# ON LIU´S CRITERION IN ASADA´S MODEL OF MONETARY AND FISCAL POLICY

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**Abstract**

*It is known that a simple Hopf bifurcation in a dynamic model can arise if the Jacobian matrix of the model has a pair of purely imaginary eigenvalues and others have negative real parts. Liu´s criterion gives conditions under which the eigenvalues have required properties. In this paper, a six-dimensional dynamic model of Asada (2014), describing the development of the firms´ private debt, the output, the expected rate of inflation, the rate of interest, the government expenditures, and the government bond is introduced. There are found conditions on the parameters of the model under which Liu´s criterion is satisfied. A numerical example illustrates the reached result.*

***Key words**: macroeconomic dynamic model, equilibrium, Jacobian matrix, Liu´s conditions*

***JEL Codes:** E 30, E 32 ,G180*

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## 1. Introduction

In recent times, the credibility of Minsky’s (1982,1986) financial instability hypothesis, which means that financially dominated capitalist economy is inherently unstable, is rapidly increasing. It seems that recent turbulence of the world economy proved it. For example, Japanese economy experienced the serious deflationary depression in the 1990s and the 2000s, and the serious financial crisis that started in USA by mortgage crisis in 2008 rapidly propagated to the other part of the world such as European and Asian countries. But Minsky did not think that such an inherent instability is uncontrollable by the government and the central bank. He emphasized that it is important to “stabilize an unstable economy” by means of the proper macroeconomic stabilization policies by the government and the central bank. In this respect, Minsky inherits Keynes' spirit (1936).

Although Minsky himself did not formulate the mathematical models that support his basic idea, some economists developed the mathematical dynamic models that describe his financial instability hypothesis. In particular, as a reaction especially to the deflationary depression in the Japanese economy, Asada (2012,2014) set up a series of three Keynesian/Minskian mathematical macrodynamic models that contribute to the theoretical analysis of financial instability and macroeconomic stabilization policies. They is two-dimensional model of fixed prices without active macroeconomic policy, four-dimensional model of flexible prices with central bank´s stabilization policy, and six-dimensional model of flexible prices with monetary and fiscal policy mix. In these papers Asada investigated the question of the existence of an equilibrium and its stability and sketched the possibility of the existence of cycles around this equilibrium. But a rigorous treating of the existence of cycles around an equilibrium in these Asada´s models has not been performed yet.

In this paper we investigate the Asada´s six-dimensional model with monetary and fiscal policy mix with the aim to perform some basic investigations which must be carried out at the investigation of the existence of cycles around its equilibrium. It is in particular a question of the existence of an equilibrium and finding conditions under which the eigenvalues of the corresponding Jacobian matrix have a pair of purely imaginary eigenvalues and others have negative real parts. The eigenvalues with this property are necessary for the existence of the simple Hopf bifurcation in the model. To find conditions for receiving eigenvalues of this kind is rather demanding work especially if dealing with high-dimensional models. Liu´s criterion (Liu,1994) shows to be an effective tool in this domain.

The paper is arranged as follows. Section 2 introduces the model. In section 3, utilizing Liu´s criterion, there are found conditions which guarantee that the eigenvalues of the Jacobian matrix of the model have a pair of purely imaginary eigenvalues and others have negative real parts. In section 4 a numerical example is presented by means of numerical simulations. Section 5 summarizes the achieved results of the paper and point out other possibilities in the analysis of this model.

## 2. The model

The first five equations of the model have the form

$$
\begin{aligned}
&d = \Phi(g) - s_f\left(r - i(\rho, d)d\right) - (g+\pi)d,\ 0 < s_f < 1,\\
&y = \alpha\left(c + \Phi(g) + v - y\right),\ \alpha > 0,\\
&\pi^e = \gamma\left[\xi\left(\bar{\pi} - \pi^e\right) + (1-\xi)\left(\pi - \pi^e\right)\right] = F_3\left(\pi, \pi^e\right),\quad \gamma > 0,\ 0 \le \xi \le 1,\\
&\rho = \begin{cases} \beta_1(\pi - \bar{\pi}) + \beta_2(y - \bar{y}) & \text{if } \rho > 0 \\ \max\left[0, \beta_1(\pi - \bar{\pi}) + \beta_2(y - \bar{y})\right] & \text{if } \rho = 0, \end{cases}\\
&v = \sigma\left[\theta(\bar{y} - y) + (1-\theta)\left(\bar{b} - b\right)\right],\ \sigma > 0, 0 < \theta \le 1,
\end{aligned}
\tag{1}
$$

where the meanings of the symbols are as follows: $D$ - stock of firms´ nominal private debt, $p$ - price level, $K$ - real capital stock, $d = D / pK$ - private debt-capital ratio, $\pi = p / p$ - rate of price inflation, $\pi^e$ - expected rate of price inflation, $g = K / K$ - rate of capital accumulation, $\Phi(g)$ - adjustment cost function of investment that has the properties

$\Phi_g = \partial\Phi/\partial g \geq 1, \Phi_{gg} = \partial^2\Phi/\partial g^2 > 0,$ see Uzawa (1969), $P$ - real profit, $r = P/K$ - rate of profit, $i(\rho, d)$ - nominal rate of interest which is applied to firms′ private debt, $\rho$ - nominal rate of interest of the government bond, $\rho - \pi^e$ - expected real rate of interest of the government bond, $s_f$ - firm′s internal retention rate that is issumed to be constant, $Y$ - real output (real national income), $y = Y/K$ - output-capital ratio, which is a surrogate variable of the rate of capital utilization and the rate of labor employment, $G$ - real government expenditure, $v = G/K$ - government expenditure-capital ratio, $B$ - stock of nominal government bond, $b = B/pK$ - government bond-capital ratio, $\bar{b}$ - the target value of $b$ that is set by the government, $\alpha$ - quantity adjustment speed of the disequilibrium in the goods market, $C$ - real private consumption expenditure, $c = C/K$ - private consumption expenditure-capital ratio, $T$ - real tax, $\tau = T/K$ - tax-capital ratio, $\bar{\pi}$ - target rate of inflation that is set by the central bank, $\bar{y}$ - normal value of output corresponding to the natural level of employment, $\xi$ - credibility parameter, $\gamma$ - adjustment speed of the adaptive inflation expectation, $\dot{(*)} = d(*)/dt$.

The first equation in (1) describes the dynamic law of the firms′ private debt.

The second equation in (1) expresses the Keynesian quantity adjustment process of the disequilibrium in the goods market.

The third equation in (1) is a mixed type inflation hypothesis. This is a mixture of the ′forward looking′ and the ′backward looking′ inflation expectations. In case of $\xi = 0$ it is reduced to $\pi^e = \gamma(\pi - \pi^e)$ which is a purely adaptive inflation expectation hypothesis. On the other hand, in case of $\xi = 1$ it is reduced to $\pi^e = \gamma(\bar{\pi} - \pi^e)$, which means that the publics′ expected rate of inflation gravitates towards the target rate of inflation that is set and announced by the central bank. We can consider that the parameter value $\xi$ is a measure of the ′degree of the credibility′ of the central bank′s inflation targeting, so that we call it the ′credibility parameter′.

The fourth equation in (1) formalizes an interest rate monetary policy rule by the central bank, which is a variant of the ′Taylor rule′ type monetary policy that considers both of the rate of inflation and the level of real output which is a surrogate variable of labor employment. In this formulation, the zero bound of the nominal interest rate is explicitly considered. We can consider that this is a type of the flexible inflation targeting monetary policy rule, and $\bar{\pi}$ is the target rate of inflation that is set by the central bank.

The fifth equation in (1) formalizes the government′s fiscal policy rule. This equation means that the changes of the real government expenditure respond to both of the real national income (employment) and the level of the public debt. The parameter $\theta$ is the weight of the employment consideration rather than the public debt consideration in government′s fiscal policy.

Consider the following functions and parameters:

$$\begin{aligned}
&\Phi(g) = ag^2, a > 0, \text{and such that } \Phi_g(g) = 2ag \geq 1, \Phi_{gg}(g) > 0,\\
&g = g(r, \rho - \pi^e, d) = \frac{\kappa}{1 + e^q}, q = md + o(\rho - \pi^e) - nr, r = \beta y, \beta = \frac{P}{Y},\\
&0 < \beta < 1, \kappa, m, o, n - \text{positive parameters}, \pi = \varepsilon(y - \bar{y}) + \pi^e,\\
&c = (1 - s_1)[(y - r + (1 - s_f)r - \tau(y))] + (1 - s_2)i(\rho, d)d + (1 - s_3)\rho b,\\
&0 < s_1 < 1, 0 < s_2, s_3 \leq 1, i(\rho, d) = \rho + i_1 d, i_1 > 0, \tau(y) = \tau_1 y - T_0,
\end{aligned} \tag{2}$$

$0 < \tau_1 < 1, T_0 > 0.$

Substituting (2) into (1), we receive system (3) which consists of equations

$$d = a\left(\frac{\kappa}{1+e^q}\right)^2 - s_f\left[\beta y - \left(\rho + i_1 d\right)d\right] - \left[\frac{\kappa}{1+e^q} + \varepsilon\left(y - \bar{y}\right) + \pi^e\right]d = F_1\left(d, y, \pi^e, \rho\right), \quad (3.1)$$

$$\begin{aligned} y = \alpha\Big\{&\left(1 - s_1\right)\left[\left(1 - s_f\beta - \tau_1\right)y + T_0\right] + \left(1 - s_2\right)\left(\rho + i_1 d\right)d + \left(1 - s_3\right)\rho b + \\ &a\frac{\kappa}{1+e^q} + v - y\Big\} = \alpha F_2\left(d, y, \pi^e, \rho, v, b\right), \end{aligned} \quad (3.2)$$

$$\pi^e = \gamma\left[\xi\left(\bar{\pi} - \pi^e\right) + \left(1 - \xi\right)\varepsilon\left(y - \bar{y}\right)\right] = F_3\left(y, \pi^e\right), \quad (3.3)$$

$$\rho = F_4\left(y, \pi^e\right) = \begin{cases} \beta_1\left(\pi^e - \bar{\pi}\right) + \left(\beta_1\varepsilon + \beta_2\right)\left(y - \bar{y}\right) & if\ \rho > 0 \\ \max\left[0, \beta_1\left(\pi^e - \bar{\pi}\right) + \left(\beta_1\varepsilon + \beta_2\right)\left(y - \bar{y}\right)\right] & if\ \rho = 0, \end{cases} \quad (3.4)$$

$$v = \sigma\left[\theta\left(\bar{y} - y\right) + \left(1 - \theta\right)\left(\bar{b} - b\right)\right] = F_5\left(y, b\right), \quad \sigma > 0, 0 < \theta < 1. \quad (3.5)$$

Let us set up now the sixth equation of the model which will describe the development of the government bond. Differentiating the definitional expression $b = B / pK$ with respect to time we receive after small arrangement

$$\frac{b}{b} = \frac{B\,pK - B\left(pK + pK\right)}{\left(pK\right)^2}\frac{pK}{B} = \frac{B}{B} - \frac{p}{p} - \frac{K}{K}. \quad (4)$$

Consider further relations:

$$\frac{M}{pK} = \frac{m(\rho)H}{pK} = l(y, \rho) = \varphi(\rho)y, \quad m_\rho = \frac{d\,m}{d\,\rho} > 0, \quad \varphi_\rho = \frac{d\,\varphi}{d\,\rho} < 0, \quad (5)$$

$$pT + B + H = pG + \rho B, \quad (6)$$

where $M = m(\rho)H$ - nominal money stock, $H$ - high powered money that is issued by the central bank, $m$ - money multiplier, $m > 1$.

Eq. (5) is the $LM$ equation that describes the equilibrium condition for the money market. The function $l(y, \rho) = \varphi(\rho)y$ is a particular form of the standard Keynesian real money demand function. We can express high powered money-capital ratio $h = H / pK$ in the following way: From (5) we have $m(\rho)h = l(y, \rho) = \varphi(\rho)y.$ From this we get $h = (\varphi(\rho) / m(\rho))y.$ Put $\psi(\rho) = \varphi(\rho) / m(\rho).$ Then

$$h = \psi(\rho)y, \quad \psi'(\rho) = \frac{\varphi'(\rho)m(\rho) - \varphi(\rho)m'(\rho)}{\left(m(\rho)\right)^2} < 0. \quad (7)$$

Eq. (6) is the budget constraint of the ′consolidated government′ that includes the central bank. This equation means that the goverment expenditure including the interest payment of the government bond $pG + \rho B$ must be financed by tax $pT$, bond financing $B$, or by money financing by the central bank $H$.

Substituting Eg. (6) into (4), we obtain

$$\frac{b}{b} = \frac{B}{B} - \pi - g\left(\beta y, \rho - \pi^e, d\right). \tag{8}$$

From (6) we have $B = pG + \rho B - pT - H.$ Substituting this into (8) we obtain

$$b = \left(\frac{pG - pT - H}{B}\right) b + \left[\rho - \pi - g\left(\beta y, \rho - \pi^e, d\right)\right] b. \tag{9}$$

Taking into account that

$$\frac{pG - pT}{B} b = \frac{pG - pT}{B} \frac{B}{pK} = \frac{G - T}{K} = v - \tau(y), \quad \tau = \frac{T}{K} = \tau(y), \quad \frac{H}{B} b = \frac{H}{B} \frac{B}{pK} = \frac{H}{pK},$$

we receive from (9)

$$b = v - \tau(y) - \frac{H}{pK} + \left[\rho - \pi - g\left(\beta y, \rho - \pi^e, d\right)\right] b. \tag{10}$$

Next, differentiating the definitional expression $h = H / pK$ with respect to time, we obtain the following relation

$$h = \frac{HpK - H\left(pK + pK\right)}{\left(pK\right)^2} = \frac{H}{pK} - \frac{p}{p} \frac{H}{pK} - \frac{H}{pK} \frac{K}{K} = \frac{H}{pK} - \pi h - gh.$$

From this we get

$$\frac{H}{pK} = \left[\pi + g\left(\beta y, \rho - \pi^e, d\right)\right] h + h. \tag{11}$$

Differentiating Eg. (7) with respect to time, we obtain

$$h = \psi'(\rho) y \rho + \psi(\rho) y.$$

Substituting into this equation corresponding equations from (3) we receive

$$h = \psi'(\rho) y F_4\left(y, \pi^e\right) + \psi(\rho) \alpha F_2\left(d, y, \pi^e, \rho, v, b\right). \tag{12}$$

Substituting equations (7), (11), and (12) into Eq. (10), we obtain the equation for the development of the government bond

$$b = v - \tau(y) - \left[\varepsilon\left(y - \bar{y}\right) + \pi^e + g\left(\beta y, \rho - \pi^e, d\right)\right] \psi(\rho) y - \psi'(\rho) y F_4\left(y, \pi^e\right) - \psi(\rho) \alpha$$
$$F_2\left(d, y, \pi^e, \rho, v, b\right) + \left[\rho - \varepsilon\left(y - \bar{y}\right) - \pi^e - g\left(\beta y, \rho - \pi^e, d\right)\right] b = F_6\left(d, y, \pi^e, \rho, v, b\right) \tag{13}$$

Adding equation (13) to equations (3) we get the model

$$d = F_1\left(d, y, \pi^e, \rho\right),$$
$$y = \alpha F_2\left(d, y, \pi^e, \rho, v, b\right),$$
$$\pi^e = F_3\left(y, \pi^e\right),$$

$$\rho = F_4\left(y, \pi^e\right),$$
$$\nu = F_5\left(y, b\right),$$
$$b = F_6\left(d, y, \pi^e, \rho, \nu, b\right). \tag{14}$$

### 3. Liu´s criterion

An equilibrium $E = \left(d^*, y^*, \pi^{e*}, \rho^*, \nu^*, b^*\right)$ of model (14) satisfies the condition $d = y = \pi^e = \rho = \nu = b = 0.$ From (3.3) and (3.5) we get $y^* = \bar{y}, \pi^{e*} = \bar{\pi}, b^* = \bar{b}.$ Taking this into account we receive from the relation $\pi = \varepsilon\left(y - \bar{y}\right) + \pi^e$ that at equilibrium $\pi* = \bar{\pi}.$ Therefore at equilibrium if we neglect nonnegative constraint of $\rho$ the condition $\pi^e = \rho = \nu = 0$ is satisfied. The values $d^*, \rho^*,$ and $\nu^*$ are determined by the system of equations

$$F_1\left(d^*, \bar{y}, \bar{\pi}, \rho^*\right) = 0$$
$$F_2\left(d^*, \bar{y}, \bar{\pi}, \rho^*, \nu^*, \bar{b}\right) = 0$$
$$\nu^* = \tau(\bar{y}) + \left[\pi^e + g\left(\beta\,\bar{y}, \rho^* - \bar{\pi}, d^*\right)\right]\psi\left(\rho^*\right)\bar{y} + \left[g\left(\beta\bar{y}, \rho * - \bar{\pi}, d^*\right) + \bar{\pi} - \rho^*\right]\bar{b},$$

with respect to $d^*, \rho^*,$ and $\nu^*.$ Assume, that among the solutions of these equations there is a triple $\left(d^*, \rho^*, \nu^*\right), d^* > 0, \rho^* > 0, \nu^* > 0.$ Under this assumption we have the equilibrium of model (14)

$$E = \left(d^*, y^*, \pi^{e*}, \rho^*, \nu^*, b^*\right) = \left(d^*, \bar{y}, \bar{\pi}, \rho^*, \nu^*, \bar{b}\right), d^* > 0, \bar{y} > 0, \bar{\pi} > 0, \rho^* > 0, \nu^* > 0, \bar{b} > 0.$$

It is worth to note that parameters $\alpha, \gamma, \sigma, \varepsilon, \xi, \beta_1, \beta_2, \theta$ do not have any influence on the values of an equilibrium.

Let us translate the equilibrium $E = \left(d^*, y^*, \pi^{e*}, \rho^*, \nu^*, b^*\right)$ into the origin $E_0 = (0,0,0,0,0,0)$ by translation

$$d_1 = d - d^*, y_1 = y - y^*, \pi_1^e = \pi^e - \bar{\pi}, \rho_1 = \rho - \rho^*, \nu_1 = \nu - \nu^*, b_1 = b - b^*.$$

Having done it model (14) takes the form

$$
\begin{aligned}
d_1 &= a\left(\frac{\kappa}{1+e^{q_1}}\right)^2 - s_f\left[\beta\left(y_1+\bar{y}\right)-\left(\left(\rho_1+\rho^*\right)+i_1\left(d_1+d^*\right)\right)\left(d_1+d^*\right)\right] \\
&\quad -\left[\frac{\kappa}{1+e^{q_1}}+\varepsilon\left(\left(y_1+\bar{y}\right)-\bar{y}\right)+\left(\pi_1^e+\bar{\pi}\right)\right]\left(d_1+d^*\right)=f_1\left(d_1,y_1,\pi_1^e,\rho_1\right), \\
&\quad q_1 = m\left(d_1+d^*\right)+o\left[\left(\rho_1+\rho^*\right)-\left(\pi_1^e+\bar{\pi}\right)\right]-n\beta\left(y_1+\bar{y}\right), \\
y_1 &= \alpha\Big\{\left(1-s_1\right)\left[\left(1-s_f\beta-\tau_1\right)\left(y_1+\bar{y}\right)+T_0\right]+\left(1-s_2\right)\left(\left(\rho_1+\rho^*\right)+i_1\left(d_1+d^*\right)\right) \\
&\quad \left(d_1+d^*\right)+\left(1-s_3\right)\left(\rho_1+\rho^*\right)\left(b_1+b^*\right)+a\frac{\kappa}{1+e^{q_1}}+v_1+v^*-\left(y_1+\bar{y}\right)\Big\} \\
&\quad =\alpha f_2\left(d_1,y_1,\pi_1^e,\rho_1,v_1,b_1\right), \\
\pi_1^e &= \gamma\left[\xi\left(\bar{\pi}-\left(\pi_1^e+\bar{\pi}\right)\right)+\left(1-\xi\right)\varepsilon\left(\left(y_1+\bar{y}\right)-\bar{y}\right)\right]=f_3\left(y_1,\pi_1^e\right), \\
\rho_1 &= f_4\left(y_1,\pi_1^e\right)=\begin{cases}\beta_1\left(\left(\pi_1^e+\bar{\pi}\right)-\bar{\pi}\right)+\left(\beta_1\varepsilon+\beta_2\right)\left(\left(y_1+\bar{y}\right)-\bar{y}\right) & \text{if } \rho>0 \\ \max\left[0,\beta_1\left(\left(\pi_1^e+\bar{\pi}\right)-\bar{\pi}\right)+\left(\beta_1\varepsilon+\beta_2\right)\left(\left(y_1+\bar{y}\right)-\bar{y}\right)\right] & \text{if } \rho=0,\end{cases} \\
v_1 &= \sigma\left[\theta\left(\bar{y}-\left(y_1+\bar{y}\right)\right)+\left(1-\theta\right)\left(\bar{b}-\left(b_1+\bar{b}\right)\right)\right]=f_5\left(y_1,b_1\right),\quad \sigma>0, 0<\theta<1, \\
b_1 &= v_1+v^*-\tau_1\left(y_1+\bar{y}\right)+T_0-\left[\varepsilon\left(\left(y_1+\bar{y}\right)-\bar{y}\right)+\pi_1^e+\bar{\pi}+\frac{\kappa}{1+e^{q_1}}\right]\psi\left(\rho_1+\rho^*\right)\left(y_1+\bar{y}\right) \\
&\quad +\psi'\left(\rho_1+\rho^*\right)\left(y_1+y^*\right)f_4\left(y_1,\pi_1^e\right)-\psi\left(\rho_1+\rho^*\right)\alpha f_2\left(d_1,y_1,\pi_1^e,\rho_1,v_1,b_1\right) \\
&\quad +\left[\left(\rho_1+\rho^*\right)-\varepsilon\left(\left(y_1+\bar{y}\right)-\bar{y}\right)-\pi_1^e-\bar{\pi}-\frac{\kappa}{1+e^{q_1}}\right]\left(b_1+\bar{b}\right)=f_6\left(d_1,y_1,\pi_1^e,\rho_1,v_1,b_1\right).
\end{aligned}
\tag{15}
$$

The Jacobian matrix of model (15) at the equilibrium $E_0$ is

$$
J\left(E_0\right)=\begin{pmatrix}
f_{11} & f_{12} & f_{13} & f_{14} & 0 & 0 \\
\alpha f_{21} & \alpha f_{22} & \alpha f_{23} & \alpha f_{24} & \alpha & \alpha\left(1-s_3\right)\rho^* \\
0 & \gamma\varepsilon\left(1-\xi\right) & -\gamma\xi & 0 & 0 & 0 \\
0 & \beta_1\varepsilon+\beta_2 & \beta_1 & 0 & 0 & 0 \\
0 & -\sigma\theta & 0 & 0 & 0 & -\sigma\left(1-\theta\right) \\
f_{61} & f_{62} & f_{63} & f_{64} & f_{65} & f_{66}
\end{pmatrix},
\tag{16}
$$

where

$$
\begin{aligned}
f_{11} &= -mA + md^* B - C + s_f\left(\rho^* + 2i_1 d^*\right) - \overline{\pi},\\
f_{12} &= n\beta A - n\beta d^* B - s_f \beta - \varepsilon d^*,\\
f_{13} &= oA \ - od^* B - d^*,\\
f_{14} &= -oA + od^* B + s_f d^*,\\
f_{21} &= -mA + \left(1 - s_2\right)\left(\rho^* + 2i_1 d^*\right),\\
f_{22} &= n\beta A + \left(1 - s_1\right)\left(1 - s_f \beta - \tau_1\right) - 1,\\
f_{23} &= oA,\\
f_{24} &= -oA + \left(1 - s_2\right)d^* + \left(1 - s_3\right)\overline{b},\\
f_{61} &= mB\left[\psi\left(\rho^*\right)\overline{y} + \overline{b}\right] - \psi\left(\rho^*\right)\alpha f_{21},\\
f_{62} &= -\tau_1 - \left(\varepsilon + n\beta\right)\left[\psi\left(\rho^*\right)\overline{y} + \overline{b}\right] - \psi'\left(\rho^*\right)\left[f_4(0,0) + \overline{y}\left(\beta_1\varepsilon + \beta_2\right)\right]\\
&\quad - \psi\left(\rho^*\right)\left(\overline{\pi} + C + \alpha f_{22}\right),\\
f_{63} &= -\left(1 + oB\right)\left[\psi\left(\rho^*\right)\overline{y} + \overline{b}\right] - \psi'\left(\rho^*\right)\overline{y}\beta_1 - \psi\left(\rho^*\right)\alpha f_{23},\\
f_{64} &= oB\left[\psi\left(\rho^*\right)\overline{y} + \overline{b}\right] + b - \left(\overline{\pi} + C\right)\psi'\left(\rho^*\right)\overline{y} - \psi''\left(\rho^*\right)\overline{y} f_4(0,0)\\
&\quad - \psi\left(\rho^*\right)\alpha f_2(0,0,0,0,0) - \psi\left(\rho^*\right)\alpha f_{24},\\
f_{65} &= 1 - \psi\left(\rho^*\right)\alpha,\\
f_{66} &= -\psi\left(\rho^*\right)\alpha\left(1 - s_3\right)\rho^* + \rho^* - \pi - C,\\
A &= 2a\frac{\kappa^2 e^q}{\left(1 + e^q\right)^3}, \quad B = \frac{\kappa e^q}{\left(1 + e^q\right)^2}, \quad C = \frac{\kappa}{1 + e^q}.
\end{aligned}
$$

The characteristic determinant of Jacobian matrix (16) is

$$
\left|J\left(E_0\right) - \lambda I\right| = \begin{pmatrix}
f_{11} - \lambda & f_{12} & f_{13} & f_{14} & 0 & 0\\
\alpha f_{21} & \alpha f_{22} - \lambda & \alpha f_{23} & \alpha f_{24} & \alpha & \alpha\left(1 - s_3\right)\rho^*\\
0 & \gamma\varepsilon\left(1 - \xi\right) & -\gamma\xi - \lambda & 0 & 0 & 0\\
0 & \beta_1\varepsilon + \beta_2 & \beta_1 & -\lambda & 0 & 0\\
0 & -\sigma\theta & 0 & 0 & -\lambda & -\sigma\left(1 - \theta\right)\\
f_{61} & f_{62} & f_{63} & f_{64} & f_{65} & f_{66} - \lambda
\end{pmatrix} = 0,
$$

and its characteristic equation is

$$
\lambda^6 + a_1\lambda^5 + a_2\lambda^4 + a_3\lambda^3 + a_4\lambda^2 + a_5\lambda + a_6 = 0. \tag{17}
$$

Consider the matrix which is set up of the coefficients of equation (17)

$$\Delta_6 = \begin{bmatrix} a_5 & a_6 & 0 & 0 & 0 & 0 \\ a_3 & a_4 & a_5 & a_6 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ 0 & 1 & a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & 1 & a_1 & a_2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

and its principal sub-determinants

$$\Delta_1 = a_5,\ \Delta_2 = \begin{vmatrix} a_5 & a_6 \\ a_3 & a_4 \end{vmatrix},\ \Delta_3 = \begin{vmatrix} a_5 & a_6 & 0 \\ a_3 & a_4 & a_5 \\ a_1 & a_2 & a_3 \end{vmatrix},\ \Delta_4 = \begin{vmatrix} a_5 & a_6 & 0 & 0 \\ a_3 & a_4 & a_5 & a_6 \\ a_1 & a_2 & a_3 & a_4 \\ 0 & 1 & a_1 & a_2 \end{vmatrix},$$

$$\Delta_5 = \begin{vmatrix} a_5 & a_6 & 0 & 0 & 0 \\ a_3 & a_4 & a_5 & a_6 & 0 \\ a_1 & a_2 & a_3 & a_4 & a_5 \\ 0 & 1 & a_1 & a_2 & a_3 \\ 0 & 0 & 0 & 1 & a_1 \end{vmatrix},\ \Delta_6 = \begin{vmatrix} a_5 & a_6 & 0 & 0 & 0 & 0 \\ a_3 & a_4 & a_5 & a_6 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ 0 & 1 & a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & 1 & a_1 & a_2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}. \tag{18}$$

Liu´s criterion requires

$$a_6 > 0, \Delta_1 > 0, \Delta_2 > 0, \Delta_3 > 0, \Delta_4 > 0, \Delta_5 = 0.$$

Let us firstly find conditions under which $\Delta_5 = 0$. Consider an equilibrium $E$ of model (14) at fixed value $s_3 = 1$. Denote $\Gamma = (\gamma, \varepsilon, \theta, s_3)$. Arranging $\Delta_5$ into polynomial with respect to parameter $\sigma$ on the base of (17) and (18) and potential critical parameters $\xi$ we obtain

$$\Delta_5 = d_0(\xi,\Gamma)\sigma^4 + d_1(\xi,\Gamma)\sigma^3 + d_2(\xi,\Gamma)\sigma^2 + d_3(\xi,\Gamma)\sigma + d_4(\xi,\Gamma),$$

where functions $\vartheta_j = d_j(\xi,\Gamma)$, $j = 0,1,2,3,4,$ are smooth and bounded with respect to all sets of parameters which are considered in the model. At $\varepsilon = 0, s_3 = 1, \theta = 1$ the function $d_0(\xi,\Gamma)$ is

$$d_0(\xi,\gamma,\varepsilon = 0,\theta = 1,s_3 = 1) = f_{11}f_{22}f_{66}(f_{11} + f_{66})(f_{11} - \gamma\xi)(f_{66} - \gamma\xi)\alpha^5\gamma\xi.$$

Suppose that $f_{11} \neq f_{66}$. Solve the equation $d_0(\xi,\gamma,\varepsilon = 0,\theta = 1,s_3 = 1) = 0$ with respect to $\gamma$ and $\xi$. Let $\xi = g(\gamma)$ be the solution of this equation. Consider a pair

$\tilde{\gamma}, \tilde{\xi} : d_0(\tilde{\xi}, \tilde{\gamma}, \varepsilon = 0, \theta = 1, s_3 = 1) = 0.$ Put $\delta = 1/\sigma$ and construct in a neighborhood of the values $(\tilde{\xi}, \tilde{\gamma}, \varepsilon = 0, \theta = 1, s_3 = 1, \delta = 0)$ an equation

$$F(\xi, \gamma, \varepsilon, \theta, s_3, \delta) = \delta^4 \Delta_5 = 0.$$

There is

$$F(\xi, \gamma, \varepsilon, \theta, s_3, \delta) = \delta^4 \left( d_0 \frac{1}{\delta^4} + d_1 \frac{1}{\delta^3} + d_2 \frac{1}{\delta^2} + d_3 \frac{1}{\delta} + d_4 \right) = 0$$

and

$$F(\xi, \gamma, \varepsilon, \theta, s_3, \delta) = d_0(\xi, \gamma, \varepsilon, \theta, s_3) + d_1\delta + d_2\delta^2 + d_3\delta^3 + d_4\delta^4 = 0.$$

As $F(\tilde{\xi}, \tilde{\gamma}, 0,1,1,0) = 0, \dfrac{\partial F(\tilde{\xi}, \tilde{\gamma}, 0,1,1,0)}{\partial \xi} = \dfrac{\partial d_0(\tilde{\xi}, \tilde{\gamma}, 0,1,1)}{\partial \xi} \neq 0,$ accoding to Implicit Functional Theorem there exists a function $\xi = \varphi(\gamma, \varepsilon, \theta, s_3, \delta)$ defined in a neigborhood $\Omega$ of the point $(\tilde{\gamma}, \varepsilon = 0, \theta = 1, s_3 = 1, \delta = 0)$ such that $\varphi(\tilde{\gamma}, \varepsilon = 0, \theta = 1, s_3 = 1, \delta = 0) = \tilde{\xi}$ and $F(\varphi(\gamma, \varepsilon, \theta, s_3, \delta), \gamma, \varepsilon, \theta, s_3, \delta) = \delta^4 \Delta_5 = 0.$ This means that $\Delta_5 = 0.$

Analyzing $a_6$ from (17) we find out that $a_6 > 0$ for large enough $a$ from the expression $A = 2a \dfrac{\kappa^2 e^q}{(1 + e^q)^3}$ and sufficiently small $m$ from the expression $q = md + o(\rho - \pi^e) - n\beta y.$

For $\Delta_1$ we get at the values $\varepsilon = 0, s_2 = 1, s_3 = 1, s_f = 0, \theta = 1, \beta_2 = 0$ the relation $\Delta_1 = (C + Am - Bd^* m + \bar{\pi})(C + \bar{\pi} - \rho^*)\alpha\gamma\xi\sigma$ that is positive for large enough $\kappa$ and small $m$. As all functions in the model are continuous we have that $\Delta_1 > 0$ for large enough $\kappa$, small $m, \varepsilon, s_f, \beta_2$ and for $s_2, s_3, \theta$ which are sufficiently close to the value 1.

For $\Delta_2$ we get at the values $\varepsilon = 0, s_1 = 1, s_2 = 1, s_3 = 1, s_f = 0, \theta = 1, \beta_2 = 0$ the relation

$$\Delta_2 = (-C - Am + Bd^* m - \bar{\pi})(-C - \bar{\pi} + \rho^*)\alpha\gamma\xi\sigma((-C - Am + Bd^* m - \bar{\pi})\alpha\sigma(-C - \bar{\pi} + \rho^*) +$$

$$\gamma(-(-C - Am + Bd^* m - \bar{\pi}))\alpha\xi\sigma + \alpha(-(-C - Am + Bd^* m - \bar{\pi})(-1 + An\beta)\xi - Am(An\beta - Bd^* n\beta)\xi)$$
$$(-C - \bar{\pi} + \rho^*) - \alpha\xi\sigma(-C - \bar{\pi} + \rho^*)))$$

Analyzing this relation we find out that $\Delta_2 > 0$ for large enough $\kappa$, small $\varepsilon, s_f, \beta_2, \beta$ and for $s_1, s_2, s_3, \theta$ which are sufficiently close to the value 1.

The formulae for $\Delta_3$ and $\Delta_4$ at $\varepsilon = 0, s_1 = 1, s_2 = 1, s_3 = 1, s_f = 0, \theta = 1, \beta_2 = 0$ we do not present here as they are rather big. But their analysis performed in a similar way as it was done at $\Delta_1$ and $\Delta_2$ gives that $\Delta_3 > 0$ and $\Delta_4 > 0$ for large enough $\kappa$, small $\varepsilon, s_f, \beta_2, \beta$ and for $s_1, s_2, s_3, \theta$ which are sufficiently close to the value 1.

By these considerations we found conditions under which Liu´s criteriom is satisfied.

## 4. Numerical example

**T**ake in model (14):

$\overline{y} = 0.2, \varepsilon = 1, \overline{\pi} = 0.02, \overline{b} = 0.05, \beta = 0.01, \beta_1 = 1, \beta_2 = 0.1, \tau_1 = 0.5, T_0 = 0.02, s_1 = 0.21,$
$s_2 = 0.95, s_3 = 0.5, s_f = 0.24, m = 0.5, o = 1, n = 2, \kappa = 0.2, a = 2, i_1 = 0.1, \alpha = 1, \gamma = 10, \sigma = 1,$

$\theta = 0.5, \psi(\rho) = 0.1e^{-\rho}.$

The values of the equilibrium:

$$E = \left(d^* \cong 0.169, y^* = 0.2, \pi^{e*} = 0.02, \rho^* \cong 0.028, \nu^* \cong 0.086, b^* = 0.05\right)$$

The critical value of parameter $\xi$: $\xi_0 \cong 0.484311$

Liu´s criterion:
$a_6 \cong 0.026, \Delta_1 \cong 0.232, \Delta_2 \cong 0.056, \Delta_3 \cong 0.0094, \Delta_4 \cong 0.00044$

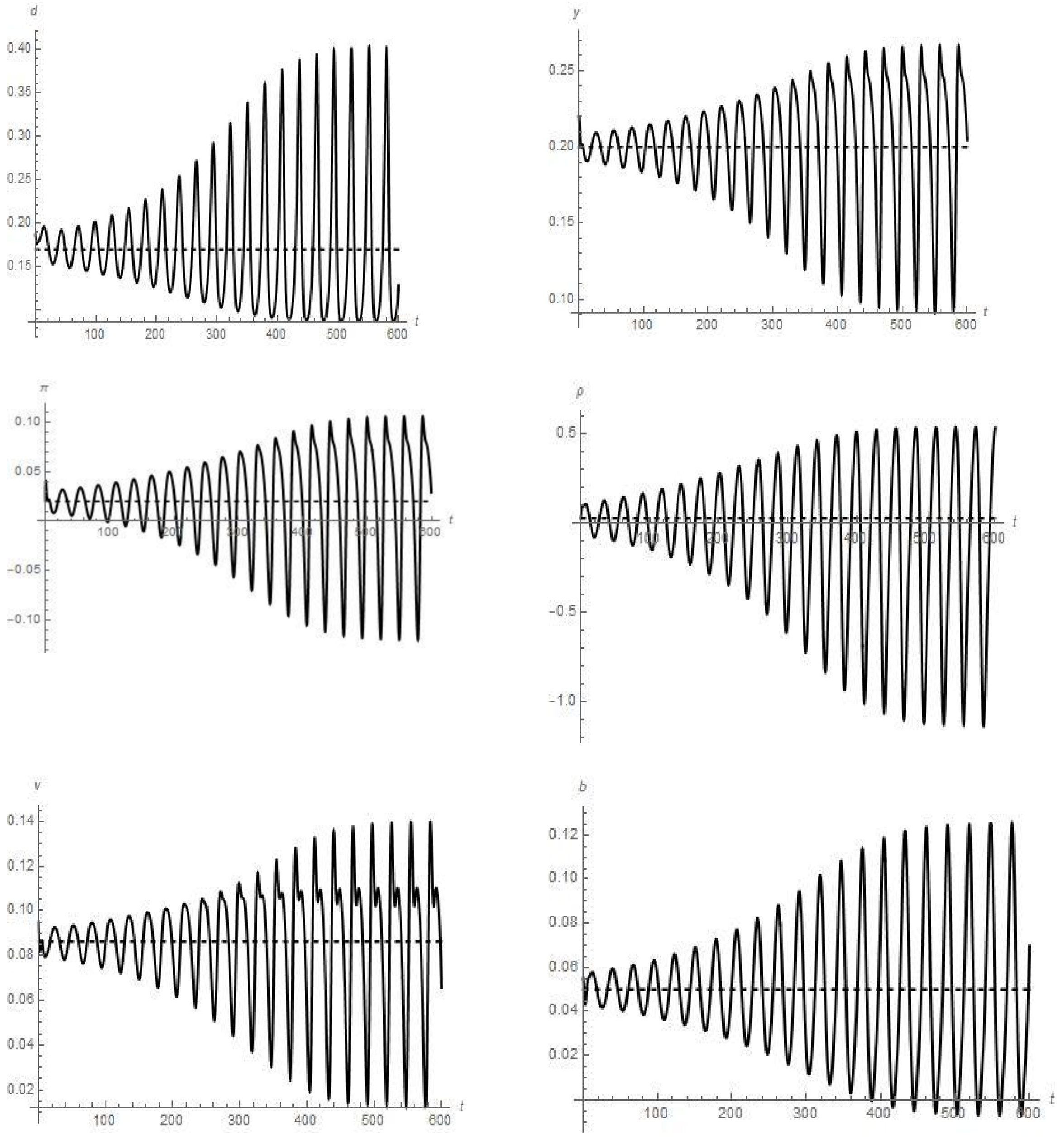
The eigenvalues of the Jacobi matrix at the critical value $\xi_0$:

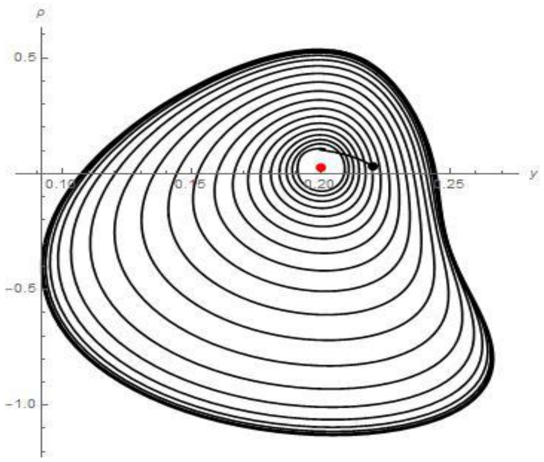
$$\left(-4.86, -0.333 + i\,0.905, -0.333 + i\,0.905, +i\,0.216, -i\,0.216, -0.114\right)$$

Remark. The exact values of all presented constants are at disposal at the corresponding author.

The structure of the eigenvalues indicates that the necessary condition for the existence of the Hopf bifurcation is satisfied. But still we cannot decide on the base of so far gained results if the Hopf bifurcation arises. For such a decision we should perform other analysis of the model and to construct its bifurcation equation. But in spite of it we can gain some knowledge on the behavior of solutions of the model in a small vicinity of the equilibrium by numerical simulations.
The pictures in figure 1 indicate that the equilibrium $E$ is unstable and solutions from its small neighborhood go to a stable cycle at the values $\xi$ that are on the left of its critical value $\xi_0$.
The pictures in figure 1 indicate that the equilibrium $E$ is unstable and solutions from its small neighborhood go to a stable cycle at the values $\xi$ that are on the left of its critical value $\xi_0$.
At the same time the pictures in figure 2 indicate that the equilibrium $E$ is stable and solutions from its small neighborhood go to it at the values $\xi$ that are on the right of its critical value $\xi_0$. On the base of this knowledge we can say that the critical value $\xi_0 \cong 0.484311$ is the bifurcation value of the model leading to the Hopf bifurcation.

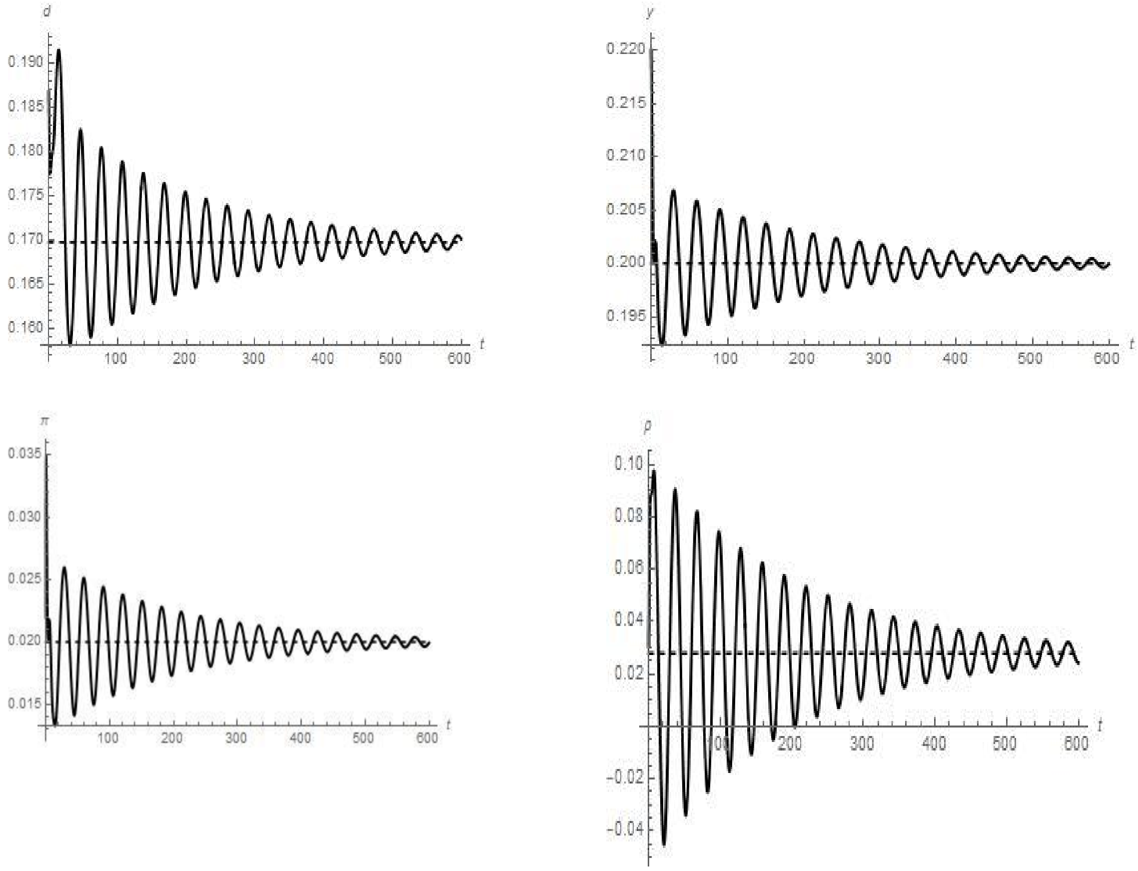
Figure 1: There are consecutively depicted the developments of the nominal firms′ private debt $d$ , the real output $y$ ,the expected rate of inflation $\pi^e$, the nominal rate of interest $\rho$, the government expenditure $v$, and the government bond $b$ of the solution of the model with the initial values $d_0 = 0.9\, d^*, y_0 = 0.9\, \bar{y},\ \ \pi_0^e = 0.9\, \bar{\pi}, \rho_0 = 0.9\, \rho^*, v_0 = 0.9\, v^*, b_0 = 0.9\, \bar{b}$ at the value $\xi = 0.9\, \xi_0$ and the projection of this solution into the plane $O_{\rho,y}$.

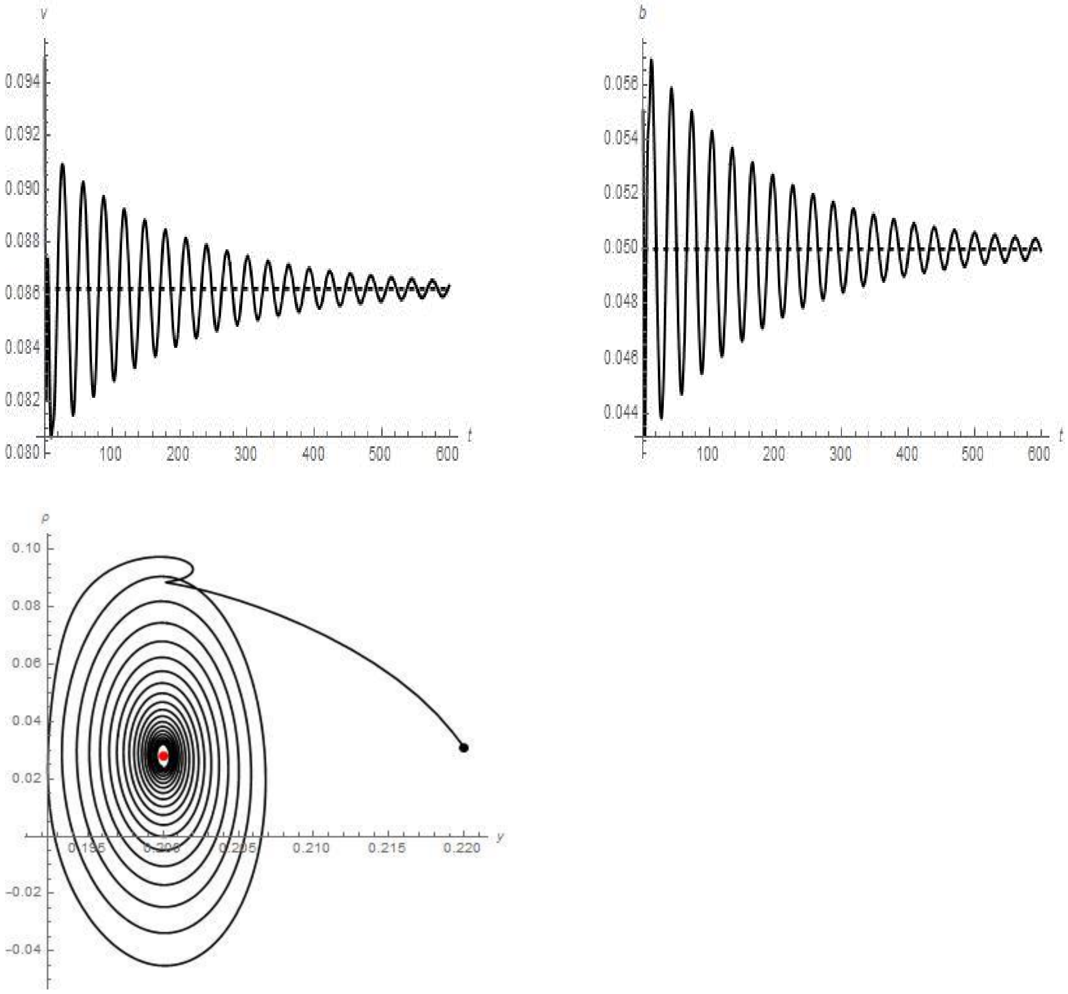





Source: the authors.

Figure 2: There are consecutively depicted the developments of the nominal firms´ private debt $d$ , the real output $y$ , the expected rate of inflation $\pi^e$, the nominal rate of interest $\rho$, the government expenditure $\nu$, and the government bond $b$ of the solution of the model with the initial values $d_0 = 1.1\,d^*, y_0 = 1.1\,\overline{y},\ \pi_0^e = 1.1\,\overline{\pi}, \rho_0 = 1.1\,\rho^*, \nu_0 = 1.1\,\nu^*, b_0 = 1.1\,\overline{b}$ at the value $\xi = 1.1\,\xi_0$ and the projection of this solution into the plane $O_{\rho,y}$.






Source: the authors.

## 5. Conclusion

In this paper, we investigated the Asada´s (2012, 2014) six-dimensional model of flexible prices with monetary and fiscal policy mix. The question of the existence of an equilibrium in this model was analyzed. Utilizing Liu´s criterion, there were found conditions under which the Jacobian matrix has the eigenvalues with one pair of purely imaginary eigenvalues and others have negative real parts. It was found that among parameters of the model the parameter $\xi$, which expresses the credibility of the activities of Central Bank in the public, is critical for the satisfaction of Liu´s criterion. The numerical example was presented and the critical value of the parameter $\xi$ was found to be $\xi_0 \cong 0.484311$. By numerical simulations there were depicted two solutions of the mode, one with the value $\xi = 0.9\,\xi_0$, and the second one with the value $\xi = 1.1\,\xi_0$. The analysis of the courses of these solutions indicates that on the left side of the critical value $\xi_0 \cong 0.484311$ the Hopf bifuracion arises. But the true assertion of this kind can be stated only after the construction of the bifurcation equation of the model. Investigation of this kind can be performed in the future.

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