M A T H E M A T I C A L   E C O N O M I C S

No. 6(13) 2010

RUIN PROBABILITY ON A FINITE TIME HORIZON

Zbigniew Michna

Abstract. In this article we investigate the classical risk process. We derive a formula for the ruin probability on a finite time horizon for zero initial capital that is Cramér's formula and for an arbitrary initial capital that is Seal's formula. Applying these formulas and the approximation of a gamma process by compound Poisson processes we obtain a formula for the supremum distribution of a gamma process with a linear drift.

Keywords: classical risk process, finite time ruin probability, Cramer's formula, Seal's formula, gamma process.

JEL Classification: G22, C02.

1. Introduction

A compound Poisson process is a basic skeleton in the risk theory. This model was introduced by Lundberg and Cramér and is called the classical Cramér–Lundberg model. Let $N(t)$ be a Poisson process with the intensity λ, $\{Y_n\}_{n=1}^{\infty}$ a sequence of positive independent random variables with the same distribution F. Moreover the sequence $\{Y_n\}_{n=1}^{\infty}$ is independent of the Poisson process $N(t)$. The process

$$R(t) = u + ct - \sum_{k=1}^{N(t)} Y_k$$

is the so called classical risk process, where $c > 0$. However

$$S(t) = u - R(t) = \sum_{k=1}^{N(t)} Y_k - ct$$

Zbigniew Michna
Department of Mathematics, Wrocław University of Economics, Komandorska Street 118/120, 53-345 Wrocław, Poland.
E-mail: zbigniew.michna@ue.wroc.pl

is the claim surplus process (see Fig. 1). The process $S(t)$ is a Lévy process.

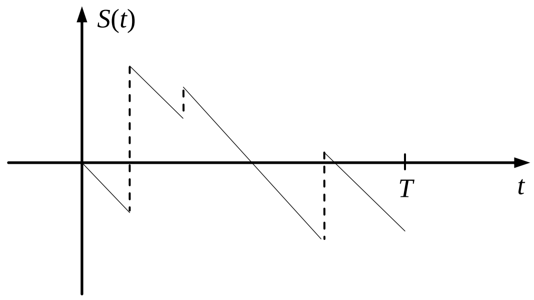

Fig. 1. A sample path of the process $S(t)$ with $c = 1$

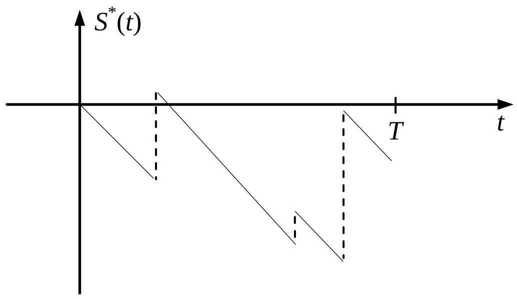

Fig. 2. A sample path of the process $S^*(t)$ – reverse to the sample path of Fig. 1

In this article we investigate the ruin probability on a finite time horizon that is the probability that the risk process goes down below the level zero or equivalently the surplus process crosses over the level u until time T. More precisely, we investigate probability

$$\psi(u,T) = P\left(\sup_{t\le T} S(t) > u\right). \tag{1}$$

Thus we are interested in the claim surplus process on a finite time horizon. Hence we will consider stochastic processes on the interval $[0, T]$.

We will need two properties of the compound Poisson process. The first property is the reversibility of a compound Poisson process with respect to time and the second one is the exchangeability of the segments for a compound Poisson process.

Let us define

$$S^*(t) = S_T - S_{T-t},$$

where $0 \le t \le T$ (see Fig. 2) and for $0 \le v \le T$ (see Figs. 3 and 4)

$$S^{(v)}(t) = \begin{cases} S_{t+v} - S_v & 0 \le t \le T - v \\ S_T - S_v + S_{t-T+v} & T - v \le t \le T \end{cases}.$$

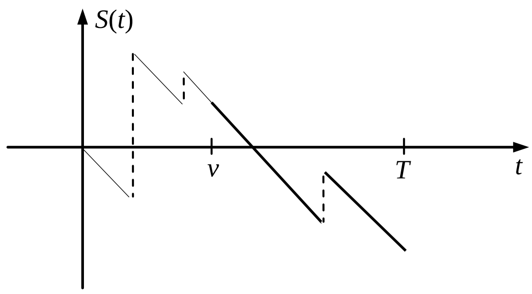

Fig. 3. A sample path of the process $S(t)$

Theorem 1.

(i) $$S^*(t) \overset{d}{=} S(t),$$

(ii) $$S^{(v)}(t) \overset{d}{=} S(t)$$

in the sense of finite dimensional distributions.

Proof. Check the conditions for a Poisson process.

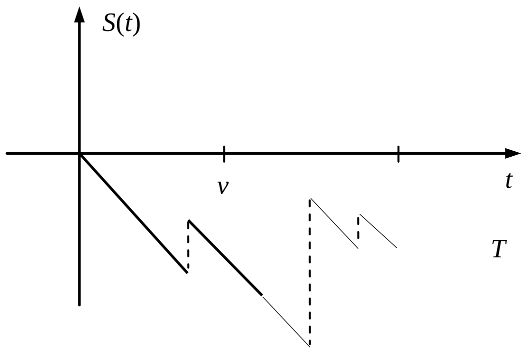

Fig. 4. A sample path of the process $S^{(v)}(t)$ – to the sample path of Fig. 3

2. Cramér's formula

Now we find a formula for the ruin probability of the classical risk process with the initial reserve equal to zero. This formula is called Cramér's formula (Cramér, 1955; Asmussen, 2000).

Theorem 2.

$$1-\psi(0,T)=P\left(\sup_{t\le T}S(t)\le 0\right)=\frac{1}{cT}\int_0^{cT}P\left(\sum_{k=1}^{N(T)}Y_k\le x\right)dx.$$

Proof. First let us assume that c = 1. Let us define

$$M(v,t)=\left\{S^{(v)}(t)\le S^{(v)}(s),\ 0\le s\le t\right\}$$

and notice that

$$\begin{aligned}1-\psi(0,T)&=P\left(S(t)\le 0,\quad 0\le t\le T\right)\\&=P\left(S^*(t)\le 0,\quad 0\le t\le T\right)\\&=P\left(S(T)\le S(T-t),\quad 0\le t\le T\right)\\&=P\left(S(T)\le S(t),\quad 0\le t\le T\right)\\&=P(M(0,T)),\end{aligned}$$

where in the second equality we use theorem 1 (i). Thus

$$1-\psi(0,T)=P\left(M(0,T)\right)=\frac{1}{T}\int_0^T P\left(M(v,T)\right)dv=\frac{1}{T}E\int_0^T I\{M(v,T)\}dv,$$

where in the second equality we used theorem 1 (ii) ($P(M(v, T))$ does not depend on v). Hence we need to investigate $\int_0^T I\{M(v,T)\}\,dv$. If $S^{(v)}(T)$ = $S(T) > 0$ then $I\{M(v, T)\} = 0$. Thus let $S(T) \le 0$. If $M(0, T)$ is fulfilled then $M(v, T) = M(0, v)$. Indeed, if

$$\begin{aligned}M(v,T)&=\left\{S(T)\le S(t+v)-S(v),0\le t\le T-v\right\}\cap\\&\qquad\left\{S(T)\le S(T)-S(v)+S(t-T+v),T-v\le t\le T\right\}\\&=\left\{S(T)\le S(t)-S(v),v\le t\le T\right\}\cap\\&\qquad\left\{S(T)\le S(T)-S(v)+S(t),0\le t\le v\right\}\\&=\left\{S(T)\le S(t)-S(v),v\le t\le T\right\}\cap M(0,v)\\&=M(0,v),\end{aligned}$$

where the last equality is satisfied under the assumption that $M(0, T)$ is fulfilled since $S(v) \le 0$ if $M(0, v)$ is satisfied thus on $M(0, T)$ the assumption $\{S(T) \le S(t) - S(v), v \le t \le T\}$ is fulfilled.

Hence if M(0, T) is fulfilled then

$$\int_0^T I\left\{M\left(v,T\right)\right\}dv = \int_0^T I\left\{M\left(0,v\right)\right\}dv = -S\left(T\right).$$

It is enough to consider $\int_0^T I\left\{M\left(v,T\right)\right\}dv$ on $M(0, T)$ because on the set $S(T) \le 0$ we exchange the segments (the first segment to the minimum of $S(t)$) and we again get $M(0, T)$. More precisely, let us define the process

$$S_M(t) = \begin{cases} S(t) & \text{if} \quad S(T) > 0 \\ S^{(\tau)}(t) & \text{if} \quad S(T) \le 0 \end{cases}$$

where τ is the epoch when the process $S(t)$ attains its minimum on $[0, T]$.
Then

$$S_M(t) \overset{d}{=} S(t).$$

Thus

$$\begin{aligned} 1-\psi(0,T) &= \frac{1}{T} E\int_0^T I\left\{M(v,T)\right\}dv \\ &= \frac{1}{T} ES^{-}(T) \\ &= \frac{1}{T}\int_0^\infty P\left(S(T) \le -x\right)dx \\ &= \frac{1}{T}\int_0^T P\left(S(T) \le -x\right)dx \\ &= \frac{1}{T}\int_0^T P\left(\sum_{k=1}^{N(T)} Y_k \le T - x\right)dx \\ &= \frac{1}{T}\int_0^T P\left(\sum_{k=1}^{N(T)} Y_k \le x\right)dx, \end{aligned}$$

where in the fourth equality we use the fact that $S^{-}(T) \le T$. The formula for any $c > 0$ we get in the following way

$$1-\psi(0,T) = P(\sup_{t\le T} \sum_{k=1}^{N(t)} Y_k - ct \le 0)$$
$$= P(\sup_{t\le T} \sum_{k=1}^{N(t)} Y_k / c - t \le 0)$$
$$= \frac{1}{T}\int_0^T P\left(\sum_{k=1}^{N(T)} Y_k / c \le x\right)dx$$
$$= \frac{1}{T}\int_0^T P\left(\sum_{k=1}^{N(T)} Y_k \le cx\right)dx$$
$$= \frac{1}{cT}\int_0^{cT} P\left(\sum_{k=1}^{N(T)} Y_k \le x\right)dx.$$

3. Seal's formula

We generalize the previous formula for any value of the initial capital $u > 0$. The next theorem is called Seal's formula (Seal, 1974) and was originated by Prabhu (1961); see also (Asmussen, 2000).

Theorem 3. *Let* $T > 0$, $c > 0$ *and* $u > 0$ *then*

$$\psi(u,T) = P\left(\sup_{t\le T} S(t) > u\right)$$
$$= P\big(S(T) > u\big) + c\int_0^T \big(1-\psi(0,T-s)\big)P\big(S(s)\in[u,u+ds]\big)$$
$$= P\big(S(T) > u\big) + c\int_0^T \big(1-\psi(0,T-s)\big) f(u+cs,s)ds,$$

where $f(x, s)$ *is the density function of the random variable* $\sum_{k=1}^{N(s)} Y_k$ *provided that it exists and* $\psi(0,T-s)$ *is given in Th.* 2.

Proof. Let $c = 1$. The event $\{S(T) \le u\}$ can happen in two ways: until the moment T the ruin does not occur and this event happens with probability $1 - \psi(u, T)$, or the ruin occurs and we are able to define the last epoch σ when the process $S(s)$ is going down below the level u (see Fig. 5).

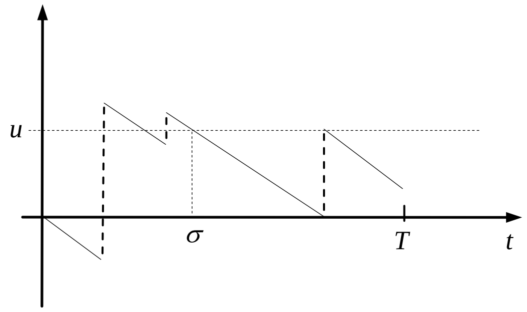

Fig. 5. The last epoch σ when the process $S(t)$ is going down below the level u

In this case $\sigma \in [s, s + ds]$ if and only if $S(s) \in [u, u + ds]$ and we do not have another upcrossing of the level u after time s which occurs with probability $1 - \psi(0, T - s)$. Thus we get

$$P\left(S(T) \le u\right) = 1 - \psi(u,T) + \int_0^T \left(1 - \psi(0,T-s)\right) P\left(S(s) \in [u, u+ds]\right),$$

which gives the assertion for $c = 1$. In order to get the formula for an arbitrary $c > 0$ it is enough to notice that

$$P\left(\sup_{t\le T} S(t) > u\right) = P\left(\sup_{t\le T} \sum_{k=1}^{N(t)} Y_k / c - t > u / c\right)$$

and use the formula for $c = 1$.

4. Formula for the ruin probability for gamma process

Gamma process is a Lévy process with one dimensional distribution being gamma.

Definition 4.1. *A stochastic process* $Z = \{Z(t), 0 \le t < \infty\}$ *is gamma process with the shape parameter a and scale parameter b if*

1. $Z(0) = 0$ *a.s.*;

2. *Z has independent increments*;

3. *Z has stationary increments*;

4. $Z(1)$ *has gamma distribution with the shape parameter* $a > 0$ *and scale parameter* $b > 0$ *that is distribution with the density function*

$$f(y)=\begin{cases}0 & \text{if } \; y\le 0\\ \dfrac{1}{b^a\Gamma(a)}y^{a-1}\exp(-\tfrac{y}{b}) & \text{if } \; y>0\end{cases} \tag{2}$$

where Γ *is the gamma function.*

The gamma process can be obtained as a limit of certain compound Poisson processes (classical cumulated claims processes).

Let $\left\{Y_k^{(p)}\right\}_{k=1}^{\infty}$ be a sequence of the size of successive claims independent of each other with a common cumulative distribution function $F^{(p)}$ dependent on the parameter $p > 0$. More precisely

$$F^{(p)}(y)=\begin{cases}0 & \text{if } \; y\le p\\ 1-\dfrac{Q(y)}{Q(p)} & \text{if } \; y>p\end{cases}, \tag{3}$$

where

$$Q(y)=\int_y^{\infty}\frac{a}{x}e^{-\frac{x}{b}}\,dx. \tag{4}$$

Moreover, let us assume that $N^{(p)}(t)$ is a Poisson process with the intensity $Q(p)$ and independent of $\left\{Y_k^{(p)}\right\}_{k=1}^{\infty}$. Let us define

$$S^{(p)}(t)=\sum_{k=1}^{N^{(p)}(t)}Y_k^{(p)}-ct. \tag{5}$$

Proposition. *If* $p\downarrow 0$ *then*

$$S^{(p)}(t)\Rightarrow Z(t)-ct$$

weakly in the Skorokhod topology where Z is a gamma process.

Proof. Since $S^{(p)}(1) \Rightarrow Z(1) - c$ (Dufresne, Gerber, Shiu (1991)) and the stochastic processes $S^{(p)}(t)$ and $Z(t) - ct$ are Lévy processes thus by Th. V. 19 (Pollard, 1984) we get the assertion.

Using Cramér's and Seal's formulas and the approximation of the gamma process by compound Poisson processes we can obtain a formula for the distribution of the supremum of gamma process with a linear drift.

Theorem 4. Let $c > 0$ and $T > 0$. Then

(i) $$\Psi(0,T) = P\left(\sup_{t\le T} Z(t) - ct > 0\right) =$$

$$= P\left(Z(T) - cT > 0\right) + \frac{ab}{c} P\left(Z(T+1/a) - cT \le 0\right);$$

(ii) $$\Psi(u,T) = P\left(Z(T) - cT > u\right) + c\int_0^T P\left(Z(T-s) \le c(T-s)\right) f(u+cs,s)ds$$

$$-ab\int_0^T P\left(Z(T-s+1/a) \le c(T-s)\right) f(u+cs,s)ds,$$

where

$$f(y,s) = \frac{1}{b^{as}\Gamma(as)} y^{as-1} \exp\left(-\frac{y}{b}\right)$$

and

$$P\left(Z(s) \le x\right) = \frac{1}{b^{as}\Gamma(as)} \int_0^x y^{as-1} \exp\left(-\frac{y}{b}\right) dy.$$

Proof. Let us notice that the functional $\sup_{t\le T} x(t)$ is continuous in the Skorohkod J_1 topology and the distribution

$$P\left(\sup_{t\le T} Z(t) - ct \le u\right)$$

is continuous. Thus we use Prop. 4.1 and get

$$\lim_{p\downarrow 0} P\left(\sup_{t\le T} S^{(p)}(t) \le u\right) = P\left(\sup_{t\le T} Z(t) - c \le u\right)$$

Hence by Th. 2 we obtain

$$1 - \psi_p(0,T) = P\left(\sup_{t\le T} S^{(p)}(t) \le 0\right) = \frac{1}{cT}\int_0^{cT} P\left(\sum_{k=1}^{N(T)} Y_k^{(p)} \le x\right) dx.$$

Taking $p \downarrow 0$ we have

$$P\left(\sup_{t\le T} Z(t) - ct \le 0\right) = \frac{1}{cT}\int_0^{cT} P\left(Z(T) \le x\right)dx$$

$$= \frac{1}{cTb^{aT}\Gamma(aT)}\int_0^{cT} dx \int_0^{x} y^{aT-1}\exp\left(-\frac{y}{b}\right)dy$$

$$= \frac{1}{b^{aT}\Gamma(aT)}\int_0^{cT} y^{aT-1}\exp\left(-\frac{y}{b}\right)dy - \frac{1}{cTb^{aT}\Gamma(aT)}\int_0^{cT} y^{a(T-1/a)-1}\exp\left(-\frac{y}{b}\right)dy$$

$$= P\left(Z(T) - cT \le 0\right) - \frac{ab}{c}P(Z(T+1/a) - cT) \le 0,$$

where in the second equality we use the density function of the random variable $Z(T)$ and in the third we integrate by parts. Thus we obtain the assertion (i).

Similarly in order to get the formula (ii) we use Th. 3 and we let $p \downarrow 0$.

It is possible to get a formula for the supremum distribution of spectrally positive α-stable Lévy process (Michna (2011a)) and more generally for any spectrally positive Lévy process (Michna (2011b)).

Literature

Asmussen S. (2000). *Ruin Probabilities*. World Scientific Publishing. Singapore.

Cramér H. (1955). *Collective risk theory.* The Jubilee volume of Forsakringsbolaget. Skandia. Stockholm.

Dickson D.C.M., Waters H.R. (1993). *Gamma processes and finite time survival probabilities*. Astin Bulletin 23. Pp. 259-272.

Dufresne F., Gerber H.U., Shiu E.S.W. (1991). *Risk theory with the gamma process*. Astin Bulletin 21. Pp. 177-192.

Michna Z. (2002). *Modele graniczne w teorii ryzyka ubezpieczeniowego*. Publishing House of Wrocław University of Economics. Wrocław.

Michna Z. (2011a). *Formula for the supremum distribution of a spectrally positive α-stable Lévy process*. Statistics and Probability Letters 81. Pp. 231-235.

Michna Z. (2011b). *Formula for the supremum distribution of a spectrally positive Lévy process*. Arxiv preprint arXiv:1104.1976, 2011. arxiv.org.

Pollard D. (1984). *Convergence of Stochastic Processes.* Springer-Verlag. New York.

Prabhu N.U. (1961). *On the ruin problem of collective risk theory*. The Annals of Mathematical Statistics 32. Pp. 757-764.

Seal H.L. (1974). *The numerical calculation of* $U(w, t)$, *the probability of non ruin in an interval* $(0, t)$. Scandinavian Actuarial Journal. Pp. 121-139.