RISK PROCESSES WITH DEPENDENT CLAIM SIZE AND CLAIM OCCURRENCE TIMES*

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Summary: The paper is devoted to the risk process, when the claim amount and the inter-claim times may be dependent. The impact of the degree of dependence on the probability of ruin is investigated. The three cases are studied: the case when the joint distribution has the bivariate exponential distribution and when the dependent structure is described by FGM and Clayton copulas. The comparison with the previous study is made too.

Keywords: risk process, dependence, probability of ruin, copula.

1. Introduction

The paper is devoted to the risk process, when the claim amounts and the claim occurrence times may be dependent. In the classical approach [Rolski et al. 1999; Ostasiewicz 2000] we assume that all random variables and processes are independent. This assumption is very nice and useful from the mathematical point of view, but it is often unrealistic. In practice some random variables may be dependent. For example, in investigating natural catastrophic events, we meet such variables. The total claim amount on the occurrence of a catastrophe, e.g. the earthquake, and the time elapsed since the previous catastrophe are often dependent [Cossette et al. 2008]. Bigger damages occur when the time between the claims is longer [Boudreault et al. 2006]. Also in auto insurances the size of the next claim amount depends on the time elapsed since the last claim [Ambagaspitiya 2009]. The degree of such dependencies is rather small, but it exists.

In the paper we will investigate the probability of ruin for the above dependent risk model. The influence of the degree of dependence on the ruin probability is studied. This paper is a continuation of

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the previous works done by the author. He has studied the risk model where the claim amounts might be dependent [Heilpern 2010a, 2010b] or the interclaim times might be dependent [Heilpern 2012]. In these cases we observe that the relation between the probability of ruin and the degree of dependence may be irregular and nonmonotonic. It can be essentially dependent on the value of the initial capital and the probability of ruin may take positive value for a limit case, when the initial capital tends to infinity.

This paper is organized as follows. In section 2 we present the dependent risk process. The risk model with the dependent claim amounts is briefly investigated in section 3. Next section 4 is devoted to the risk model with dependent interclaim times. In section 4, the main section in our paper, we study the case when the interclaim times W_i and the claim amounts X_i may be dependent. We study three cases. First, we assume, that the joint distribution of such random variables is the bivariate exponential distribution. Second, the distribution structure is described by the FGM copula and third by the Clayton copula. In the last case we use the simulation methods to compute the probability of ruin.

2. Risk process

Now we introduce the main notions and properties connected with the risk process studied in this paper. This process takes the following form:

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i,$$

where $u > 0$ is an initial capital, $c > 0$ is a premium rate, X_i is a claim amounts and $N(t)$ is a claim number process. If W_i is the i-th interclaim time and $T_n = \sum_{i=1}^{n} W_i$ is the claim arrival time, then the claim number process takes the form $N(t) = \sum_{k=1}^{\infty} \mathbf{1}_{T_k \le t}$, where $\mathbf{1}_A$ is the indicator of event A.

We assume that the claims X_1, X_2, … are the identically distributed random variables with the means $E(X_i) = \frac{1}{\beta}$. The interclaim times W_1, W_2, … are the identically distributed random variables, too and

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$E(W_i) = \frac{1}{\lambda}$. The random vectors $\{(W_i, X_i)\}$, where $i = 1, 2, \ldots$, are independent, but the components W_i, X_i may be dependent, contrary to the classical approach [Rolski et al. 1999; Ostasiewicz 2000].

The time of ruin T is a moment when the risk process $U(t)$ takes a negative value for the first time, i.e. $T = \inf\{t: \ U(t) < 0\}$. We will study the probability of ruin in the infinite horizon. Such a probability is determined by the formula

$$\psi(u) = P(T < \infty | \ U(0) = u)$$

in our paper. We know, that if $c\beta \leq \lambda$, then a ruin is a certain event for any value of initial capital u in this case, i.e. $\psi(u) = 1$. So, we assume that $c\beta > \lambda$ in our paper.

In the classical risk models [Rolski et al. 1999; Ostasiewicz 2000] we have the assumptions that all the random variables: the interclaim times $W_1, W_2, \ldots$ and the claim amounts $X_1, X_2, \ldots$ are independent and $N(t)$ is Poisson process. So, the interclaim times W_i have the exponential distribution with the cumulative distribution function (c.d.f.) $F_W(t) = 1 - e^{-\lambda t}$. For the extreme cases, when the initial capital u is equal to zero or to infinity, we obtain the following values:

$$\psi(0) = \frac{\lambda}{c\beta} \qquad \psi(\infty) = 0.$$

When the interclaim times W_i have any distribution, not only exponential, we obtain so called the Sparre Andersen model [Rolski et al. 1999].

3. Dependent claims and dependent interclaim times

When the claim amounts X_i are dependent we obtain some irregular situation [Heilpern 2010a, 2010b]. For instance, when the dependent structure is described by Archimedean copula, the probability of ruin for initial capital which equals zero takes the greatest value when the claims are independent, i.e. $\psi_I(0) \geq \psi(0)$, where $\psi_I(u)$ is a probability of ruin for independent case. But, for other extreme case, when the initial capital u tends to infinity, we obtain the reverse situation: $\psi_I(\infty) \leq \psi(\infty)$. The probability of ruin may be greater than zero in this case.

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In the paper [Heilpern 2001a] the author studies the case when the dependent structure is done by the Clayton copula: $C_\alpha(u_1, \dots, u_n) = (u_1^{-\alpha} + \dots + u_n^{-\alpha})^{-1/\alpha}$ and the claim amounts X_i have Pareto distribution. The graphs of the probability of ruin for the different degree of dependence are presented in Figure 1. Four cases of the Kendall τ coefficient of correlation: 0 (independence), 0.25, 0.5 and 1 (strict dependence) are investigated in this figure.

We can see that the relation between probability of ruin and the degree of dependence depends on the value of the initial capital. For small values of initial capital u, the smaller degree of dependence gives the greater probability of ruin. But for the bigger values of initial capital u we observe reverse relation and the middle values of initial capital give the nonmonotonic situation. We have the smallest probability of ruin for the independent case. The ruin probability increases when the degree of dependence increases; it obtains maximum and next decreases.

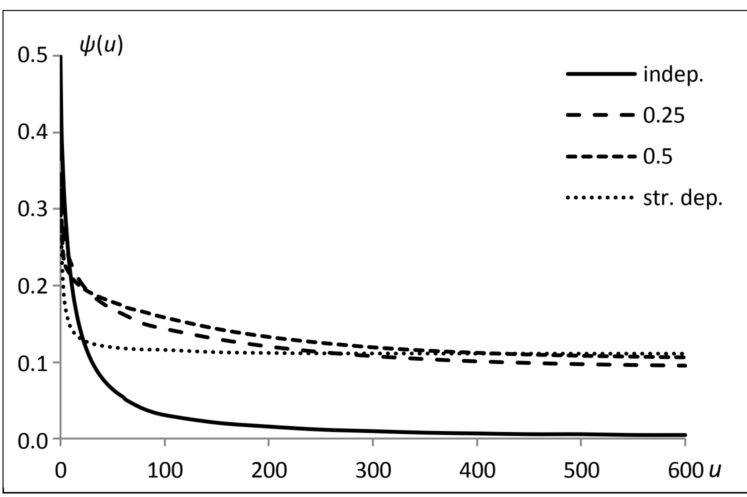

Figure 1. Probability of ruin for different degrees of dependence of claims for the dependent claims

Source: [Heilpern 2010a].

The claims X_i, which take only two values, are studied in [Heilpern 2010b]. We have a similar situation as in the case when the claims have Pareto distribution. Only, for the greater values of initial capital we obtain the nonmonotonic situation, too.

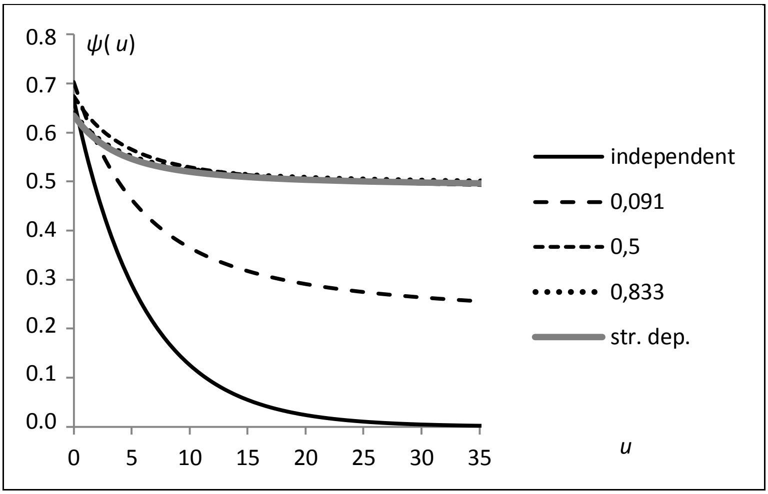

Figure 2. Probability of ruin for different degrees of dependence of interclaims times

Source: [Heilpern 2012].

The case when the interclaim times were dependent was studied in [Heilpern 2012]. We can observe the nonmonotonic situation for every value of initial capital u in this case. Figure 2 presents the probability of ruin where the interclaim times W_i and the claim amounts X_i have the exponential distributions and the dependent structure of random variables W_i is described by Clayton copula. The probability of ruin may be positive even for infinite initial capital, i.e. $\psi(\infty) > 0$, in this case, too.

4. Dependent claim size and claim occurrence times

The case when the interclaim times and the claim sizes are dependent are studied in many papers. For instance the papers [Albrecher, Boxma 2004; Ambagaspitiya 2009; Boudreault et al. 2006; Cheung et al. 2010; Cossette et al. 2008, 2010] are devoted to such a subject. We will present only two of these works in our paper: [Ambagaspitiya 2009; Cossette et al. 2010] and we will investigate the analysis of the influence of the degree of dependence on the ruin probability.

First we introduce some additional notions and properties. Let $Y_i = X_i - cW_i$ be the decrements of the risk process and $M =$

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$\max_{n\geq 1}\sum_{k=1}^{n} Y_i$ be a maximal decrement. Then, the probability of ruin can be done by the following formula:

$$\psi(u) = P(M > u).$$

In our paper we will investigate the cases when the claim amounts X_i and the interclaim times W_i have the exponential distribution with c.d.f. $F_X(x) = 1 - e^{-\beta x}$ and $F_W(t) = 1 - e^{-\beta t}$ only. When the random variables X_i, W_i are independent, the probability of ruin is done by the known formula [Rolski 1999; Ostasiewicz 2000]:

$$\psi(u) = \frac{\lambda}{c\beta} e^{-(c\beta-\lambda)u/c}.$$

In the case, when variables X_i and W_i are strictly dependent, i.e. $X_i = aW_i$ where $a = \frac{\lambda}{\beta}$, the probability of ruin is equal 0 or 1 only. It depends on value of a. If $a = \frac{\lambda}{\beta} < c$, then all decrements $Y_i = (a - c)W_i$ are negative and we obtain the ruin probability $\psi(u) = 0$ for every value of initial capital u. In the other case the ruin is a certain event. i.e. $\psi(u) = 1$ for every value of u.

R.S. Ambagaspitiya studies the ruin probability [2009] when the (W_i, X_i) have the bivariate gamma distribution. He obtains the exact formula for such a probability. For special case of such distribution: the bivariate exponential distribution, with the moment generating function (m.g.f.):

$$M_{W,X}(s_1, s_2) = \frac{1}{\left(1-\frac{s_1}{\lambda}\right)\left(1-\frac{s_2}{\beta}\right)-\rho\frac{s_1 s_2}{\lambda\beta}},$$

the ruin probability is equal

$$\psi(u) = Be^{-zu},$$

where

$$z = \frac{\lambda\beta}{(1-\rho)c}\left(\frac{c}{\lambda} - \frac{1}{\beta}\right), \qquad B = 1 - \frac{2\left(\frac{c}{\lambda}-\frac{1}{\beta}\right)}{\left(\frac{c}{\lambda}-\frac{1}{\beta}\right)+\sqrt{\left(\frac{c}{\lambda}-\frac{1}{\beta}\right)^2+4(1-\rho)\frac{c}{\lambda\beta}}}.$$

The margins of such joint distribution have the exponential distribution with means equal $1/\lambda$ and $1/\beta$. Let us recall that m.g.f of the random vector (W, X) is defined by formula

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$$M_{W,X}(s_1, s_2) = E(e^{s_1 W} e^{s_2 X})$$
$$= \int_0^\infty \int_0^\infty e^{s_1 t} e^{s_2 x} f_{W,X}(t, x) dt dx,$$

where $f_{W,X}(t, x)$ is a joint probability density function.

Parameter ρ is a Pearson correlation coefficient between W_i and X_i [Ambagaspitiya 2009]. We can study the influence of the degree of dependence on the probability of ruin changing the value of this parameter. Statement B is a decreasing function of ρ and z is an increasing function, so the ruin probability is the decreasing function of the value of correlation coefficient ρ. Greater degree of dependence gives the smaller ruin probability for every value of the initial capital. We see that contrary to previous cases we obtain the regular, monotonic situation in this case.

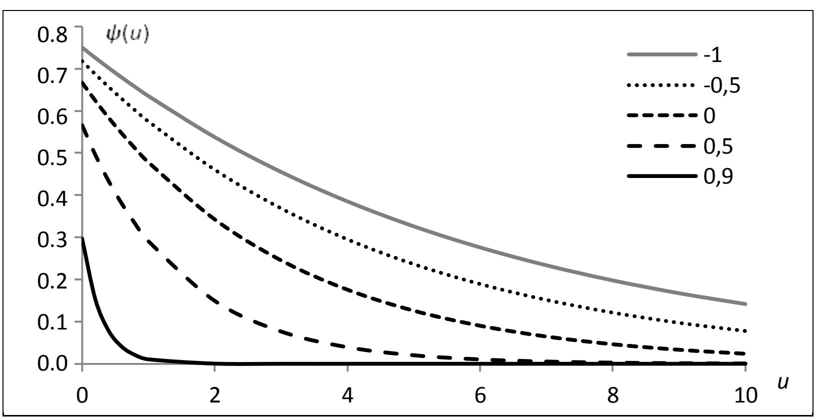

Figure 3. Probability of ruin for different values of the coefficient of correlation ρ

Source: own calculations.

Example 1. Let premium rate c = 1.5 and random variables W_i, X_i have the exponential distribution with mean equal to one, i.e. $\lambda = \beta = 1$. Figure 3 presents the graphs of probability of ruin for different values of the degree of dependence done by the five values of Pearson coefficient of correlation ρ: –1, –0.5, 0, 0.5 and 0.9. We recall, that for $\rho = 1$ we have $\psi(u) = 0$ for every value of initial capital u.

H. Cossete, E. Marceau and F. Marri [Cossette et. al 2010] presented the model, where the dependence structure is done by Farlie-Gumbel-Morgenstern (FGM) copula. Then joint c.d.f. of (W, X) is equal:

$$F_{W,X}(t, x) = C_\theta^{FGM}(F_W(t), F_X(t)),$$

where $C_\theta^{FGM}(u,v) = uv + \theta uv(1-u)(1-v)$ and $-1 \le \theta \le 1$. FGM copula reflects the small dependencies only, because the Kendall coefficient of correlation takes the form $\tau_\theta = \frac{2\theta}{9}$ and $\tau_\theta \in \left[-\frac{2}{9}, \frac{2}{9}\right]$. However, we meet such dependencies mainly in practice.

The Laplace transform of the ruin probability function $\psi(u)$ is equal [Cossette et. al 2010]

$$\hat{\psi}(s) = \int_0^\infty e^{-u}\psi(u)du = \frac{\beta_1(s) + \beta_2(s)}{h_1(s) - h_2(s)},$$

where

$$h_1(s) = \left(\frac{\lambda}{c} - s\right)\left(\frac{2\lambda}{c} - s\right),$$

$$h_2(s) = \frac{\lambda}{c}\left(\frac{\beta}{s+\beta}\left(\frac{2\lambda}{c} - s\right) - \theta s \frac{2\beta}{s+\beta}\right),$$

$$\beta_1(s) = \frac{\lambda}{c}\left(\frac{1}{s+\beta}\left(\frac{2\lambda}{c} - s\right) - \theta s\left(\frac{1}{s+2\beta} - \frac{1}{s+\beta}\right)\right),$$

$$\beta_2(s) = \frac{1}{\rho}(\beta_1(0)(s-\rho) + \beta_1(\rho)s)$$

and ρ is a root of equation $h_1(s) = h_2(s)$, for which $\mathrm{Re}(\rho) > 0$.

Example 2. Let $\lambda = \beta = 1$, $c = 1.5$ and the dependent structure is described by FGM copula. In Figure 4, we present the graphs of probability of ruin for different values of parameter θ: –2/9, –0.1, 0, 0.1 and 2/9, which reflect the degree of dependence.

We obtain the regular, monotonic situation in this case, too. The probability of ruin is the decreasing function of the degree of dependence, reflected by parameter θ, for every value of initial capital u.

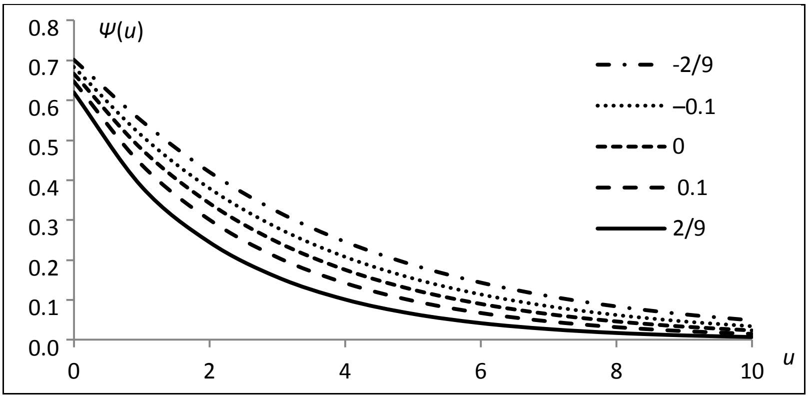

Figure 4. Probability of ruin, when the dependent structure is done by FGM copula for different values of the degree of dependence

Source: own calculations.

Now, we investigate the case when the dependent structure is described by Clayton copula. It is Archimedean copula done by simple formula: [Nelsen 1999; Heilpern 2007]

$$C_\alpha^{Cl}(u,v) = (u^{-\alpha} + v^{-\alpha} - 1)^{-1/\alpha},$$

where $\alpha \in [-1, 0) \cup (0, \infty)$. This parameter reflects the degree of dependence, because Kendall τ coefficient of correlation is equal

$$\tau_\alpha = \frac{\alpha}{\alpha+2}.$$

So, the Clayton copula can model all the values of Kendall coefficient, i.e. $\tau_\alpha \in [-1, 1]$, contrary to FGM copula. The positive value of parameter α gives us the positive dependence of the interclaim times W_i and the value of claim amounts X_i and negative value presents the negative dependence. The limit values, when such a parameter tends to zero, gives the independence of these random variables, i.e. $\lim_{\alpha \to 0} C_\alpha^{Cl}(u,v) = uv$.

The probability of ruin cannot be done by exact formula in this case. We may compute it using the simulation methods. But, there are simpler simulation procedures, where the dependent structure is described by the Archimedean copulas [Nelsen 1999; Heilpern 2007].

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Example 3. Let $\lambda = \beta = 1$, $c = 1.5$ and the dependent structure is described by Clayton copula. Figure 5 presents the graphs of probability of ruin, when the dependence structure is described by the Clayton copula, for different values of parameter α: –2/3, –1/3, –1/9, 0, 1/3, 1/2 and 2/3, which reflect the different values of degree of dependence. The values of the ruin probability were computed using a simulation method.

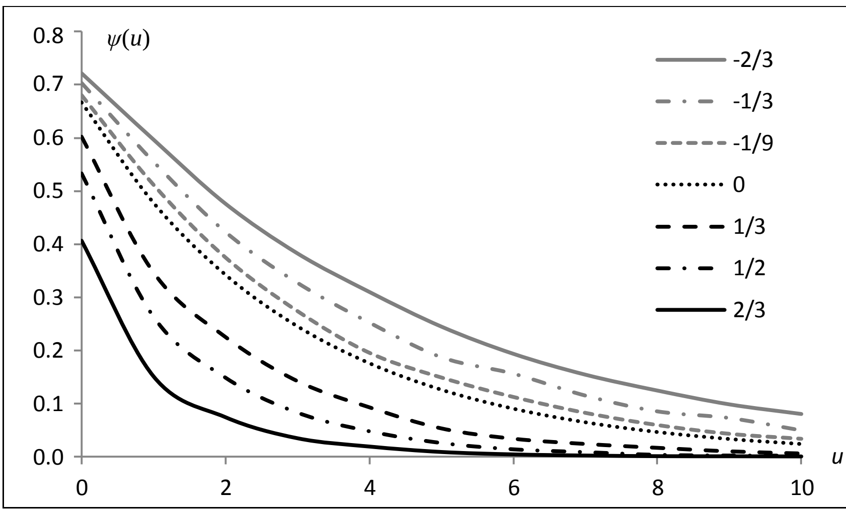

Figure 5. Probability of ruin, when the dependent structure is done by Clayton copula for different values of the degree of dependence

Source: own calculations.

We see that there is a regular, monotonic situation in this case, too. The probability of ruin decreases when the degree of dependence grows for every value of initial capital *u*.

5. Conclusion

The case of the dependent risk process investigated in this paper, based on the dependence between the claim amount and the claim occurrence times, is essentially different from other risk processes studied by the author [Heilpern 2010a; 2010b; 2012]. The examples of risk process investigated in our paper are characterized by the regular, monotonic relation between the degree of dependence and the probability of ruin. There is not such a regular relation in the risk process studied previously by the author. It essentially depends on the value of initial capital.

Maybe, there exists the risk process with the dependence between the claim size and the interclaim times, which is characterized by the above irregular relation. The solution of this problem is left for future research.

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PROCESY RYZYKA Z ZALEŻNYMI WYPŁATAMI I OKRESAMI MIĘDZY WYPŁATAMI

Streszczenie: Praca poświęcona jest procesowi ryzyka, w którym wielkości wypłat mogą być zależne od długości okresów między wypłatami. Badany jest wpływ stopnia zależności na prawdopodobieństwo ruiny. Rozpatrywane są trzy przypadki. Najpierw przyjmuje się, że rozkład łączny jest dwuwymiarowym rozkładem wykładniczym, a następnie rozpatruje się przypadki, gdy struktura zależności opisana jest funkcją łączącą (*copula*) FGM oraz Claytona. Otrzymane wyniki są porównane z wynikami otrzymanymi we wcześniejszych pracach autora.

Słowa kluczowe: proces ryzyka, zależność, prawdopodobieństwo ruiny, funkcje łączące.