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TRADE CREDIT PORTFOLIO SELECTION – A MARKOVIAN APPROACH

The application of stochastic processes to the prediction of accounts receivable and cash flow is a classic financial operations research problem. Although there is a vast related literature, some theoretical and practical problems still exist. This paper investigates a form of vector which initiates a Markovian process because the distribution of this vector is much more simple for practical reasons than is suggested in literature. A Markovian prediction when the initial vector varies from period to period but a fundamental matrix is constant, is also examined. A more general case is a model in which the fundamental matrix, as well as the initial vector, is time varying.

The main focus of this paper is to develop a criterion helpful in selecting clients on the basis of the definite risk of trade credit portfolio under Markovian model of accounts receivable.

Keywords: *application of finite Markov chains, financial liquidity management, accounts receivable management*

1. Introduction. A Review of Literature

Markovian prediction of accounts receivable and cash flow is surely one of the classic financial operations research problems. It has been mathematically [3] and empirically [1] proved that stochastic process can be used to solve problem such as the aging structure of accounts receivable, their balance at the end of a given period and at infinity, cash flow, provision for doubtful accounts and so forth. We may be tempted to say that everything is clear. Unfortunately, the fact is that several problems in the Markovian prediction of accounts receivable and cash flow still exist. Let us undertake a short review of the classic literature. The main issues of Markovian prediction are as follows:

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• heterogeneity of clients getting trade credit,
• form of initial vector,
• stationarity of initial vector and transition matrix.

R.M. Cyert, H.J. Davidson and G.L. Thompson [3] have already postulated that clients who get trade credit should be separated into homogenous groups for which individual initial vectors and transition matrices must be prepared. This procedure avoids confusing the accounts receivable of clients with different payment behaviour, and consequently errors in prediction. H. Frydman, J. Kallberg and D. Kao [4] empirically tested stationary and nonstationary (mover-stayer model) Markovian models of retail revolving credit accounts. In their empirical survey they formulated the opinion that incorporating heterogeneity into a model is more important than incorporating nonstationarity. This study is important. However, the consequences of formulating such mover-stayer models seem to be limited because:

• the economic literature lacks any application of mover-stayer models to the classic financial management problems of accounts receivable and cash flow,
• the study applied to revolving retail credit, while most models concentrate on trade credit for corporations; it is questionable whether the payment behaviours of retail clients and institutional clients are the same or even similar,
• dividing clients into homogenous groups, which are usually called credit risk classes, is very popular in practice and has a sound theoretical basis.

To sum up it might be suggested that the heterogeneity of clients does not need any further investigation.

The next problem is the form of the initial vector. Most authors in the classic literature assume that the initial vector has a full aging structure when the Markovian process is initiated. Therefore at moment t the vector which begins the process consists of current entry and past-due entries. This case is true only when at any moment t, when credit is offered, a total number of transitions is realized. For example, if an aging schedule has five age categories (excluding bad debt category) then at the moment the process begins, the accounts receivable vector has full age categories. Assuming stationarity, such a vector initiates the process every time trade credit is offered. But when in one period, $t - t + 1$, for example one month, only one transition is possible, it is impossible for accounts receivable to have full age categories apart from current age category. In mathematical notation we have the initial vector:

$$\overline{\overline{X_t}} = [\, X_t \;\; X_{t-1} \;\; X_{t-2} \ldots]\,. \tag{1}$$

The age structure for $\overline{\overline{X_t^*}}$ as the total balance of accounts receivable is:

$$[\, x_t \;\; x_{t-1} \;\; x_{t-2} \ldots] = \left[\frac{X_t}{\overline{\overline{X_t^*}}} \;\; \frac{X_{t-1}}{\overline{\overline{X_t^*}}} \;\; \frac{X_{t-2}}{\overline{\overline{X_t^*}}} \ldots \right]. \tag{2}$$

In period $t - t + 1$ when credit is offered, accounts receivable goes through only one transition, from the state “no credit” to the state “current credit”, therefore:

$$x = [\,1\ \ 0 \ldots 0\,]\,. \tag{3}$$

The initial vector has entries as follows:

$$\overline{\overline{X_t^*}} \cdot [\,1\ \ 0 \ldots . 0\,] = [\,X_t\ \ 0 \ldots . 0\,]\,. \tag{4}$$

The above remarks have some practical and theoretical consequences.

Because the current receivables' vector always initiates the stochastic process, it is therefore stationary like the transition matrix, as the classic assumption demands. An initial vector and fundamental matrix (N) are used to find a steady-state age distribution of receivables. Therefore the steady-state distribution is:

$$\hat{X}_{t+1} = \overline{\overline{X_t^*}} \cdot x \cdot N\,. \tag{5}$$

The initial vector has only current entry, so the distribution depends on the first row of the fundamental matrix N:

$$\overline{\overline{X_t^*}} x N = [X_t\ \ 0 \ldots 0] \begin{bmatrix} q_{1,1} & \cdots & q_{1,j} \\ \vdots & & \vdots \\ q_{j,1} & \cdots & q_{j,j} \end{bmatrix} = [X_t q_{1,1} + \ldots + 0 \quad \ldots \quad X_t q_{1,j} + \ldots + 0]\,. \tag{6}$$

This simplifies practical computation.

If the initial vector has a full age distribution, then the amount of receivables in past-due categories is overvalued every time a transition is made and likewise from a steady-state perspective. After each transition from one nonabsorbing state to another nonabsorbing state, every age entry will be overvalued to the following extent:

$$F = [0\ \ X_{t-1}\ \ X_{t-2} \ldots] \cdot \mathrm{Q} \tag{7}$$

where Q is a matrix of the transition probabilities from one to another nonabsorbing state.

After one transition the distribution of receivables is $[\mathrm{X}_t\ \ 0 \ldots 0] \cdot \mathrm{Q}$ because an initial vector has only a current entry. The distribution for a full age vector is $[\mathrm{X}_t\ \ \mathrm{X}_{t-1}\ \ \mathrm{X}_{t-2} \ldots] \cdot \mathrm{Q}$. Let F be the difference between these vectors, under the assumption that they have the same dimensions, then:

$$\mathrm{F} = [\mathrm{X}_t\ \ \mathrm{X}_{t-1}\ \ \mathrm{X}_{t-2} \ldots] \cdot \mathrm{Q}\ -\ [\mathrm{X}_t\ \ 0 \ldots 0] \cdot \mathrm{Q}\ =\ [0\ \ \mathrm{X}_{t-1}\ \ \mathrm{X}_{t-2} \ldots] \cdot \mathrm{Q}\,. \tag{8}$$

For the steady-state distribution every age category is overvalued on amount:

$$\mathrm{F}' = [0\ \ \mathrm{X}_{t-1}\ \ \mathrm{X}_{t-2} \ldots] \cdot \mathrm{N}\,. \tag{9}$$

The proof is analogous to the previous but instead of the matrix Q we have a fundamental matrix N. Therefore

$$\mathrm{F}' = [\mathrm{X}_t \;\; \mathrm{X}_{t-1} \;\; \mathrm{X}_{t-2} \ldots]\cdot \mathrm{N} - \; [\mathrm{X}_t \;\; 0 \ldots 0]\cdot \mathrm{N} = \; [0 \;\; \mathrm{X}_{t-1} \;\; \mathrm{X}_{t-2} \ldots]\cdot \mathrm{N} \qquad (10)$$

Values F and F′ depend on the amount of past-due receivables and the fundamental matrix. The less the sum of past-due receivables is, and the closer to zero the probabilities of the fundamental matrix, the lower the F and F′ are, so that the distribution of the initial vector is less important.

The problems considered above are independent of the method used to prepare the aging schedule. "The total balance" method, and "the partial balance" method, are both suitable for Markovian prediction. In most classic literature the first one is preferred, but the second one is actually much more popular now. The difference lies in the amount of receivables distributed over age categories before absorption. A little correction is needed to balance these amounts [9]. For practical reasons in this paper "the partial balance" method will be used.

The final problem to be dealt with is the stationarity of the initial vector and transition matrix. In general, credit sales and accounts receivable change from period to period because of trends, seasonal and accidental variations. Uncertainty conditions influence the stochastic process in two ways:

1. It is possible that, while payment behaviours are constant in time, accounts receivable and their distribution change because of credit sale fluctuations. This means that the transition matrix is constant but the initial vector varies,

2. A more common case is where both the initial vector and the transition matrix change, along with changes in credit sales, accounts receivable and the financial liquidity of clients.

The problem is that the stationarity assumption is totally contrary to business practice.

The most meaningful approach has been presented by W. Corcoran [1]. He proposed formulating an age distribution for a short one-month period on an initial vector, and a transition matrix taken from a current period. Therefore, the stochastic process always has only one transition, because the initial vector and transition matrix are restored before a new transition. This matrix is a combination of a current matrix and an exponentially-smoothed matrix. Corcoran's approach is very simple and easy to apply. It is also suitable for solving problems of changes in the initial vector and transition matrix. But there are some small drawbacks with it:

- a one-time dependent initial vector and transition matrix are difficult to use over a period longer than a month. As Corcoran [1] suggests, a sales projection is necessary but it does not eliminate errors in cash flow prediction,
- if the initial vector is constant over time, no optimal solution for infinity can be found,
- the smoothing technique is connected with time, like other statistical tools, not with the essence of payment behaviour as in a classic Markovian prediction.

The problems described above need further investigation in order to formulate better criterion for trade credit portfolio selection primarily formulated by Cyert and Thompson [3]. Therefore *the main purpose of this paper is to formulate improved criterion if cash flow from accounts receivable is modelled as classic, Markovian process.*

2. Markovian prediction under a variable initial vector and invariable transition matrix

When the initial vector has only a current receivables entry, the transition matrix is simplified in comparison with the fully distributed vector. In "the partial balance" method, the transition matrix from nonabsorbing states to nonabsorbing states is similar to the matrix presented below:

$$\mathrm{Q} = \begin{bmatrix} 0 & p_{0,1} & \dots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & p_{\bar{k}-1,\bar{k}} \\ 0 & \dots & \dots & 0 \end{bmatrix} \tag{11}$$

where $k = 0, ..., \bar{k}$ is the number of age categories excluding a bad debts category. To find the solution to the distribution of accounts receivable when the initial vector is variable, and the transition matrix is invariable, it is necessary to formulate the following lemma.

Lemma 1

If in matrix Q, which is a transition matrix from nonabsorbing states to nonabsorbing states, the last *r* columns are substituted with nulls, the entries of the last columns of the fundamental matrix apart from diagonal ones, are nulls.

Proof:

Q has a specific form and $(\mathrm{I} - \mathrm{Q})^{-1} = \mathrm{N}$, therefore for *r* columns substituted with nulls, we have:

$$I - \mathrm{Q} = \begin{bmatrix} 1 & \dots & \dots & 0 \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ 0 & \dots & \dots & 1 \end{bmatrix} - \begin{bmatrix} 0 & p_{0,1} & \dots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & 0 & \dots & 0 \end{bmatrix}, \tag{12}$$

$$\begin{bmatrix} n_{1,1} & n_{1,2} & \cdots & n_{1,i} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ n_{j,1} & n_{j,2} & \cdots & n_{j,i} \end{bmatrix} \cdot \begin{bmatrix} 1 & 1-p_{0,1} & \ldots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & 0 & \ldots & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix} \tag{13}$$

and finally,

$$\mathrm{N} = \begin{bmatrix} 1 & 1-p_{0,1} & \ldots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & \ldots & 1 \end{bmatrix}. \tag{14}$$

By lemma 1, a theorem can be proposed which formulates a Markovian prediction if the initial vector is time varying but the fundamental matrix is invariable.

Theorem 1

If the fundamental matrix has arisen by the substitution of the last r columns of the matrix Q with nulls, and both the initial vector and the fundamental matrix are invariable with time, then the stationary age distribution of accounts receivable is as follows:

$$\mathrm{X}''_{t+1} = \mathrm{X}''^{*}_{t,(k)} \cdot x'' \cdot (\mathrm{N}_{r-1} - \mathrm{N}_r)_k\,, \tag{15}$$

where N_{r-1} and N_r are fundamental matrices for age category k and $\mathrm{X}''^{*}_{t,(k)} \cdot x''$, is the variable initial vector.

Proof:

Accounts receivable vector X''_{t+1} at moment $t + 1$ consists of receivables which have arisen in current and previous periods, hence:

$$\mathrm{X}''_{t+1} = \mathrm{X}_{t,t} + \mathrm{X}_{t,t-1} + \mathrm{X}_{t,t-2} + \ldots = \mathrm{X}_t + \mathrm{X}_{t-1} \cdot \mathrm{Q} + \mathrm{X}_{t-2} \cdot \mathrm{Q}^2 + \ldots. \tag{16}$$

Because the initial vector is variable with time, then every age category arises from a different initial vector. The total balance of accounts receivable is the sum of current receivables in a given period times the number of transitions from nonabsorbing states to nonabsorbing states.

The value of each age category depends on matrix Q and fundamental matrix N constant in time. If the number of columns $r = \widehat{k}$ in the matrix is substituted with nulls, then $\mathrm{Q_r} = 0$ and $\mathrm{N_r} = \mathrm{I}$, thus:

$$\mathrm{X}_{t,t} = \mathrm{X}''^{*}_{t} \cdot x'' \cdot (\mathrm{I} + 0 + \ldots + 0) - \mathrm{X}''^{*}_{t} \cdot x'' \cdot (0 + \ldots + 0)\,. \tag{17}$$

If all columns except the last two are substituted, then $X_{t,\,t-1}$ is:

$$X_{t,t-1} = X''^{*}_{t-1} \cdot x'' \cdot (I + Q + 0 + ... + 0) - X''^{*}_{t-1} \cdot x'' \cdot (I + 0 + ... + 0) \tag{18}$$

and when all columns are substituted except the second and third, we have:

$$X_{t,t-2} = X''^{*}_{t-2} \cdot x'' \cdot (I + Q \;\; + Q^2 + ... + 0) - X''^{*}_{t-2} \cdot x'' \cdot (I + Q + 0 + ... + 0) \tag{19}$$

In lemma 1 $(I + 0 + ... + 0)$ there is a fundamental matrix N, in which $\widehat{k} - 1$ columns are substituted ($N_{\widehat{k}-1}$), and $(0 + ... + 0)$ is a matrix N in which all columns are substituted ($N_{\widehat{k}}$). Both matrices are of age category *k*, therefore:

$$X''^{*}_{t} \cdot x'' \cdot (N_{\widehat{k}-1} - N_{\widehat{k}})_0 \,. \tag{20}$$

For the matrix $I + Q + 0 + ... + 0$ $\widehat{k} - 2$ columns are substituted in age category $k = 1$, hence:

$$X''^{*}_{t-1} \cdot x'' \cdot (N_{\widehat{k}-2} - N_{\widehat{k}-1})_1 \tag{21}$$

and

$$X''^{*}_{t-2} \cdot x'' \cdot (N_{\widehat{k}-3} - N_{\widehat{k}-2})_2 \,, \tag{22}$$

$$\vdots$$

Substituting (18)–(20) to (14) we get:

$$X''^{*}_{t+1} = X^{*}_{t} \cdot x'' \cdot (N_{\widehat{k}} - 0) + X^{*}_{t-1} \cdot x'' \cdot (N_{\widehat{k}-1} - N_{\widehat{k}}) + X^{*}_{t-2} \cdot x'' \cdot (N_{\widehat{k}-2} - N_{\widehat{k}-1}) + ... \tag{23}$$

For the respective matrices denoted by N_{r-1} and N_r for age category *k* then $(N_{r-1} - N_r)_k$ and the initial vector is $X''^{*}_{t,(k)} \cdot x''$.

If both the initial vector and the fundamental matrix are time varying, a Markovian prediction is more complicated.

3. Markovian prediction when the initial vector and fundamental matrix are invariable with time

A change in the range of probabilities in a fundamental matrix depends on the characteristics of credit sales and payment behaviours, i.e., trend, seasonal and occasional variations. No matter which kind of change is experienced, the same solution for all cases can be proposed.

Lemma 2

If the averaged or smoothed variance of fundamental matrices $\overline{\mathrm{N}}$ is invariable with time, then a predicted fundamental matrix N^* is a solution of the equations:

$$\begin{cases} n_{1,1}^* \cdot (n_{1,1}^* - 1) = 0 & \cdots \quad n_{1,j}^* \cdot (2n_{j,i}^* - n_{1,j}^* - 1) = \overline{n}_{1,j} \\ \vdots & \vdots \\ n_{j,1}^* \cdot (2n_{1,1}^* - n_{j,1}^* - 1) = 0 & \cdots \quad n_{j,j}^* \cdot (n_{j,j}^* - 1) = 0 \end{cases} \tag{24}$$

where $n_{j,j}^*$ is an entry of matrix N^* for each $j \times j$.

Proof:

Let the fundamental matrices $\overline{\mathrm{N}}$ change from period to period in periods of present time. If a different variance exists for each matrix, then there exists an averaged or smoothed matrix $\mathrm{Av}(\mathrm{Var}[\overline{\mathrm{N}}])$.

The variance of matrix $\overline{\mathrm{N}}$ for a given moment is:

$$\mathrm{Var}[\overline{\mathrm{N}}] = \overline{\mathrm{N}} \cdot (2\overline{\mathrm{N}}_{dg} - \mathrm{I}) - \overline{\mathrm{N}}_{sq} \tag{25}$$

where $\overline{\mathrm{N}}_{dg}$ is a diagonal matrix of fundamental matrix $\overline{\mathrm{N}}$ at a given moment, and $\overline{\mathrm{N}}_{sq}$ is a squared matrix. If a present averaged or smoothed variance $\mathrm{Av}(\mathrm{Var}[\overline{\mathrm{N}}])$ is invariable with time, then this variance must be equal to the variance of the predicted fundamental matrix N^* which is to be found:

$$\mathrm{Av}(\mathrm{Var}[\overline{\mathrm{N}}]) = \mathrm{Var}[\mathrm{N}^*] \tag{26}$$

$$\begin{bmatrix} 0 & \cdots & \overline{n}_{1,j} \\ \vdots & & \overline{n}_{j-1,j} \\ 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} n_{1,1} & \cdots & n_{1,j} \\ \vdots & & \vdots \\ n_{j,1} & \cdots & n_{j,j} \end{bmatrix} \cdot \left\{ \begin{bmatrix} 2n_{1,1} & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 2n_{j,j} \end{bmatrix} - \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \right\} - \begin{bmatrix} n_{1,1}^2 & \cdots & n_{1,j}^2 \\ \vdots & & \vdots \\ n_{j,1}^2 & \cdots & n_{j,j}^2 \end{bmatrix}. \tag{27}$$

The averaged or smoothed variance is invariable with time because it determines the range of probabilities of each $\overline{\mathrm{N}}$ for seasonal, just as for occasional changes. Therefore this variance is also constant for future periods on condition that the range of changes is the same. A trend in credit sales or balance of accounts receivable will be incorporated into the value of an initial vector changing from period to period.

From the equality of matrices we get:

$$\begin{bmatrix} 0 & \dots & \bar{n}_{1,j} \\ \vdots & & \bar{n}_{j-1,j} \\ 0 & \dots & 0 \end{bmatrix} = \begin{bmatrix} n^*_{1,1} \cdot (n^*_{1,1} - 1) & \dots & n^*_{1,j} \cdot (2n^*_{j,j} - n^*_{1,j} - 1) \\ \vdots & & \vdots \\ n^*_{j,1} \cdot (2n^*_{1,1} - n^*_{j,1} - 1) & \dots & n^*_{j,j} \cdot (n^*_{j,j} - 1) \end{bmatrix}. \tag{28}$$

Thus, the proof is complete.

For a given fundamental matrix N^*, a steady-state age distribution under a variable initial vector can be found.

Theorem 2

If an initial vector and a fundamental matrix are variable in time, then the steady-state distribution of accounts receivable is equal to:

$$X'''_{t+1} = X''^{*}_{t,(k)} \cdot x'' \cdot (N^*_{r-1} - N^*_r)_k \tag{29}$$

Proof:

If in equation (15) the fundamental matrix is variable, then so is matrix N^*, therefore we have $(N^*_{r-1} - N^*_r)_k$. Finally, we get (29).

From equation (24) it is obvious that each entry in matrix N^* can have at least two values and that therefore a lot of predicted fundamental matrices will exist. This problem can be solved by choosing only two extreme matrices – the most and the least favourable. An average time of absorption is helpful in searching for such matrices.

The average time of absorption can be found in a very similar way to that of G.Gallinger and B.Healey [5]. A slight modification of their model is necessary when age categories have different lengths of time.

Lemma 3

If the time of staying in a given state of absorption is different, then the total average time of absorption (β) is equal to:

$$\beta = N^*_0 \cdot T_{dg} \cdot I \tag{30}$$

where N^*_0 is a fundamental matrix N^* having arisen by substituting any columns, then T_{dg} is a diagonal matrix of time length of age categories, and I is an identity column vector.

Proof:

The number of transitions from each nonabsorbing state to absorption is equal to $N^* \cdot I$ [7]. The time of remaining in any nonabsorbing state can be different, because age categories often vary; for example, the current age may be 14, the first past-due category 30, and so on. Therefore the total time of absorption is dependent on the time length of each age category. Let T_{dg} be a matrix whose diagonal entries are subsequent time lengths, while the other entries are nulls. Finally, $N_0^* \cdot T_{dg}$ is the time of absorption for each age category, and $\beta = N_0^* \cdot T_{dg} \cdot I$ is the total time.

If $\lambda_{\min}$ is the shortest time of absorption, then $N_{\max}^*$ is the most favourable fundamental matrix for a corporation and $X_{t+1}'''(\max)$ is the best age distribution of accounts receivable. If $\lambda_{\max}$ is the longest time of absorption, then $N_{\min}^*$ is the least favourable fundamental matrix and $X_{t+1}'''(\min)$ is the worst age distribution. Therefore:

$$\begin{aligned} &N_{\max}^* \quad \text{for} \quad \lambda_{\min} = \hat{I} \cdot \beta' \\ &N_{\min}^* \quad \text{for} \quad \lambda_{\max} = \hat{I} \cdot \beta'' \end{aligned} \tag{31}$$

The conclusion is that for Markovian prediction under a variable initial vector and fundamental matrix we get “optimistic” and “pessimistic” age distribution, or a range of minimal and maximal probabilities.

The models in theorems 1 and 2 enable the computation of future cash flow, bad debts from accounts receivable and also provision for doubtful accounts receivable in the way shown by R.M. Cyert, H.J. Davidson and G.L. Thompson [2].

Theorems 1 and 2 can be used to select a credit portfolio for a corporation.

4. Selection of credit portfolio

An age distribution of accounts receivable shows the amount of receivables in each age category. In the end receivables can either be paid or become bad debts. The ratio of bad debts to total credit sales is called the credit risk ratio or probability of default. This ratio is very important because it:

- represents for every client the probability of insolvency or for homogenous groups of clients called credit risk classes,
- is a fundamental variable in trade credit management because it shows future cash flow from accounts receivable or the total balance of bad debts (as well as provision for doubtful accounts receivable),
- takes into account constraints on financial standing in view of the financial and marketing strategy of a corporation through the financial standing of its clients.

If all clients are divided into credit classes (where $a = 1, ..., a'$), then for each class a credit risk ratio can be computed. An effective accounts receivable policy lies in granting or refusing credit to clients with a known risk level. Since credit policy concentrates on risk classes, it is therefore a portfolio risk measure that is important. If the cash flow from the credit portfolio of a-th risk class can be predictable, then a measure of risk level can be dispersion coefficient of that class.

In general, provision for doubtful accounts receivable is taken from a binomial distribution [2]. Let $\overline{\mathrm{Q}}_a$ be an averaged or smoothed matrix of transition from a non-absorbing state to an absorbing state under a variable initial vector and an invariable fundamental matrix. W is a two-entry zero-one vector. Therefore the total balance of bad debts is as follows:

$$\mathrm{B}_{t+1}^{(a)} = \mathrm{X}_{t,(a)}^{\prime\prime *} \cdot x_a'' \cdot (\mathrm{N}_{r-1}^{(a)} - \mathrm{N}_r^{(a)})_k \cdot \overline{\mathrm{Q}}_a \cdot W \tag{32}$$

where $\mathrm{X}_{t,(a)}^{\prime\prime *}$ are the total receivables of an a-th risk class.

Since not only the initial vector but also the fundamental matrix are variable with time, we have two values of bad debts – the most ("optimistic") and the least ("pessimistic") favourable. Therefore for matrices $\hat{\mathrm{Q}}_a$ and $\hat{\mathrm{Q}}_a$ the bad debts vectors are:

$$\mathrm{B}_{t+1}^{(a)}(\max) = \mathrm{X}_{t,(a)}^{\prime\prime *} \cdot x_a'' \cdot (\mathrm{N}_{r-1}^{*(a)}(\max) - \mathrm{N}_r^{*(a)}(\max))_k \cdot \hat{\mathrm{Q}}_a \cdot W \tag{33}$$

$$\mathrm{B}_{t+1}^{(a)}(\min) = \mathrm{X}_{t,(a)}^{\prime\prime *} \cdot x_a'' \cdot (\mathrm{N}_{r-1}^{*(a)}(\min) - \mathrm{N}_r^{*(a)}(\min))_k \cdot \hat{\mathrm{Q}}_a \cdot W \tag{34}$$

Because provision for doubtful accounts receivable can be computed as the sum of bad debts and the variance of this amount (or alternatively the variance of cash flow) taken from the binomial distribution, then we have values for provision in two investigated cases:

$$\mathrm{RB}_{t+1}^{(a)} = \mathrm{B}_{t+1}^{(a)} \cdot \left(1 + \frac{1}{\sqrt{\mathrm{B}_{t+1}^{(a)}}} \cdot \sqrt{1 - \frac{\mathrm{B}_{t+1}^{(a)}}{\mathrm{X}_{t,(a)}^{\prime\prime *}}} \right) \quad \text{or} \tag{35}$$

$$\mathrm{RB}_{t+1}^{(a)}(\max) = \mathrm{B}_{t+1}^{(a)}(\max) \cdot \left(1 + \frac{1}{\sqrt{\mathrm{B}_{t+1}^{(a)}(\max)}} \cdot \sqrt{1 - \frac{\mathrm{B}_{t+1}^{(a)}(\max)}{\mathrm{X}_{t,(a)}^{\prime\prime *}}} \right) \quad \text{and} \tag{36}$$

$$\mathrm{RB}_{t+1}^{(a)}(\min) = \mathrm{B}_{t+1}^{(a)}(\min) \cdot \left(1 + \frac{1}{\sqrt{\mathrm{B}_{t+1}^{(a)}(\min)}} \cdot \sqrt{1 - \frac{\mathrm{B}_{t+1}^{(a)}(\min)}{\mathrm{X}_{t,(a)}^{\prime\prime *}}} \right). \tag{37}$$

The expected cash flow from an a-th credit portfolio for the first case is:

$$\mathrm{E}_a = \mathrm{X}_{t,(a)}^{\prime\prime *} \cdot \mathrm{x}_a'' \cdot (\mathrm{N}_{r-1}^{(a)} - \mathrm{N}_r^{(a)})_k \cdot \overline{\mathrm{Q}}_a \cdot (\xi - W), \tag{38}$$

where ξ is an identity column vector. The expected values in the second case are obvious.

From the binomial distribution, the variance of an a-th credit portfolio [3] is equal to:

$$\mathrm{V}_a = \mathrm{X}''^{*}_{t,(a)} \cdot x''_a \cdot (\mathrm{N}^{(a)}_{r-1} - \mathrm{N}^{(a)}_r)_k \cdot \overline{\mathrm{Q}}_a \cdot (\xi - W) \cdot (1 - x''_a \cdot (\mathrm{N}^{(a)}_{r-1} - \mathrm{N}^{(a)}_r) \cdot \overline{\mathrm{Q}}_a \cdot (\xi - W)) \quad (39)$$

and similarly for $\mathrm{V}_a(\max)$ and $\mathrm{V}_a(\min)$.

The dispersion coefficient for an a-th class is the risk measure of a single portfolio. Therefore the sum of expected cash flow and the sum of its variances we the risk measure of the total portfolio for two investigated cases:

$$\mathrm{V}_\varepsilon = \frac{\sum_{a=1}^{a'} \sqrt{\mathrm{V}_a}}{\sum_{a=1}^{a'} \mathrm{E}_a} \quad \text{or} \quad \mathrm{V}_\varepsilon(\max) = \frac{\sum_{a=1}^{a'} \sqrt{\mathrm{V}_a(\max)}}{\sum_{a=1}^{a'} \mathrm{E}_a(\max)} \quad \text{and} \quad \mathrm{V}_\varepsilon(\min) = \frac{\sum_{a=1}^{a'} \sqrt{\mathrm{V}_a(\min)}}{\sum_{a=1}^{a'} \mathrm{E}_a(\min)}. \quad (40)$$

We have considered portfolio risk under present clients' structure, i.e., the risk is given up to this moment. But if this structure changes, then the total risk will change. It is necessary to control the overall risk while accepting or rejecting clients. Such a criterion has been proposed by R. Cyert and G.T hompson [3]. However, it can be developed yet further. The new criterion more precisely characterises the way the clients are to be selected.

Theorem 3

If current risk level of a total credit portfolio for $\overline{\mathrm{b}}$ number of clients, where $\overline{\mathrm{b}} = 1, \ldots, \mathrm{b}'$, and the structure of *f* portfolio are given, then the new portfolio structure should satisfy the following condition:

$$\frac{\sum_{a=1}^{a'} \sum_{\bar{b}=1}^{b'} \sqrt{\mathrm{V}'_{a,(\bar{b})}}}{\sum_{a=1}^{a'} \sum_{\bar{b}=1}^{b'} \mathrm{E}'_{a,(\bar{b})}} \le \Omega \cdot \mathrm{V}_\varepsilon \quad \text{or} \quad (41)$$

$$\frac{\sum_{a=1}^{a'} \sum_{\bar{b}=1}^{b'} \sqrt{\mathrm{V}'_{a,(\bar{b})}(\max)}}{\sum_{a=1}^{a'} \sum_{\bar{b}=1}^{b'} \mathrm{E}'_{a,(\bar{b})}(\max)} \le \Omega(\max \cdot \mathrm{V}_\varepsilon(\max) \quad \text{and} \quad \frac{\sum_{a=1}^{a'} \sum_{\bar{b}=1}^{b'} \sqrt{\mathrm{V}'_{a,(\bar{b})}(\min)}}{\sum_{a=1}^{a'} \sum_{\bar{b}=1}^{b'} \mathrm{E}'_{a,(\bar{b})}(\min)} \le \Omega(\min) \cdot \mathrm{V}_\varepsilon(\min) \quad (42)$$

in such a way that for the maximum number of clients the dispersion coefficient of the new portfolio is, at most, equal to control values: $\Omega \cdot \mathrm{V}_\varepsilon$ or $\Omega(\max) \cdot \mathrm{V}_\varepsilon(\max)$ and $\Omega(\min) \cdot \mathrm{V}_\varepsilon(\min)$, where Ω is a control factor, and:

1) clients with the least credit risk are accepted in first turn,

2) the choice of present and potential clients is related to decreasing value of credit sales for these clients.

Proof:

The structure of a portfolio changes when:

1) the number of clients decreases, usually by getting rid of bad clients,

2) the number of clients increases, often by accepting high-risk marginal clients.

A control factor Ω is a value which expresses the value of the future credit policy of a corporation. An increase in this value is suitable for a more “aggressive” credit policy. A decrease in the control factor means a more “conservative” policy. The most reasonable policy is to accept clients with the lowest credit risk and with the biggest credit purchase. Together they should increase the market value added (MVA) of the corporation on condition that NPV of credit sales is greater than zero. It should be emphasized that accepting low-risk clients also diminishes the discount rate and so the overall required rate of return of corporation. Of course, no one can ensure that the value of purchases declared by clients will be fulfilled.

A very new client (more precisely his/her credit purchase) will change the expected cash flow, its variance and the dispersion coefficient up to the control value. New clients should not alter the credit risk of classes to which they have been classified provided their financial standing is similar to that of the present clients of the corporation. Consequently, respective fundamental matrices and other matrices (excluding the initial vectors) should not change either. Expected cash flows and their variances for a-th class and $\overline{\mathrm{b}}$ client are as follows:

$$\mathrm{E}'_{a,(\bar{b})} = \sum_{\bar{b}=1}^{b'} \mathrm{X}''^{*}_{t,(a,\bar{b})} \cdot x''_a \cdot (\mathrm{N}^{(a)}_{r-1} - \mathrm{N}^{(a)}_{r})_k \cdot \overline{\mathrm{Q}}_a \cdot (\,\xi - W\,)\,, \tag{43}$$

$$\mathrm{V}'_{a,(b)} = \left(\sum_{\bar{b}=1}^{b'} \mathrm{X}''^{*}_{t,(a,\bar{b})} \right) \cdot x''_a \cdot (\mathrm{N}^{(a)}_{r-1} - \mathrm{N}^{(a)}_{r})_k \cdot \overline{\mathrm{Q}}_a \cdot (\xi - W) \cdot (1 - x''_a \cdot (\mathrm{N}^{(a)}_{r-1} - \mathrm{N}^{(a)}_{r})_k \cdot \overline{\mathrm{Q}}_a \cdot (\xi - W)) \tag{44}$$

By summing the new expected cash flows and their variances through all credit risk classes, and substituting their respective values into the dispersion coefficient we reach conditions (4.40) and (4.41).

Summary

The Markovian prediction of accounts receivable and cash flow constitutes one of classic financial operation research tools. In this paper, I have discussed some theoretical and practical problems of this approach.

First, it has been stated that if only one transition from nonabsorbing state to any other state is possible in a period, then the initial vector is made up only of current accounts receivable. In other words, in a one-period transition, for example, one month, there is no way for the initial vector to have full age categories. This simplifies computation and has further theoretical consequences.

The second investigated problem has been prediction with either the initial vector or both initial vector and transition matrices changing with time.

The total balance of accounts receivable at any moment is the sum of current receivables and past-due receivables. If accounts receivable of each age category has risen from a different initial vector, then future age distribution depends on the value of each vector which initiates the stochastic process and the fundamental matrix constant in time. In the paper a suitable prediction model has been proposed.

If a fundamental matrix is also variable with time, it has been suggested that we use a more complicated model. Because of seasonal and occasional variations in credit sales and accounts receivable, payment behaviours change, and a fundamental matrix changes, too. For such matrices variance can be computed at any moment in time. The averaged or smoothed variance for the total current period is stable in time because it expresses a definite range of variations. This variance can be employed to find a new fundamental matrix. On account of the great number of such matrices, a criterion of minimal and maximal total time of absorption is necessary. Finally a Markovian model under the initial vector and fundamental matrix variable with time has been formulated.

A provision for accounts receivable, as opposed to total cash flow from each credit risk category, can be computed from Markovian models. A ratio of credit risk is fundamental in accounts receivable management. The dispersion coefficient is a risk measure of risk classes. Accepting new clients or rejecting present clients changes the risk of each credit portfolio and the risk of the total portfolio. We have proposed defining control values of dispersion and a way of selecting clients up to control limits.

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Kryterium wyboru portfela kredytów handlowych w ujęciu procesu Markowa

Wykorzystanie procesów stochastycznych do prognozowania stanu należności z tytułu dostaw i usług (kredytów handlowych) oraz przepływów pieniężnych z tych należności należy do klasycznych problemów badań operacyjnych. Pomimo licznej literatury przedmiotu na ten temat wciąż można wskazać pewne problemy teoretyczne i aplikacyjne. W artykule omówiono strukturę wektora inicjującego proces Markowa, ponieważ jest ona dużo prostsza niż sugeruje się w literaturze przedmiotu. Rozważa się również przypadek, w którym wektor inicjujący zmienia się z okresu na okres przy stałej macierzy przejść. Ostatecznie prezentowane jest uogólnienie odpowiednie dla sytuacji, w której zarówno macierz przejść, jak i wektor inicjujący nie są stacjonarne.

Model Markowa służy sformułowaniu kryterium wyboru portfela kredytów handlowych przedsiębiorstwa przy określonym ryzyku kredytowym tych portfeli. Opiera się ono na:

- predykcji należności nieściągalnych i przepływów pieniężnych z wykorzystaniem procesu Markowa, w którym wektor inicjujący i macierz przejść nie są stacjonarne,
- analizie scenariuszowej dostarczającej kryterium brzegowego dla różnych portfeli kredytów handlowych.

Słowa kluczowe: *zastosowanie łańcuchów Markowa, zarządzanie płynnością finansową, zarządzanie należnościami*