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THE PRIMER ON ARBITRAGE CONCEPTIONS IN ECONOMICS: THEIR LOGICS, ROOTS AND SOME FORMAL MODELS (HISTORICAL AND BIBLIOGRAPHICAL NOTES)

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Abstract. The paper makes up the first part of a larger study devoted to arbitrage ideas, models and pricing methodology in spirit of “no arbitrage” (or fairness or transparency) demands.The work – as a whole – is entitled “Arbitrage in Economics and Elsewhere – Facts Well Known and Less Known” and consists of three papers. In the present essay we intentionally interweave “loose (informal) variations on themes” (of arbitrage theories, their applications and connotations) with (brief) demonstrations of selected formal models and some more rigorous mathematical technicalities. Some efforts are made to highlight significant economic aspects as well as to reveal a piece of mathematical “machinery” hidden behind the stories told. Nevertheless, the introductory character of the current paper causes the descriptive, philosophical and historical elements to prevail: we invoke very old roots such as Aristotle’s or Aquinata’s thoughts and then follow Cournot, Walras and Keynes works, up to the crucial paper of Miller, Modigliani. Along the way the very deep considerations on the coherency of subjective probability systems are mentioned – “the probabilistic core” of an arbitrage/no arbitrage questions (thoughts of Ramsey and de Finetti). Subsequently, the basics (finite state-space) of the modern, martingale (no arbitrage) modeling (originated by Harrison, Kreps, Pliska) is presented, as well as the “factor-type” schema of the arbitrage pricing theorem (Ross’s conception). The role played by the supplemented bibliography should be also pointed out. It significantly enters the planned communication. The author’s aim was to provide the (selected) basis, and “vocabulary” which will be useful for reading the entirety of the “trilogy” – the presented foreword really constitutes a kind of “a bibliographical note”.

Keywords: arbitrage, Walrasian law of one price, Cournot’s exchange cycle, free lunch, linear pricing, martingale, cornucopia.

JEL Classification: C02, C60, C70.

“Buy low, sell high: that is arbitrage”
D. Ellerman (1984), Arbitrage Theory:
A Mathematical Introduction

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1. Introduction

P. Dybvig and S. Ross (1987) began the explanation of the term "arbitrage" with these sentences: "An arbitrage opportunity is an investment strategy that guarantees a positive payoff in some contingency with no possibility of a negative payoff and with no net investment. By assumption, it is possible to run arbitrage possibility at arbitrary scale; in other words, an arbitrage opportunity represents a money pump". The word "risk" is absent in the cited description, but, one may add: "it is possible to attain a gain (in a market game) without any risk of incurring losses".

On the other hand, H. Varian (1987) in his brilliant, instructive article "The Arbitrage Principle in Financial Economics" (published in 1987 in the *Journal of Economic Perspective*) tells the anecdote on the fictitious arbitrage-type accident in which a Yankee farmer outsmarts an economics professor, arranging the evidently asymmetric game (unjust to the professor). The author concludes with an introductory definition of arbitrage as "arranging a transaction involving no cash outlay that results in a sure profit". In the above formulation the word risk does not appear – *explicite* – either, but it is "hidden" (in a sense), because the element of uncertainty is contained in the rules of the game (solving a riddle could not be a priori assumed). What is of more importance, he next adds the following remarks: "(...) opportunities for arbitrage do occasionally arise. But in a well-developed market with rational, profit-seeking individuals such opportunities should be very rare indeed, since profit-maximizing agents will attempt to exploit arbitrage opportunities as soon as they arise. It is generally felt that part of the definition of equilibrium in a perfect market is that no opportunities for pure arbitrage exist".

Y. Kabanov (2001), who describes the arbitrage/no arbitrage questions in the stochastic finance framework, introduces, by contrast, the element of risk at once. He writes: "We shall consider models where an investor, acting on a financial market with random price movements and having T as his time horizon, transforms his initial endowment ξ into a certain resulting wealth; let R_T^{ξ} denote the set of all final wealth corresponding to possible investment strategies. The natural question is, whether the investor has an arbitrage opportunities, i.e. whether he can get non-risky profits".

It is commonly known (and felt) that "stochastic arbitrage framework" contains, as a special case, "the deterministic" one.

Before doing a little deeper (logical and mathematical) analyses of the subject, let us present the main theses of the article. In the essay some problems belonging to the field of the arbitrage are discussed (and, what is genuinely more significant, non arbitrage conditions) in economics and elsewhere. Firstly some "prehistory" of the subject will be mentioned, accompanied by some "illuminative", general remarks.

At the subsequent step we "land" in the 19th century and remark "arbitrage-type" land topics of A. Cournot, L. Walras and G. Kirchhoff. Next we "enter" the 20th century and mention F.P. Ramsey's and B. de Finetti's statements, who discovered the natural connection linking "economic arbitrage" with the so-called coherency condition – a requirement on a system of subjective probability weights to sum to the unit (otherwise the intrinsic contradiction appears, and "money pump" effects would be observed – in "lottery-wise" formulation of modelled circumstances).

We close this part of the article by quoting the famous first theorem of Miller and Modigliani, stating the independency of firms' evaluation on structure of its capital. In a short comment we point out the significance of the perfectness of the market assumption, enabling making use of arbitrage, equalizing the procedure to determine the equilibrium price (an assumption of no taxation and lack of transaction costs were made too).

Then we pass to presenting the mainstreams of arbitrage considerations "after M.-M.". Several important contributions may be noted including Savage, Samuelson's "discovery" of the very early, "propheting" dissertation of L. Bachelier, the development of the CAPM theory and methodology (Treynar, Sharp, Littner, Mossin), E. Famas considerations of a market's efficiency leading straight to the martingale framework in (stochastic) finance markets area, and Black, Sholes, Merton's works, beginning (symbolically) the Golden Age of stochastic finance. Remember that they at first systematically made use of stochastic (diffusion) processes with continuous time, to modelling the movement of asset prices (but pricing itself based on the arbitrage benchmark). The above mentioned research studies will not be elaborated here.

However, no one but the two "scientific undertakings" can be thought of as a "proper opening" of the chapter of modern arbitrage (non-arbitrage) theory and applications. We mean, first, the decisive construction of S.A. Ross (1976) of Arbitrage Pricing Theory. As it is known, APT made an alternative model to CAPM, which helped to explain the behaviour of market participants in conditions of equilibrium (attained thanks to arbitrage approaching) on stochastic environment (via expression of returns as linear combinations of "factors" – plus residual noise). The second one ought to be referred to the scientific activity of M. Harrison, D. Kreps and S. Pliska at the turn of the 1970s and 80s (Harrison, Kreps, 1979; Kreps, 1981; Harrison, Pliska, 1981). They were the first who founded contemporary, martingale-style modelling of no arbitrage of prices (as well as – returns). The both above mentioned concepts will be discussed (shortly) in the paper.

The next (final) part of the article will be devoted to signaling the ways of generalizations of the announced ideas and models as well as to indicating some specializations of problems. Secondly, the opportunity will be made to outline a piece of mathematics hidden behind the told stories (its roots may be found into algebraic and geometric investigation of linear objects by German mathematicians from the turn of 19th and 20th centuries).

If we add, at the beginning, that the matter of much importance (for economists as well as for mathematicians) is to formulate no arbitrage conditions and connecting the absence of arbitrage with the law of one price, the prohibition of "free lunch" opportunities and linear pricing of assets, then it may be stated that the general ideas (of the paper) have been signalled.

The author's aim is to present some relations and examples connecting various problems (of a practical, theoretical and even ethical nature) when arbitrage ideas appear. We can start by going back to Aristotle's imperative and Thomas Aquinas' considerations on just pricing (Taylor, 1991). Remember that in the famous principle: "*pecunia pecuniam parere non potest*" not merely the prohibition of interest demanding is contained. Behind its main idea, the condemnation of assigning two different prices to "virtually the same" goods is hidden. In Aquinata's reasoning, in turn, the conceptions of "*equalitas valoris*", "*bonitas rei*" and "*valor intrinsecus*" (of goods, services, labor) leads him to the notion of "*iustum pretium*", which may be thought as an ingenous "antecedent" of Walrasian law of one price. Let us remark, at the time the famous forward-type speculation of the ancient Greek philosopher and mathematician, Tales from Milet. The legend (told by Aristotle (Bernstein, 1998)) claims that Tales had bought out the priority rights for an olive press in the region, and next, just before the olive season loaned the press with profit. One may ask if this (commonly regarded as a prototype of the options) clever undertaking may be – also – regarded as a kind of (pioneering arbitrage). Rather not, in the author's opinion. The same remarks might be formulated with respect to all spread-like activities (first of all banks, lombards) and even about so-called "ants" – people earning as border traders (at a very small scale: the permanent repeat of a large number, little "buy-sell" operations, in general – on foot). Why? The reason is, obviously, the violation of the law of one price, which in turn provides the arbitrage opportunities. The interesting example of functioning (temporarily) such a mechanism is given by P. Bernstein (1998): the journalist of the Wall Street Journal showed that at the beginning of the1980s the very high dollar/pound exchange created the "laboratory" situation of an arbitrage opportunity. The excursion from New York to London, staying there a few days at a good hotel, eating at expensive restaurants, buying several sweaters, whisky and porcelain was so cheap that the differences of price in England and the United States compensated (venturesome tourists) all expenditure (in fact, the investment brought more than was paid for it). So after some time, the prices equalized, the "gap pound-dollar" was reduced to the equilibrium level and then such an excursion to Harrods stopped to be profitable – one could buy the same at the 5th Avenue for the same price.

Ending this short (pre)historic remarks, it may be appropriate to invoke Ricardo's theory of comparative advantages (the famous story about Portuguese wines and English cloth). In the author's opinion, Ricardo's reasoning should be treated (also) as "propheting" discoveries, ancestoring by more than 150 years the contemporary ideas of macroeconomic arbitrage. What should be noted is the qualitative difference beetwen just described international operations and described "touristic arbitrage".

Despite the scale, the essence of arbitrage opportunities makes differences among prices of the same goods in the same time. They allow borrowing and lending at no cost (at different rates of interest), buying and selling the same things at two different prices. But such a disparity between the two rates cannot exist in the long term. The "money pump" mentioned at the beginning is not, in any way, a "*perpetuum mobile* device". Just the opposite. Arbitrageurs themselves will drive the rates together and act to annihilation. Similarly, earning by "buying low and selling high" is a short-term job. "The laws of supply and demand, cause arbitrage tends to eliminate its own possibility by reducing price discrepencies" (Ellerman, 1984). So, somewhat paradoxically, the arbitrage opportunities cause clearing of markets: sharp operators produce just prices! By the way, even if "propensity to earning by arbitrage" would turn out to be the common ("intrinsic") feature of the nature of people (acting in socio-economical circumstances) and all of us might be qualified as "arbitreurs", it is nothing to be ashamed of. Independently on "ultimate" or "profound" judgements concerning cleverness, cunning or avariciousness of Mankind as a whole, such behaviour can be simply explained as "Darwinian", "rational" and – in a sense – "deliberate" (if we accept the idea of the existence of an Invisible Hand governing (meta) markets). If the assumptions on "regularity" are made (markets are, at least approximately, perfect, (information) efficient and (eventually) complete), then arbitrage opportunities cannot exist (or – if they "accidentally" appear, they immediately vanish). "The modern study of the arbitrage is the study of the implications of assuming that no arbitrage opportunities are available" (Dybvig, Ross, 1987). Mathematically, it concerns the conditions on "ideal" processes of prices and properties of spaces of financial instruments that enable the non-arbitrage (fair, proper) pricing. Such prices guarantee attaining equilibrium on the (financial) market (also in dynamic as well as stochastic settings).

2. Further remarks: some history and connotations of arbitrage

The entirety of the issues connected with arbitrage ideas, "non--arbitrage" stochastic models and arbitrage pricing methodology may be regarded (somewhat narrowly) as the "modern classics" of the theory of finance. The above mentioned period counts at present more than fifty years (including the "prenatal stage" of the scientific activity of the above mentioned "Godfathers": M. Miller and F. Modigliani, and then R. Merton, F. Black and M. Sholes). The subject though is far from the state of a typical closed theory. There are quite opposite circumstances: the permanent evolution of notions is observed as well as the increase of the generality of considerations. The domain of the introduced models becomes richer and richer. The subject matter is, by its nature, interdisciplinary – not only within micro- and macroeconomics. It spreads from the fundamentals of mathematical economics, equilibrium theory, resources allocation, subjective probability, stochastic analysis, financial risk modelling and management, optimisation, approximation, graph theory and even physics, social choice problems and business ethics in its widest sense.

In a more detailed plan, the stress should be placed on the strict, natural connections linking no-arbitrage conditions with perfectness, efficiency and completeness of financial market models (as was mentioned in the preceding point). One may add such properties as fairness of market games, exclusion of "free lunch" possibilities (and its refined, modern formulations) as well as the Walras' law of one price. From our previous, non-formal, discussion, it follows that the violation of this rule implies the appearing of arbitrage opportunities. So – by logical law of transposition – holding of the law of one price (LOP) makes necessary conditions for the exclusion of arbitrage opportunities (but, generally LOP has “a little” weaker requirements, than no arbitrage, see (Pliska, 2005)).

This selected review of history and some crucial ideas and formal statements from the field can be also regarded as a bibliographical note, which anyway does not pretend to be complete (first of all by its “tightness” and subjectivity of choice). However, there are quoted about 100 titles of scientific articles and books, which outline the most important achievements in the area (not merely in the author’s opinion!).

The notions and statements discussed below are systemized historically – in some streams of development. We begin (in the following fragments) with a description of so-called Cournot’s cycle (on the exchange rates of currencies), shortly signal the 2nd Kirchhoff’s “voltage law” (following (Ellerman, 1984)) and enter the 20th century, remembering the fundamentals of the subjective probability theory (posed by F.P. Ramsey and B. de Finetti), which – seemingly, quite unexpectedly – united the necessary postulates of coherency of probability systems with no arbitrage demands.

Let us start with a presentation of the multiplicative arbitrage theorem, formulated about 40 years before the famous law on one price of L. Walras (Cournot, 1838; Walras, 1874-1877/1926).

Theorem on the Cournot Cycle (Ellerman, 1984). *There exists a system of absolute prices for commodities such that the exchange rates (relative prices) are prices (absolute) ratios*

if and only if

the exchange rates are arbitrage-free.

We can explain the meaning of the above statement by the example: international currency market (without transaction costs). Let us assume there is an international currency market where any currency can be transformed into any other currency. Assume, moreover, that there are m currencies and denote by r_{ij} exchange rates. The quantity r_{ij} $(i,\ j = 1, ..., n)$ informs of cost (in j-th currency) of a unit i-th currency. If we “superpone” these operations, the costs of such compositions become products of one-step transformations: so the rates multiply along any “path of exchange”. If there exists an absolute price system $p_1, p_2, ..., p_m$, then “natural way” of obtaining exchange rates is dividing (respective) prices. Thus $r_{ij} = \dfrac{p_i}{p_j}$ by definition. It is clear that such determining of quantities r_{ij} is “consistent” in the sense that if we start with currency no 1 and pass over all the cycle returning to the initial currency, then composite ratios multiply to 1: the exchange is “fair”, “proper”, arbitrage-free.

The contents of Cournot’s thesis may be written (with the use of introduced symbols) as:

$$\left\{ \exists \{ p_1, ..., p_m \} : \forall_{i,j} \; r_{i,j} = \frac{p_i}{p_j} \right\} \Leftrightarrow \left(\Pi r_{ij} = 1 \right), \qquad \text{(C.C.)}$$

around the whole cycle. So the quotient-like exchange rates are the only non-arbitrage rates. In the opposite case, we have “final” (or “global”) rate r, around the cycle:

$$\prod_{i,j} r_{ij} = r > 1,$$

and the above sharp inequality informs us about the presence of arbitrage.

The second “canonical” example concerns the “rationality requirement” of arranging of lotteries. Roughy speaking, a lottery admitting possibility of certain gain (for players or bookmakers), independently of the result of the game, is called incoherent in the sense of de Finetti (see (de Finetti, 1937; Ramsey, 1926)). Such determination of probability structure falls even on the intuition or “reasonabless” level – it allows “making a book” (against oneself). Below we give an illustriation of the phenomenon (in the theorem of total probability framework (de Finetti, 1937; Ellerman, 1984; Clark, 1993). Let

$$E_1, E_2, ..., E_n \text{ be random events,}$$

mutually exclusive and such that

$$E_1 \cup E_2 \cup ... \cup E_n = \Omega \text{ (whole space of elementary events).}$$

Let us assign probabilities “for” above events

$$p_1, p_2, ..., p_n \quad \text{where} \quad p_k = \Pr(E_k); \quad k = 1, ..., n.$$

Imagine lottery with stakes (positive or negative) $S_1, ..., S_n$ paid when events $E_1, ..., E_n$ (respectively) occur. Let us note that p_k may be interpreted as “stake” for the unit of “whole” stake S_k or the price of one unit ticket in attribute to E_k.

Define gains (for player) by formula:

$$G_h = S_h - \sum_{i=1}^{n} p_i S_i ; \quad h = 1, ..., n . \tag{*}$$

Let S_h be "unknown". We will investigate the possibilities of attaining given a priori gains.

(*) represents a system of linear equations, with determinant equal

$$D = \begin{vmatrix} 1-p_1 & -p_2 & ... & -p_4 \\ -p_1 & 1-p_2 & ... & -p_n \\ ... & ... & ... & ... \\ -p_1 & -p_2 & ... & 1-p_n \end{vmatrix} = 1 - \left(p_1 + ... p_n \right) . \tag{dFD}$$

If $D \neq 0$, then it is possible to adjust stakes S_h to obtain any values for G_h. So if $\sum_{i=1}^{n} p_1 \neq 1$, then we meet the incoherency of the "proposed" probability weights, which causes opportunity for arbitrage. Any system of gains is attainable, so the arranged lottery represents a typical "money pump". Therefore, it is necessary to demand "coherency condition" equivalent with no arbitrage restriction:

$$\sum_{i=1}^{n} p_1 = 1 .$$

In the illuminating paper of D. Ellerman (1984) some interesting examples of "arbitrage/no arbitrage" situations are given, which help reveal a common denominator of such mechanisms. The author reports an important statement of physics, the 2nd "voltage" law of Kirchoff, and argues (succesfully!) that its essence is in fact in the spirit of no arbitrage requirement. It is worth noting that Kirchhoff's discovery was preceded by ten years by Cournot's work (Kirchhoff, 1847) (the situation which is similar to the sequences of Bachelier, Einstein, Smoluchowski, Wiener in the context of the "history of Wiener process"). The mentioned Kirchhoff's law may be stated (in additive form – contrary to the multiplicative formulation of Cournot's statements):

Kirchhoff's Second Law. *There exists a system of potentials at the nodes of a circuit such that the voltages on the wires between nodes are the potential differences if the voltages add to zero around any cycle.*

In other words, the algebraic sum (and, more generally, respective directed integral around the arc) of drops of voltage at elements of (closed) electrical circuit equals to the algebraic sum (integral) of electrical forces in the circuit. The "physical no-arbitrage condition" can be described as follows: the balance of voltages around arbitrary closed circuit (cycle) is zero.

We have already mentioned that circumstances in which two perfect substitutes must trade at the same prices (LOP) are implied by the no-arbitrage condition. It should be, however, remembered that LOP is less restrictive than the absence of arbitrage because it deals only with the case in which two assets are identical but have different prices. Nevertheless, it seems to be necessary for accomplishing the introductory consideration to remember the parity theory of forward rate exchange based on the LOP, which was first formulated by Keynes and developed by Einzig (Keynes, 1923; Einzig, 1937). We will follow Dybvig, Ross (1987).

Let *s* denote the current spot price of say, Deutschmarks, in terms of dollars, and let *f* denote the forward price of marks one year in the future. The forward price is the price at which agreements can be struck currently for the future delivery of the marks with no money changing hands today. Also, let $r_\$$ and r_m denote the one-year dollar and mark interest rates, respectively. To prevent an arbitrage possibility from developing, these four prices must stand in particular relations. We will not report the details, and argumentation (replicating Keynes' original reasoning). Instead of doing so, we formulate the conclusion, given by the following statement: the prevention of arbitrage will enforce the forward parity demand (equation):

$$(1+r_\$)/(1+r_m) = f/s. \tag{KP}$$

The article of D. Ellerman is devoted, mainly, to graph theory and algebraic (general) approach to arbitrage. But he gave one more example of (additive!) arbitrage in economics – the Koopmans and Reiter "transportation model" (Koopmans, Reiter, 1951). We will not cite their concept (despite their virtual importance) and suggest the interested reader see Ellerman's original text.

It is quite impossible to continue the discussion of contemporary research in arbitrage area without mentioning Miller, Modigliani's works (Modigliani, Miller, 1958). The detailed analysis of the content of their famous article (concerning, roughly speaking, problems of valuation of a firm) is not our task here. From our point of interest, the significance of the article "The Cost of Capital, Corporation Finance and the Theory of Investment" consists of its methodological aspect – the pioneering (strict and formalized) reasoning based on arbitrage phenomenon and techniques. The revolutionary character of the article (which involved the virtual "scientific storm" both at the US universities as well as among practitioners of finance "from Wall Street") was born in "valuation sphere": the brave thesis of the authors, claimed the independence of a value of a firm on the structure of its capital, namely, the proportion of its own capital and the debt. They argued through the arbitrage-effect, which implies equalizing of asset prices characterized by the same level of risk. When the assumptions of perfectness of market are fulfilled, then the prices of firms with identical risk levels and generating the same incomes equalize because the investors will sell overvalued shares and, at (almost) the same time, buy the undervalued ones. This is the cause of decreasing the prices of the former asset, and increasing the prices of the later ones – up to equalizing their levels. This may be seen as a kind of "*quasi-tâtonnement*" procedure determining the fair prices – equilibrium prices. The role that the markets perfectness plays here makes this assumption the inherent property. Kuziak (2000) quoted the interesting (somewhat "perverse") opinion, defining perfect markets, as "markets, for which the first theorem of Miller, Modigliani holds".

One may say that the work of Miller, Modigliani summarized the "old epoque" and opened the "new age" of research studies in the field of valuation and pricing in economics. We close this fragment by quoting the famous Miller's remark about Modigliani (at the beginning of their cooperation (Bernstein, 1998)): "Franko has a mind of arbitrageur, the Italian profiteer of financial market". Their cooperation will turn out to be fruitful in a much more serious sense: it greatly influenced the whole philosophy and methodology of pricing goods (and, especially, primary and derivative assets).

3. The main streams of arbitrage pricing modelling – the outline of the "modern classics"

In the series of articles, beginning with the paper (Ross, 1976), S. Ross presented the alternative with regard to the then prevailing methodology of CAPM – as a description of equilibrium on capital markets. It was Arbitrage Pricing Theory which also attempted to "good modelling" of the stochastic structure of capital assets' returns. G. Huberman (1982) writes: "(...) every investor believes that the stochastic properties (of capital assets' returns) are consistent with factor structure. He (S. Ross) heuristically argues that if the equilibrium prices offer no arbitrage opportunities, then the expected returns on those capital assets are approximately linearly related to the factor loadings (which, in turn, are proportional to the returns' covariances with factors)". It is worth noting a "graphical" similarity of CAPM and APT: both models are one-period models, in both models the linearity of expression explaining the returns' behaviour is assumed. Nevertheless, there are fundamental differences between them, lying in the interpretation of explanatory variables and "their" coefficients. At the "operational" part, APT is based on (statistical) factor analysis ideology, which concerns both, determining factors as well as corresponding loadings (in contrast to the regression philosophy of Sharp and others models). In the APT approach much weaker assumptions are made on properties of random variables entering the model, which, in turn, enable to derive generalizations in spirit of approximation theory (in Hilbert spaces and beyond (Huberman, 1982; Reisman, 1988; Clark, 1993)). Finally, the detailed analysis of APT methodology reveals that, in fact, we are not dealing with the single problem, but a denumerable family of models is defined expanding by succesive cumulation of "new compenents".

The following formal statement of the APT model may be found in any textbook on stochastic finance or capital markets field (see (Huberman, 1982)).

The first assumption of the model is that investors make, in a sense, a homogenic collection. All of them believe that $N \times 1$ vector of single period random returns on capital assets r satisfies the "generating equation":

$$r = A + Bf + e, \qquad \text{(RAP)}$$

where r and e are $N \times 1$ vector of random variable, (factors), A is $N \times 1$ vector and B is an $N \times K$ matrix. With no loss of generality one can normalize this condition, to make

$$E(f) = E(e) = 0$$

(where E denotes the expectation operator) to obtain $E(r) = A$.

The APT asserts the existence of $(K+1)\times 1$ vector of risk premia u, and $N \times N$ positive definite matrix Z, and a constant such that

$$(A - Cu)Z^{-1}(A - Cu) \le a, \qquad (**)$$

where the $N \times (K+1)$ matrix C is defined by putting together some $N \times 1$ (column) vector i with matrix B: $C = (i, B)$. The positive definite matrix

Z is often covariance matrix $E\left(e \cdot e^T\right)$. The exact arbitrage pricing obtains if (**) is replaced by *A* = *Cu*: it means that each component of the vector *A* is linear combination on the corresponding row of the matrix *B*. An approximate arbitrage is indicated by (**): the smaller constant a, the better approximation. The exact arbitrage corresponds, of course, to the case *a* = 0.

According to the taken "essayistic formula" of the present primer, we will not discuss the technicalities (such as an algorithms or proofs). Some remarks on (very interesting) ways of generalizations (modern, abstract setting) we put off to the next point. Here we merely point out – for the second time – that the postulated shape of relations among financial variables is implied by the assumption of absence of arbitrage (and functioning of an equilibrium prices on markets), which was formally proved by Ross's followers in the field (including Ross himself (Kreps, 1981; Connor, 1984; Huberman,1982)). The second fact of theoretical importance is linearity of pricing formulas as well as recognizing that *N*-dimensional objects (variables, states) are to be approximated by the objects belonging to some *K*-dimensional space: the "ideal" would be attained when "the small *K*-space" span "the large *N* space".

Now we leave "the Ross' path" and enter "the stream of Harrison, Kreps and Pliska."

So let us pass to the general formulation of stochastic (dynamic) modelling of asset processes admitting fair (no arbitrage) pricing. The most general (and, at the same time, the most primitive) schema looks like this (Kabanov, 2001). We monitor the price movement at two moments: at the starting point $t = 0$ and at the end of the time horizon $t = T$. Let us assume, in addition, that the market is frictionless (taxation, transaction costs) and without constraints (short sale). Making use of the symbols introduced at the beginning, we notice that R_T^ξ and R_T^0 are linear subspaces of the space of *L*, real random variables and $R_T^\xi = \xi + R_T^0$, where by ξ the initial endowment was denoted. Intuitively, the demand of an absence of the arbitrage opportunity lies in excluding positive gain from zero endowment. Denote the set of all nonnegative (almost sure) "potential random outputs" at T by symbol L_+^ξ. Then the only element fulfiling the above restriction is constant variable Θ, which is zero with probability one. The above may be shortly written in symbols as a (formal) No Arbitrage Opportunities (NA) condition

$$R_T^0 \cap L_T^\xi = \{\Theta\} \qquad \text{(NA)}$$

(we operate with elements of function space) L°.

Let us remain, for a moment, at this one-period case: $t \in \{0, T\}$. We introduce a few notions which are necessary for carrying out more detailed discussions: the model that we will consider still could have been qualified as a trivial one, however the framework of stochastic processes will appear (Courtault et al., 2004; Pliska, 2005; Jakubowski et al., 2006). Let ξ denote (from now) the *d*-dimensional vector of price increments of d basic securities, the *numeraire* is a traded asset. The portfolio strategy is just a deterministic vector $h \in R^d$ and the portfolio increment is the scalar product $h\xi$. In such a setting the market is said to fulfill no arbitrage requirement if the following equivalency holds:

$$h\xi \geq 0 \Leftrightarrow h \cdot \xi = 0. \qquad \text{(NA')}$$

In the simplest case – the finite state – space Ω – the (NA) property is equivalent to the existence of the scalar random variable Z such that

(α) $Z > 0$ (almost sure, with respect to given prob. *P*),

(β) $EZ = 1$,

(γ) $E(Z\xi) = \Theta$ ($Z\xi$ is random vector).

Denote the induced by Z probability measure ZP by $\overline{P}$. The economists named the triple $\left(\Omega, \mathbf{F}, \overline{P}\right)$ "risk-neutral world", but probabilists would prefer "martingale world" or "NA equivalent martingale measure world". The reason will be clear soon. Three papers (Harrison, Kreps, 1979; Kreps, 1981; Harrison, Pliska, 1981) founded the "modern martingale model world" in arbitrage considerations of the price movement in the "real world". Several remarks might be made at this moment:

(i) In the simplest case the (NA) condition is equivalent to the existence of a certain separating hiperplane, or (pricing, linear) functional or (discrete) probability measure m (in the last – measure framework – the requirement of NA may be formulated by statement: $\exists$ "separating" probability measure m such that

$$E_m(\eta) = 0 \qquad \forall \eta \in R_T^0,$$

which holds also for more general cases).

(ii) In the simplest case the main idea of arbitrage considerations ("beyond the deterministic or stochastic setting") can be revealed. The definitions in LeRoy, Werner (2001) say: "a strong arbitrage is a portfolio *h* that satisfies $hX \geq 0$ and $ph < 0$ (*X* – vector of increments, $p = [p_1, \ldots, p_n]$ – prices), and an arbitrage is a portfolio h that satisfies $hX \geq 0$ and $ph \leq 0$

with at least one strict inequality". Hence, the definition of no arbitrage is "graphically identical" with the introduced earlier – in "a stochastic world".

(iii) The previously mentioned observation – on linearity of pricing – may be formulated as a theorem (LeRoy, Werner, 2001, p. 26): "the payoff pricing functional is linear and strictly positive if there is no arbitrage".

(iv) In the paper devoted to resolving a valuation problem in the foundations of arbitrage price theory, Clark (1993) writes: "An arbitrage opportunity is essentially a feasible contingent claim with positive net return across all states of nature. In other words an arbitrage is a 'free lunch'". As a matter of fact both the terms: "arbitrage opportunity" and "free lunch possibility" have almost the same meaning because the later (FL) contains a bit more subtle (topological) conditions. The notion was introduced by D. Kreps (1981) in context of infinite state spaces and, further, refined to the aims of description of arbitrage phenomena for processes in continuous time (see also (Delbaen, Schachermayer, 1994)). We will not comment on these sophisticated problems in this introductory essay.

Let us present the general statement of the problem of characterizing of NA stochastic dynamics in the multi-period discrete-time, finite horizon setting. For the finite state space this problem was solved by M. Harrison and S. Pliska (1981) and next generalized to the arbitrary state space by R. Dalang, A. Morton and W. Willinger (1990). The formulation became (in a sense) standard. We follow Kabanov, Stricker (2001) and will choose some terminology, symbols and the main thesis from their paper (see also (Pliska 2005)).

Let $(\Omega, \mathbf{F}, P)$ be a probability space acquired with a finite discrete-time filtration $(\mathbf{F}_t;\ \ t = 0, 1, ..., T;\ \mathbf{F}_T = \mathbf{F})$, and let $S = (S_t;\ \ t = 0, 1, ..., T)$ be an adapted d-dimensional process (the definitions of the above used terms, from stochastic analysis, may be found, e.g., in Jakubowski et al. (2006). Let $R_T := \{\xi : \xi = H \cdot S_T,\ \ H \in \mathbf{P}\}$, where $\mathbf{P}$ is the set of all predictable d-dimensional processes (i.e., H_t is $\mathbf{F}_{t-1}$ – measurable) and

$$H \cdot S_T = \sum_{t=1}^{T} H_t \Delta S_t, \qquad \Delta S_t := S_t - S_{t-1}\,.$$

Put $A_T := R_T - L_+^0$; $\bar{A}_T$ is the closure of A_T in probability, L_+^0 is the set of non-negative random variables.

Theorem (EFTMF, EFTAP). *Somewhat untypically, we preceed the formulation of the theorem by an explanation of the "mysterious" abbreviations, placed in brackets:*

(i) *FFTMF = The First Fundamental Theorem of Mathematical Finance,*

(ii) *FFTAP = The First Fundamental Theorem of Asset Pricing*.

Both these terms are in use now. Sometimes it is noted that the second one obeys, in fact, the larger area of problems rather than "single statement" (originated by Ross's idea), which in any case is not wrong – from the general point of view. However, at first glance it may be literary interpreted as a variant of Ross's model, which in turn, seems to be misleading (e.g. see (Delbaen, Schachermayer, 2006)). So the FFTMF might be regarded as a bit "safe" and unambiguous. Let us pass to the formulation of the theorem.

The following conditions are equivalent:

(a) $A_T \cap L_+^0 = \{\Theta\}$,

(b) $A_T \cap L_+^0 = \{\Theta\}$ and $A_T = \bar{A}_T$,

(c) $\bar{A}_T \cap L_+^0 = \{\Theta\}$,

(d) there is a probability $\bar{P} \sim P$ with $\frac{d\bar{P}}{dP} \in L^\infty$ such that S is $\bar{P}$ – martingale.

"Translate" the above to the "language" of mathematical finance. In this case the "*numéraire*" is traded security, S describes the movement of prices of risky assets, and $H \cdot S_T$ is the terminal value of self financing portfolio (there is no "exogenic inflow" of capital, all increments are earning from changes of prices of assets held in the portfolio). Condition (a) is interpreted (directly) as the absence of arbitrage (it appeared in the earlier discussed one-period setting). Its intuitive (equivalent) form is the implication:

$$H \cdot S_T \geq 0 \Rightarrow H \cdot S_T = 0 . \qquad \text{(NA')}$$

The condition (d) makes the statement of special importance for the meaning of "mechanics" of dynamics of no-arbitrage markets. It claims the existence of so-called equivalent martingale measure, which enables treating market as fair game, which forces "fair pricing". One may regard it as a "crucial discovery" – from the probabilistic point of view.

It is also worth noting the fact that a martingale structure of stochastic evolutions expresses some form of (stochastic) dynamic equilibrium and "quasi-stationarity". Moreover, it is strictly connected with the informational efficiency of markets *a'la* E. Fama. There are also (indirect) links between such models of markets' functioning and validity of rational expectations hypothesis (of Muth, Lucas and others).

Let us make one more observation. The condition (a) is related ("in spirit") to the standard assumption (or – postulate) on the (scalar or vector) production function $f : R_+^n \to R_+^m$: impossibility of obtaining anything from nothing, known, in mathematical economics "jargon" as a prohibition (or lack of) the cornucopia phenomenon:

$$f(\Theta) = \Theta .$$

In fact, the (NA) demand is regarded as a generalization of this classic postulate (Malawski, Ćwięczek, 2005).

Let us mention, at the end of the current point, the formal statements of the law of one price. In its "classical form" the LOP may be stated as follows: "all portfolios with the same payoff have the same price. In symbols: $\forall h, h' \in R^n$ (portfolios, investing strategies) the implication below holds"

$$hX = hX' \Rightarrow ph = ph' \text{ (LeRoy, Werner, 2001, p. 15).}$$

In the stochastic setting (just discussed) the idea of LOP is expressed as follows: we say that the model satisfies the law on one price at the date $t = 0$ if the equality:

$$\xi + H \cdot S_T = \xi' + H' S_T$$

(where $H, H' \in \mathbf{P}, \quad \xi, \xi' \in L^0(\mathbf{F}_0)$) implies that $\xi = \xi'$ (a.s.).

In the article by Courtault et al. (2004), it is stressed that the above condition can be written in an alternative (equivalent) manner as

$$R_T \cap L^0(\mathbf{F}_0) = \{\Theta\}.$$

The similarity of the above formula with condition (a) in FFTFM is not accidental. The notions (NA) and (LOP) are related, and (in some – formal – circumstances) coincide (Courtault et al. 2004).

4. A few sentences on extensions, generalizations, as well as specifications of told stories. "Table of personages"

Now we present a handful of information on ways of contemporary elaboration in the discussed field. Some generalizations and connotations will be signalled, including also the elements of mathematical skeleton, supporting the most significant statements of the arbitrage theories. These several facts were not chosen randomly, but they constitute a selected choice. The contents of the 2nd paper of the series consists (mainly) of the expanding and more detailed investigations of the ideas announced at the current point. At the end (of the fragment), we outline the kind of table of personage – in the rectangles we place clusters of "VIPs" in a historical (as well as) subject perspective (we have in mind scientists whose thoughts and topics influenced – directly or indirectly – the discussed problems).

Let us begin with the 2nd FTFM. To this aim, notions like completeness of the (financial) markets are needed (each asset is attainable as a result of portfolio consisting of basic assets – replicability). Then not only arbitrage (martingale equivalent) measures exist, but there exists only one such a measure (guaranteeing (NA) condition). In the above, "naive" formulation the "second fundamental theorem" holds for simple cases. In more general situations (the space of states, time) more subtle restrictions turned out to be necessary (sometimes – sufficient too). D. Kreps, followed by F. Delbaen and W. Schachermayer (Kreps, 1981; Delbaen, 2002; Delbaen, Schachermayer, 1994), introduced such (sophisticated) notions as"No Free Lunch", "No Free Lunch with Bounded Risk", "No Free Lunch with Vanishing Risk" (NFL, NFLBR, NFLVR (Kabanov, 2001)).

The second remark concerns the fact that the martingale framework turned out not to suffice for the adequate description of the observed phenomena: the semi-martingale setting appeared. Similarly, "classic" diffusion process must been have substituted by general Levy and stable processes (Kabanov, 2000).

Along the "Ross's stream" the significant development and generalizations have been observed since the 1980s. D. Kreps, D. Reissman, S. Clark (among others) reworked Ross's ideas in the very abstract setting: Hibert spaces, Banach spaces, Hansdorff, locally convex, topological lattices were taken as spaces of contingent claims (and marketed-factor-claim subspaces) (Kreps, 1981; Chamberlain, 1983; Reisman, 1988; Clark, 1993). Such a level of generality enables the unique treatement of all kinds of "arbitrage problems" – so it is not merely "*ars pro arte*". Y. Kabanov and D. Kramkov (1994) initiated research on so-called large financial markets. They were the first who consequently exploited the vital idea of Ross's original proposition: one deal in fact, with a whole sequence of arbitrage pricing (by factors' combinations) problems, which appear through a successive cummulation (consistent extensions) of "small" factor spaces. At Kabanov, Kramkov's construction the kind of a "bridge linking two main streams of arbitrage problems" was built (they appealed the notion of contiguity of sequences of probability measures introduced by Le Cam in 1960 (see (Roussas, 1976)).

Tables of personages

Aristotle Thomas Aquinas

D. Ricardo A.A. Cournot L. Valras

F.P. Famsey, B. de Finetti J.M. Keynes

K.J. Arrow, G. Debreu (1954) M. Miller, F. Modigliani (1958)

F. Black, M. Scholes, R. Merton (1973) The (symbolic) date: beginning of the "Golden Age"of Modern Mathematical Stochastic Finance S. Ross (1976) – APT J. Cox, S. Ross, R. Roll, J. Ingersoll, M. Rubinstein (1979) Simplification and refinement of Ross

M. Harrison, D. Kreps (1979) Arbitrage and martingales D. Kreps (1981) Arbitrage and equilibrium – infinity many commodities, free lunch M. Harrison, S. Pliska (1981) Stochastic analysis, FTAP-s, continuous time, finite state space D. Duffie (1986) Stochastic equilibria, spanning number of "the martingales space" of market

G. Huberman, G. Chamberlaine, G. Connors, H. Reisman, S. Clark (1982-1993) Correct mathematical formulation and generalization of Ross's conception (in spirit of factor analysis, factor Hilbert space and beyond) S. Le Roy, J. Ingersoll, F. Milne, T. Page, J. Werner (1980-2000) Arbitrage equilibrium (mathematical economics "mixed" with stochastic models)

K. Back, S. Pliska (1990-1991) Arbitrage for general process R. Dalang, R. Morton, W. Willingerr (1990) Theorem of Harrison-Pliska for general state space F. Delbaen, W. Schachermayer (1994, 1998, 1999) Series of significant generalizations of FTAP-s Ch. Stricker,H. Fölmer, Yu. Kabanov, D. Kramkov (1994, 1997, 1998, …, 2000s) Optional decomposition of semi-martingales, large financial markets, generalizations "for Stochastic Finance" E. Jouni, H. Kallai (1995) FTAP-s with transaction costs M. Fritelli, J. Cochrane, J. Saá-Requejo (1999-) The theory of value coherent with No-arbitrage, good deal criterion – "beyond the arbitrage"

Yu. Kabanov Festschrift (2008-2009) – "Symbolic resumé of the epoque" Conference and Jubilee Volume: F. Delbaen, M. Rasonyi, Ch. Stricker (Eds.), *Optimality and Risk – Modern Trends in Mathematical Finance*, Springer, 2009, The Kabanov Festschrift

"ARBITRAGE AND GEOMETRY" P. Gordan (1873) The earliest "alternative theorem" J. Farkas (1901) Matrices and hiperplanes, LEMMA J. Minkowski (1910, 2nd ed.) Linear inequalities E. Stiemke (1915) Separating hiperplanes F. Riesz (1909, 1910) Representation of linear functionals H. Hahn (1927), S. Banach (1929) On extending of linear functionals; "corrolary" → Theorem on separating of convex sets in locally convex topological linear spaces J.A. Yan (1980)-D. Kreps (1981) "Specialization" above topics to the semi-martingale analysis

Some authors pointed out the connections of arbitrage problems with stochastic dominance (which is intuitive and easy to see via the violation of the law of one price, at least in the simplest, one-period, finite-state case (Pliska, 2005; Jarrow, 1986)). R. Green and S. Srivastava noted the relation between risk aversion and arbitrage (Green, Srivastava, 1986), D. Kesley and F. Milne (1995) considered APT in the non-expected utility preferences case, D. Kesley and E. Yalcin (2007) studied the APT with incomplete preferences. Problems of (non) arbitrage pricing appear, in a natural way, in insurance mathematics and in the theory of term structure (examples will be shown in the next papers of the series). Now we propose to have a look at the tables below. Coming back to the fundamentals, we repeat (after (Kabanov, 2001)) that the geometric essence of the simplest case is hidden in problems of the separation of convex sets (in R^n), and next: identification of the separating functional as a finite (eventually – probability) measure. These themes were elaborated by P. Gordan, J. Farkas, J. Minkowski and E. Stiemke and next (in different framework) by F. Riesz, H. Hahn and S. Banach. The specialization of the above topics from functional analysis to stochastic analysis is due to J. Yan (1980) and D. Kreps (1981).

5. Conclusions

We could observe the rich history of arbitrage-type thinking, these ideas appearing at the stage of classical economics and the "explosion" of research studies – invoking, directly or indirectly, this logic – after the Modiglini-Miller contributions. This seemingly virtually "non academic" problems induced very serious investigations: in the equilibrium theory (economics), generalized (abstract) factor analysis (statistics, capital markets theory), "geometrical" stochastic analysis (martingales, stochastic finance). The paper constitutes an informal introduction to the rest of the series on arbitrage as well as the outline of the evolution and development of the field and the bibliographical (selected) notes. The author's aim was also to point out the many-sidedness of connotations of no-arbitrage (pricing) models: inside economics, inside mathematics (especially, stochastic processes) as well as the interdisciplinary aspects of these problems.

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