882771c_{12} - 86.952949034c_{22} + 68.839933587c_{23} - 42.851892702k_{12} + \\
&-42.64556801k_{22} - 30.117723245k_{23} + 2485.1476848m_{22} - 74.812247397\mu = -1640.2708271 \\
&186.55882771c_{13} - 86.952949034c_{23} + 68.839933587c_{33} - 42.851892702k_{13} + \\
&-42.64556801k_{23} - 30.117723245k_{33} - 1221.465389m_{33} - 102.18340477\mu = 1856.0546867 \\
&149.36690756c_{11} - 100.57628124c_{12} + 83.250993827c_{13} - 51.612557515k_{11} + \\
&-51.020714688k_{12} - 36.033326961k_{13} + 9892.7843911m_{11} - 193.79297583\mu = -2396.0602879
\end{aligned}
\tag{35}
$$

The solution of system of equation (35) yielded the following results:

$$
\begin{aligned}
&m_{11} = 10.724\,\mathrm{kg}, \quad m_{22} = 10.134\,\mathrm{kg}, \quad m_{33} = 20.32\,\mathrm{kg}, \\
&k_{11} = 4200\frac{\mathrm{N}}{\mathrm{m}}, \quad k_{12} = -2100\frac{\mathrm{N}}{\mathrm{m}}, \quad k_{13} = 0 \\
&k_{22} = 4200\frac{\mathrm{N}}{\mathrm{m}}, \quad k_{23} = -2100\frac{\mathrm{N}}{\mathrm{m}}, \quad k_{33} = 2100\frac{\mathrm{N}}{\mathrm{m}} \\
&c_{11} = 4.8104\frac{\mathrm{N\cdot s}}{\mathrm{m}}, \quad c_{12} = -1.9898\frac{\mathrm{N\cdot s}}{\mathrm{m}}, \quad c_{13} = 0 \\
&c_{22} = 4.7647\frac{\mathrm{N\cdot s}}{\mathrm{m}}, \quad c_{23} = -1.9898\frac{\mathrm{N\cdot s}}{\mathrm{m}}, \quad c_{33} = 3.5639\frac{\mathrm{N\cdot s}}{\mathrm{m}} \\
&\mu = 0.0774659\,\mathrm{s}^{-1}
\end{aligned}
\tag{36}
$$

Using results (36) and relation (23), parameter k of the assumed Rayleigh damping model was calculated:

$$
\begin{gathered}
\mu m_{11} + \kappa k_{11} = c_{11} \\
0.0774659 \cdot 10.724 + \kappa \cdot 4200 = 4.8104 \quad \Rightarrow \quad \kappa = 0.00094753\,\mathrm{s}
\end{gathered}
\tag{37}
$$

Also this time all the obtained results (see 36 and 37) perfectly agree with the data assumed for the computational model (see 26 and tab. 2.). Taking into account the fact that the real earthquake signal is usually a function of many harmonic components, based on the obtained results, it is possible to draw a fundamental conclusion – the method of system identification proposed in the paper is effective for a complex description of kinematic excitation. The discrete Fourier transform performed on the real signal of the 2006 Kiholo Bay earthquake (see Fig. 3) indicates a significant influence of dozen harmonic components causing vibrations of the analysed model. In this sense, the complexity of the signal should be understood as a finite sum of harmonic functions modelling a given excitation. Despite such complexity of the kinematic excitation, the proposed identification method allows obtaining accurate M, C and K matrices, consistent with the assumed numerical model.

5 Conclusions

In this paper, the theoretical considerations regarding the possibility of full identification of a numerical model's parameters by means of the input–output method with system momentum change have been presented. The aim of the work is to identify the matrices M, C and K, describing the fully analysed discrete dynamic system. The proposed system identification method is based on the knowledge of the vibration excitation (the input signal) and the structure's dynamic response to the applied excitation. The reverse problem defined in this paper consists of determining the coefficients of matrices **M**, **C** and **K**, which in the case of kinematic excitation of the system vibrations is possible only when a known change in the momentum of the analysed object is made. In order to obtain a full description of the excitation vector (see eq. 2), the mass matrix must be known. When this matrix is not known, the system must be modified in order for the reverse problem to be solvable. Adding known masses to the analysed system is equivalent to introducing additional known forces to the equation of motion. As a result, this procedure leads to obtaining a linear, non-homogeneous, algebraic system of equations, from which the unknown coefficients of the individual matrices M, C, K are determined. If at the same time one takes into account the fact that the equation of motion of the computational model (see eq. 1, 2) defined in the time domain must be satisfied in any time point, one will get a set of systems of equations since in each discrete time point for which the values of the input signal and the output signal are known it is possible to write a system of algebraic equations.

In order to verify the correctness and effectiveness of the proposed system identification method, numerical analyses were carried out for a shear building model. The results presented in Sections 4.2–4.4 in each of the analysed cases of kinematic excitation confirmed the effectiveness of the proposed method. For both harmonic excitation, the sum of three harmonic components and the real earthquake, accurate values of the numerical model parameters were determined, consistent with the assumed values. Such precise results were probably obtained because the identified object was a numerical model whose dynamic response (the output signal) can be obtained with practically any precision. According to the description of the numerical analyses, presented in Section 4, first all the necessary geometric-material data relating to the computational model were assumed and an equation of motion was written in the base of the assumed generalized Lagrangian coordinates. Then, series of calculations, consisting in determining the model's dynamic response to the applied kinematic excitations while modifying the system masses in the particular experiment steps, were performed. Any results obtained in this way depend on the assumed calculation accuracy, which is limited only by the memory capacity of the computer performing the calculations. This is both an advantage and a disadvantage of the proposed method, as in experimental investigations conducted on real building structures, it may be difficult or simply impossible to attain the proper accuracy.

The theoretical considerations and numerical analyses performed in this paper allow to formulate a general conclusion that the proposed method can be useful in experimental studies carried out on laboratory models of building structures in order to determine the appropriate damping model applied to specific structures subjected to kinematic excitations. Many research centres in the world investigating problems relating to the dynamics of building structures have suitable measuring instrumentation and shake tables. If all the requirements of the proposed input–output method with system momentum change are satisfied, this method can be successfully used to identify model parameters. One should emphasize here the main advantage of the proposed method: mainly, the full identification of system parameters. Knowing mass matrix **M** and stiffness matrix **K**, one can determine the eigenfrequencies and eigenvectors of the analysed model. The identification of damping matrix **C**, besides making it possible to determine the damping coefficients, opens up possibilities of defining the most accurate damping model for the given building structure. Thus, the proposed method makes it possible to determine an optimal damping model (e.g., mass damping, material damping, the Rayleigh model, the constant damping decrement model) in laboratory investigations, to be used in numerical analyses, for example, at the stage of designing a real building structure. Taking into account the fact that the proposed method refers to linear systems in which the damping matrix preserves symmetry and is proportional to the M and K matrices, it is possible to show what extent the identified damping matrix is correlated with the mass matrix and to what extent with the stiffness matrix. In this sense, the potential of the presented method is limited to the identification of basic damping models, such as the Rayleigh model.

The advantages and disadvantages of the proposed input–output method with system momentum change need to be further verified in experimental investigations on laboratory models of building structures. The numerical analyses the results of which are reported in this paper have corroborated the theoretical considerations regarding the correctness of the proposed method formulation. Therefore, as part of further verification of the method, experimental investigations on laboratory models with the use of a shake table and a video measurement system are to be carried out. The results of the planned experimental investigations will be presented in the next publications.

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