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A COLLABORATIVE STRATEGY FOR A THREE ECHELON SUPPLY CHAIN WITH RAMP TYPE DEMAND, DETERIORATION AND INFLATION

A supply chain system has been investigated in which a single manufacturer procures raw materials from a single supplier, processes them to produce finished products, and then delivers the products to a single retailer. The customer's demand rate is assumed to be time-sensitive in nature (ramp type) that allows two-phase variation in the demand and production rate. Our adoption of ramp type demand reflects a real market demand for a newly launched product. Shortages are allowed with partial backlogging of demand (only for the retailer), i.e. the rest represent lost sales. The effects of inflation of the cost parameters and deterioration are also considered separately. We show that the total cost function is convex. Using this convexity, a simple algorithm is presented to determine the optimal order quantity and optimal cycle time for the total cost function. The results are discussed with numerical examples and particular cases of the model discussed briefly. A sensitivity analysis of the optimal solution with respect to the parameters of the system is carried out.

Keywords: *three echelon supply chain, ramp type demand, deterioration, backlogging, inflation*

1. Introduction

There has been a growing interest in supply chain management in recent years. The supply chain which is also referred to as the logistic network, consists of supplier distribution centres and retailer outlets, raw materials, work in process inventory, as well as finished goods that flow between the facilities. Quite a lot of researchers have shown interest in this field of study and many companies have also invested a lot capital in improving their supply chain management system.

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Historically, the three key members of the supply chain: the supplier, distributor and retailer have been managed independently buffered by large inventories. Increasing competitive pressure and decreasing marginal profitability are forcing firms to develop supply chains that can quickly respond to customer needs and furthermore reduce the cost of holding inventory. Through co-ordination between these members, the number of deliveries is derived in co-operation with each other to achieve the minimum overall integrated cost. Clark and Scarf [3] were the first authors to consider a multi-echelon supply chain in inventory research and the assumption of a constant demand rate is usually valid in the mature stage of a product life cycle. In the growth and end stage of the life cycle, the demand rate may be approximated by a linear function well. Resh et al. [21] and Donaldson [8] were the first who studied a model with linearly time varying demand. In most papers, two types of time varying demand rate have been considered in the supply chain of an inventory model: (i) Linear positive/negative trend in demand rate (ii) Exponentially increasing/decreasing demand rate. However, demand cannot increase (or decresse) continuously over time. Hill [13] proposed an inventory model with increasing demand followed by a constant demand. After that, several authors discussed time dependent demand in EOQ/EPQ (economic order quantity/economic production quantity) inventory models, as well as in models of a multi-echelon supply chain inventory, e.g. Goyal and Gunasekaran [12] considered an integrated production–inventory marketing model to determine the economic production quantity and economic order quantity for raw materials in a multi-echelon production system.

Research into deterioration and shortage of inventory are becoming more important. This is because in real life, decay and deterioration occur in almost all products, such as medicines, fruits and vegetables. Models of deteriorating inventory have been widely studied by several authors in recent years. Ghare and Schrader [10] were the first researchers to consider exponentially decaying inventory when the demand is constant. Covert and Philip [4] extended the model to consider deterioration with the Weibull distribution. Wee [29] derived model that takes into account integration between the vendor and buyer and the deterioration of items. Wu [30] investigated an inventory model with a ramp type demand rate, Weibull distributed deterioration rate and partial backlogging. Iida [15] considered a dynamic multi-echelon inventory model with non-stationary demands. Yang and Wee [31] analyzed a single vendor, multiple-buyers production inventory policy for deteriorating items with a constant production and demand rate. Khanra and Chaudhuri [16] proposed a quadratic time dependent pattern to diminish the extraordinarily high rate of change in demand for exponential time dependent demand. Manna and Chaudhari [18] have developed a production inventory with a ramp type, two time periods classified demand pattern, where the finite production rate depends on the demand. Zhou et al. [32] addressed a model of co-ordination in a two echelon supply chain with one manufacturer and one retailer, where the demand for the product by the retailer is dependent on the on-hand inventory. Skouri et al. [22] developed an inventory model with a general ramp type demand rate, the Weibull deterioration rate and partial backlogging of unsatisfied demand. They discussed two cases in their models: according to the first there is initially no shortage and according to the second there is initially a shortage. Singh and Singh [23] discussed a supply chain model with a stochastic lead time under imprecise partial backlogging and fuzzy ramp-type demand for expiring items. He et al. [14] developed a model for a two echelon supply chain inventory of deteriorating items where goods are sold to multiple markets with different selling seasons. Singh et al. [23, 24] discussed time sensitive demand, a Pareto distribution for deterioration and backlogging under a trade credit policy. Recently, Taleizadeh et al. [28] investigated an inventory model for a multi-product, multi-chance constraint, multi-buyer and single-vendor system, considering a uniformly distributed, lot size dependent demand with a lead time and partial backlogging. Singh et al. [25] discussed shortage in an economic production lot-size model with reworking and flexibility. Galanc et al. [11] analyzed a quantitative management support model of a certain production-supply system including boundary conditions. Sarkar [26] extended an EOQ model with time-varying demand and deterioration by including discounts on purchasing costs under the environment of delay-in-payments. Sinha [27] have solved some deterministic inventory models considering a finite horizon. Goyal et al. [12] discussed a production policy for amelioration/deteriorating items with ramp type demand. Chung and Cardenas-Barron [7] simplified the solution procedure for a model with deteriorating items under stock dependent demand and two level trade credits in supply chain management.

Moreover, the effects of inflation and the changing value of money as time processes are vital in any practical environment, especially in developing countries with high inflation. Therefore, the effect of inflation and the changing value of money cannot be ignored in real situations. To relax the assumption of no inflationary effects on costs, Buzacott [1] and Mishra [19] simultaneously developed an EOQ model with a constant inflation rate for all associated costs. Bierman and Thomas (1977) then proposed an EOQ model under inflation that also incorporated the discount rate. Mishra [20] then extended the EOQ model to take into account different inflation rates for various associated costs. Lo et al. [17] developed a three echelon supply chain model with an imperfect production process and the Weibull distributed deterioration under inflation with partial backlogging for the retailer. Chern et al. [5] proposed partial backlogging inventory lot-size models for deteriorating items with fluctuating demand under inflation. Chung et al. [7] developed an inventory model with non-instantaneous receipt and exponentially deteriorating items for an integrated three layer supply chain system with two levels of trade credit.

The model proposed by the author is concerned with the integration between the supplier, manufacturer and retailer, and takes into consideration different rates of deterioration in three stages of supply chain. We consider ramp type demand and production rates in a three echelon supply chain with partial backlogging and inflation.

2. Assumptions and symbols used

The following assumptions and notations are considered to develop the model.

2.1. Assumptions

- A single supplier, single manufacturer and single retailer are considered.
- The production rate $P(t)$ is demand dependent and the demand rate $d(t)$ is a ramp type function of time given by

$$d(t) = \begin{cases} f(t), & t < \mu \\ F(\mu), & t \geq \mu' \end{cases}$$

where $f(t)$ is a positive, continuous function of t, $t \in (0, T]$ and defined by

$$f(t) = a\mathrm{e}^{bt} \qquad \text{and} \qquad P = kd(t), \qquad a, b > 0,\ k > 1$$

- The deterioration rate is constant and deteriorated items are not repaired or replaced during a given cycle.
- Partial backlogging is allowed only for the retailer. The partial backlog is replenished by the next delivery.
- The model considers the effect of inflation.
- Multiple deliveries per order are considered. The planning horizon is finite and cycles during the planning horizon are continuous. Since one cycle is considered, the items included in the first delivery are made in the previous cycle.
- Supply is instantaneous and a single good is considered.

2.2. Symbols

B – fraction of retailer's demand backordered
r – inflation rate
Q_w – quantity raw materials ordered per order
Q_m – quantity of finished goods produced by manufacturer per production cycle
Q_r – quantity received by the retailer from the manufacturer per delivery
θ_1 – deterioration rate for the raw material
θ_2 – deterioration rate for finished goods stored by the manufacture
θ_3 – deterioration rate for goods stored by the retailer

n – the number of deliveries per order
$I_w(t_i)$ – inventory level of raw materials at any time t_i, where $0 \leq t_i \leq T_i$
$I_{m_i}(t_i)$ – manufacturer's inventory of finished goods level at time t_i, $0 \leq t_i \leq T_i$, $i = 1, 2$
$I_{r_i}(t_i)$ – retailer's inventory level of finished goods at time t_i $0 \leq t_i \leq T_i$, $i = 3, 4$
C_{1w} – supplier's ordering cost per order cycle
C_{1m} – manufacturer's ordering and set-up cost per order cycle
C_{1r} – retailer's ordering cost per order cycle
C_{2w} – holding cost for a unit of raw material per unit time
C_{2m} – manufacturer's holding cost for a unit of finished goods per unit time
C_{2r} – retailer's holding cost for a unit of finished goods per unit time
C_3 – retailer's backlog cost for a unit of finished goods per unit time
C_4 – retailer's cost for lost sales of finished goods per unit time
C_w – cost of raw materials per unit
C_m – cost to manufacturer of finished goods per unit
C_r – cost to retailer of finished goods per unit
MI_m – manufacturer's maximum inventory level of finished goods
MI_r – retailer's maximum inventory level of finished goods
TC_w – present value of supplier's total cost per unit time
TC_m – present value of manufacturer's total cost per unit time
TC_r – present value of retailer's total cost per unit time
TC – present value of total cost per unit time

3. Derivation of the model

The integrated flow of materials is shown in Fig. 1. Because we focus on co-operation between the supplier, manufacturer and retailer, there are two stages in our model. The first stage is the manufacturer's production system. The manufacturer purchases raw materials from outside suppliers and delivers fixed quantities of finished goods with multiple deliveries to the retailer over a fixed time interval.

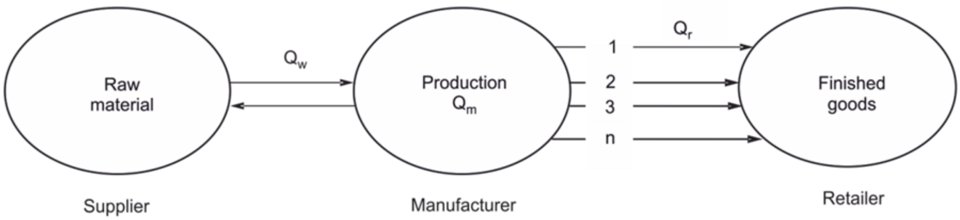

Fig. 1. Integrated flow of materials

3.1. The manufacturer's raw material inventory

A supplier procures the raw material and delivers fixed quantities Q_w to the manufacturer's warehouse at fixed time intervals. The manufacturer withdraws raw materials from the warehouse. During the time period T_1, the inventory level decreases due to both the manufacturer's demand and deterioration.

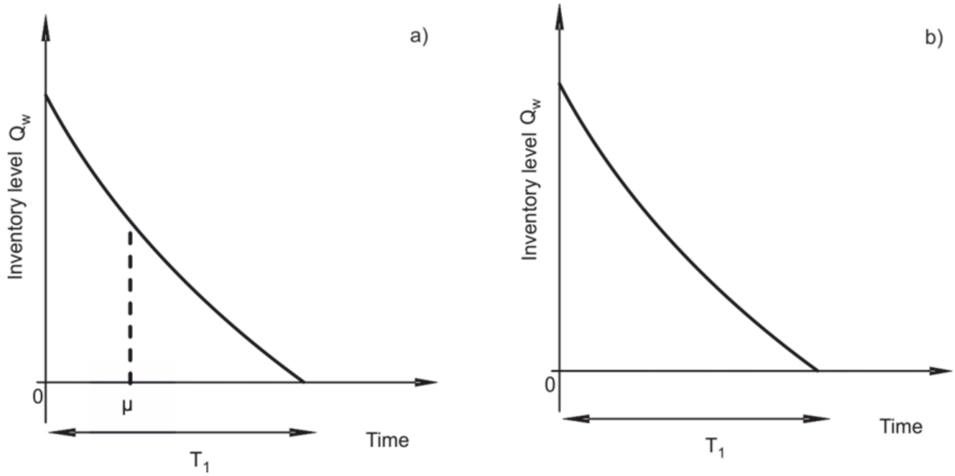

Fig. 2. Manufacturer's raw material inventory: a) $0 \le \mu \le T_1$, b) $\mu > T_1$

The manufacturer's raw materials inventory from Fig. 2a, b at any time t_1 can be represented by the following differential equation

$$\frac{dI_{wi}(t_1)}{dt_1} = -P(\mathrm{t}) - \theta_1 I_{wi}(t_1), \qquad 0 \le t_1 \le T_1 \tag{1}$$

with the boundary condition $I_{wi}(T_i) = 0$.

There are two possible relations between the parameters T_1 and μ: (i) $0 \le \mu \le T_1$ (ii) $\mu > T_1$. Each case implies a different ordering cost, holding cost and deterioration cost. Let us discuss them separately.

Case I ($0 \le \mu \le T_1$)

In this case, Eq. (1) becomes

$$\frac{dI_{w1}(t_1)}{dt_1} = -ka\mathrm{e}^{bt_1} - \theta_1 I_{w1}(t_1), \qquad 0 \le t_1 \le \mu \tag{2}$$

$$\frac{dI_{w2}(t_1)}{dt_1} = -ka\mathrm{e}^{b\mu} - \theta_1 I_{w2}(t_1), \qquad \mu \le t_1 < T_1 \tag{3}$$

with the boundary conditions, $I_{w2}(T_1) = 0$ and $I_{w1}(\mu_-) = I_{w2}(\mu_+)$.

The solutions of Eqs. (2) and (3) are

$$I_{w1}(t_1) = \frac{ka\mathrm{e}^{\mu(b+\theta_1)}}{\theta_1(b+\theta_1)}\left((b+\theta_1)\mathrm{e}^{\theta_1(T_1-\mu)} - b\right)\mathrm{e}^{-\theta_1 t_1} - \frac{ka\mathrm{e}^{bt_1}}{(b+\theta_1)}, \qquad 0 \le t_1 < \mu \tag{4}$$

$$I_{w2}(t_1) = \frac{ka\mathrm{e}^{b\mu}}{\theta_1}\left(\mathrm{e}^{\theta_1(T_1-t_1)} - 1\right), \qquad \mu \le t_1 \le T_1 \tag{5}$$

The maximum inventory level of raw materials is Q_{w1}, where $Q_{w1} = I_{w1}(0)$

$$Q_{w1} = \frac{ka\mathrm{e}^{\mu(b+\theta_1)}}{\theta_1(b+\theta_1)}\left((b+\theta_1)e^{\theta_1(T_1-\mu)} - b\right) - \frac{ka}{(b+\theta_1)} \approx ka\left(T_1 + \mu(b+\theta_1)(T_1-\mu)\right) \tag{6}$$

There is an initial ordering cost at the start of the cycle. The present value of the ordering cost is given by

$$OR_w = c_{1w} \tag{7}$$

Inventory is held during the time period T_1. The present value of holding cost is given by

$$HD_{w1} = c_{2w}\int_0^{T_1} I_w(t_1)e^{-rt_1}dt_1 = c_{2w}\left(\int_0^{\mu} I_{w1}(t_1)\,\mathrm{e}^{-rt_1}dt_1 + \int_{\mu}^{T_1} I_{w2}(t_1)\,\mathrm{e}^{-rt_1}dt_1\right)$$

$$HD_{w1} \approx c_{2w}ka\left(\mu\left(T_1 + \mu(b+\theta_1)(T_1-\mu)\right) + T_1(T_1-\mu)\right) \tag{8}$$

The costs include losses due to deterioration, as well as the cost of the items sold. Because the order is carried out at $t_1 = 0$, the present value of item cost is given by

$$IT_{w1} = c_w Q_{w1} \approx c_w ka\left(T_1 + \mu(b+\theta_1)(T_1-\mu)\right) \tag{9}$$

The present value of the total cost during the cycle is the sum of the ordering cost (OR_w), the holding cost (HD_w) and the item cost (IT_w). Hence, for the raw material, the present value of total cost per unit time is given by

$$TC_{w1} = \frac{1}{T}\left(OR_w + HD_{w1} + IT_{w1}\right) \tag{10}$$

Case II ($T_1 \le \mu \le T$)

In this case, Eq. (1) reduces to the form;

$$\frac{dI_{w1}(t_1)}{dt_1} = -ka\mathrm{e}^{bt_1} - \theta_1 I_{w1}(t_1), \qquad 0 \le t_1 < T_1 \tag{11}$$

with the boundary condition $I_{w1}(T_1) = 0.$ The solution of Eq. (11) is

$$I_{w1}(t_1) = \frac{ka}{(b+\theta_1)}\left(\mathrm{e}^{T_1(b+\theta_1)}\mathrm{e}^{-\theta_1 t_1} - \mathrm{e}^{bt_1}\right), \qquad 0 \le t_1 \le T_1 \tag{12}$$

Since $I_{w1}(0) = Q_{w2}$, then

$$Q_{w2} = \frac{ka}{(b+\theta_1)}\left(\mathrm{e}^{T_1(b+\theta_1)} - 1\right) \approx ka\left(T_1 + \frac{1}{2}(b+\theta_1)T_1^2\right) \tag{13}$$

$$HD_{w2} = c_{2w}\int_0^{T_1} I_{w1}(t_1)\mathrm{e}^{-rt_1}dt_1 \approx c_{2w}ka\left(T_1^2 + \frac{1}{2}(b+\theta_1)T_1^3\right) \tag{14}$$

$$IT_{w2} = c_w Q_{w2} \approx c_w ka\left(T_1 + \frac{1}{2}(b+\theta_1)T_1^2\right) \tag{15}$$

For the raw material, the present value of total cost per unit time is

$$TC_{w2} = \frac{1}{T}\left(OR_w + HD_{w2} + IT_{w2}\right) \tag{16}$$

3.2. Manufacturer's system for storing finished goods

The manufacture's inventory system depicted in Fig. 3a, b can be divided into two independent time intervals denoted by T_1 and T_2 (which also denote the length of these intervals). This method reduces the complexity of our problem, together with the derivation and analysis of the solution. Each phase has its own time t_i, i = 1, 2, which starts from the beginning of the phase T_i. During time period T_1, inventory builds up and hence deterioration occurs. At time $t_1 = T_1$, production stops and the inventory level increases to its maximum, MI_m. There is no production during time period T_2, and the inventory level decreases due to demand and deterioration and becomes zero at $t_2 = T_2$.

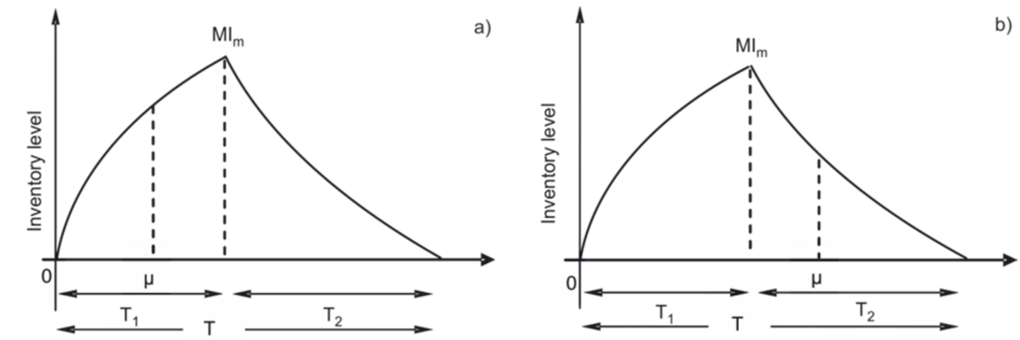

Fig. 3. Manufacturer's inventory system: a) $0 \le \mu \le T_1$, b) $T_1 \le \mu \le T$

The manufacturer's system for storing finished goods at any time t can be represented by the following differential equation (Fig. 3)

$$\frac{dI_{mi}(t_1)}{dt_1} = P(t_1) - d(t_1) - \theta_2 I_{mi}(t_1), \qquad 0 \le t_1 \le T_1 \tag{17}$$

$$\frac{dI_{mi}(t_2)}{dt_2} = -d(t_2) - \theta_2 I_{mi}(t_2), \qquad 0 \le t_2 \le T_2 \tag{18}$$

with the boundary conditions $I_{m1}(0) = 0$ and $I_{m3}(T_2) = 0$.

There are two possible relations between the parameters T_1 and μ; $0 \le \mu \le T_1$ (Fig. 3a), $T_1 \le \mu \le T$ (Fig. 3b). Each case implies a different ordering cost, holding cost and deterioration cost. Let us discuss them separately below.

Case III ($0 \le \mu \le T_1$)

In this case, Eqs. (17) and (18) become

$$\frac{dI_{m1}(t_1)}{dt_1} = (k-1)a\mathrm{e}^{bt_1} - \theta_2 I_{m1}(t_1), \qquad 0 \le t_1 \le \mu \tag{19}$$

$$\frac{dI_{m2}(t_1)}{dt_1} = (k-1)a\mathrm{e}^{b\mu} - \theta_2 I_{m2}(t_1), \qquad \mu \le t_1 \le T_1 \tag{20}$$

$$\frac{dI_{m3}(t_2)}{dt_2} = -a\mathrm{e}^{b\mu} - \theta_2 I_{m3}(t_2), \qquad 0 \le t_2 \le T_2 \tag{21}$$

with the boundary conditions $I_{m1}(0) = 0$, $I_{m1}(\mu_-) = I_{m2}(\mu_+)$, and $I_{m3}(T_2) = 0$.

The solutions of Eqs. (19)–(21) are

$$I_{m1}(t_1) = \frac{(k-1)a}{(b+\theta_2)}\left(\mathrm{e}^{bt_1} - \mathrm{e}^{-\theta_2 t_1}\right), \qquad 0 \le t_1 \le \mu \tag{22}$$

$$I_{m2}(t_1) = \frac{(k-1)a\mathrm{e}^{b\mu}}{\theta_2} - \frac{(k-1)a}{\theta_2(b+\theta_2)}\left(b\mathrm{e}^{(b+\theta_2)\mu} - \theta_2\right)\mathrm{e}^{-\theta_2 t_1}, \qquad \mu \le t_1 \le T_1 \tag{23}$$

$$I_{m3}(t_2) = \frac{a\mathrm{e}^{b\mu}}{\theta_2}\left(\mathrm{e}^{\theta_2(T_2 - t_2)} - 1\right), \qquad 0 \le t_2 \le T_2 \tag{24}$$

Based on Fig. 3a, the maximum inventory level of finished goods is

$$MI_{m1} = I_{m3}(0), \qquad MI_{m1} = \frac{a\mathrm{e}^{b\mu}}{\theta_2}\left\{\mathrm{e}^{\theta_2 T_2} - 1\right\} \approx a\mathrm{e}^{b\mu}\left(T_2 + \frac{1}{2}\theta_2 T_2^2\right) \tag{25}$$

The quantity produced in a cycle is

$$\begin{gathered} Q_{m1} = \int_0^{T_1} P(t)\,dt = \int_0^{\mu} P(t)\,dt + \int_{\mu}^{T_1} P(t)\,dt \\ Q_{m1} \approx ka\left(T_1 + bT_1\mu - \frac{1}{2}b\mu^2 + \frac{1}{2}b^2T_1\mu^2 - \frac{1}{2}b^2\mu^3\right) \end{gathered} \tag{26}$$

At the start of the cycle, the cycle has an initial production set-up cost, c_{1m}. The present value of the set-up cost is

$$SE = c_{1m} \tag{27}$$

Inventory is held during the time period T_1 and T_2. If this system does not consider the retailer, all of the holding costs belong to the manufacture. They are given by the first two terms in Eq. (28) and Eq. (29). If this system considers the retailers, the holding cost for the items that are delivered to the retailer belong to the retailer. They should be subtracted from the manufacturer's costs. They are given by the last term in Eq. (28) and Eq. (29). The present value of holding cost is (when $0 \le \mu_1 \le T_3$)

$$\begin{aligned}HD_{m1}^{1} &= c_{2m}\int_0^{T_1} I_{m1}(t_1)\mathrm{e}^{-rt_1}dt_1 + c_{2m}\int_0^{T_2} I_{m2}(t_2)\,\mathrm{e}^{-r(T_1+t_2)}dt_2 - \left(c_{2m}\int_0^{T_3} I_r(t_3)\mathrm{e}^{-rt_3}dt_3\right)\sum_{i=0}^{n-1} e^{-irT_5}\\ &= c_{2m}\left(\int_0^{\mu} I_{m1}(t_1)\,\mathrm{e}^{-rt_1}dt_1 + \int_{\mu}^{T_1} I_{m1}(t_1)\,\mathrm{e}^{-r(\mu+t_1)}dt_1\right) + c_{2m}\int_0^{T_2} I_{m2}(t_2)\mathrm{e}^{-r(T_1+t_2)}dt_2\\ &\quad - c_{2m}\left(\int_0^{\mu_1} I_{r1}(t_3)\mathrm{e}^{-rt_3}dt_3 + \int_{\mu_1}^{T_3} I_{r2}(t_3)\mathrm{e}^{-rt_3}dt_3\right)\sum_{i=0}^{n-1}\mathrm{e}^{-irT_5}\end{aligned}$$

From Sect. 3.3, Case V, and Eq. (56).

$$\begin{aligned}HD_{m1}^{1} &= c_{2m}\left(\frac{1}{2}(k-1)a\mu^2 + \frac{1}{2}(k-1)a\left(T_1^2-\mu^2\right)\mathrm{e}^{(b-r)\mu} + aT_2^2\mathrm{e}^{b\mu}\mathrm{e}^{-rT_1}\right)\\ &\quad - nc_{2m}\left(a\mu_1\left(T_3+\mu_1(b+\theta_3)(T_3-\mu_1)\right) + aT_3(T_3-\mu_1)(1+b\mu_1)\right)\end{aligned} \tag{28}$$

When $T_5 \ge \mu_1 \ge T_3$

$$\begin{aligned}HD_{m1}^{2} &= c_{2m}\left(\int_0^{\mu} I_{m1}(t_1)\mathrm{e}^{-rt_1}dt_1 + \int_{\mu}^{T_1} I_{m1}(t_1)\mathrm{e}^{-r(\mu+t_1)}dt_1\right)\\ &\quad + c_{2m}\int_0^{T_2} I_{m2}(t_2)\mathrm{e}^{-r(T_1+t_2)}dt_2 - \left(c_{2m}\int_0^{T_3} I_r(t_3)\mathrm{e}^{-rt_3}dt_3\right)\sum_{i=0}^{n-1}\mathrm{e}^{-irT_5}\\ HD_{m1}^{2} &= c_{2m}\left(\frac{1}{2}(k-1)a\mu^2 + \frac{1}{2}(k-1)a\left(T_1^2-\mu^2\right)\mathrm{e}^{(b-r)\mu} + aT_2^2\mathrm{e}^{b\mu}\mathrm{e}^{-rT_1}\right)\\ &\quad - nac_{2m}\left(T_3^2 + \frac{1}{2}(b+\theta_3)T_3^2\right)\end{aligned} \tag{29}$$

From Sect. 3.3, Case VI, and Eq. (72).

The item cost includes the loss due to deterioration, as well as the costs of the items sold. Because set up is done at $t_1 = 0$, the present value of the item cost is

$$IT_{m1} = c_m Q_{m1} \approx c_m ka\left(T_1 + b\mu T_1 - b\mu^2\right) \tag{30}$$

Therefore the present value of total cost during the cycle is the sum of the set-up cost (SE), the holding cost (HD_m) and the item cost (IT_m). The present value of total cost per unit time over the cycle is given by

$$TC_{m1}^1 = \frac{1}{T}\left(SE + HD_{m1}^1 + IT_{m1}\right), \qquad 0 \le \mu \le T_1,\ 0 < \mu_1 \le T_3 \tag{31}$$

$$TC_{m1}^2 = \frac{1}{T}\left(SE + HD_{m1}^2 + IT_{m1}\right), \qquad 0 \le \mu \le T_1,\ T_5 \ge \mu_1 \ge T_3 \tag{32}$$

Case IV ($T_1 \le \mu \le T$)

In this case, Eqs. (17) and (18) become

$$\frac{dI_{m1}\left(t_1\right)}{dt_1} = \left(k-1\right)a\mathrm{e}^{bt_1} - \theta_2 I_{m1}\left(t_1\right), \qquad 0 \le t_1 \le T_1 \qquad 33)$$

$$\frac{dI_{m2}\left(t_2\right)}{dt_2} = -a\mathrm{e}^{bt_2} - \theta_2 I_{m2}\left(t_2\right), \qquad T_1 \le t_2 \le \mu \tag{34}$$

$$\frac{dI_{m3}\left(t_2\right)}{dt_2} = -a\mathrm{e}^{b\mu} - \theta_2 I_{m3}\left(t_2\right), \qquad \mu \le t_2 \le T \tag{35}$$

with the boundary conditions $I_{m1}(0) = 0$, $I_{m2}(\mu_-) = I_{m3}(\mu_+)$, and $I_{m3}(T) = 0$.

The solutions of Eqs. (33)–(35) are

$$I_{m1}\left(t_1\right) = \frac{\left(k-1\right)a}{\left(b+\theta_2\right)}\left(\mathrm{e}^{bt_1} - \mathrm{e}^{-\theta_2 t_1}\right), \qquad 0 \le t_1 \le T_1 \tag{36}$$

$$I_{m2}\left(t_1\right) = \frac{a\mathrm{e}^{\left(b+\theta_2\right)\mu}}{\theta_2\left(b+\theta_2\right)}\left(\left(b+\theta_2\right)\mathrm{e}^{\left(T_2-\mu\right)\theta_2} - b\right)\mathrm{e}^{-\theta_2 t_2} - \frac{a\mathrm{e}^{bt_2}}{\left(b+\theta_2\right)}, \qquad T_1 \le t_1 \le \mu \tag{37}$$

$$I_{m3}\left(t_2\right)=\frac{a\mathrm{e}^{b\mu}}{\theta_2}\left(\mathrm{e}^{\theta_2\left(T_2-t_2\right)}-1\right),\qquad \mu\le t_2\le T \tag{38}$$

Based on Fig. 3b, the maximum inventory level of finished goods is $MI_{m2}=I_{m1}\left(T_1\right)$.

$$MI_{m2}=\frac{\left(k-1\right)a}{\left(b+\theta_2\right)}\left(\mathrm{e}^{bT_1}-\mathrm{e}^{-\theta_2T_1}\right)\approx\left(k-1\right)a\left(T_1+\frac{1}{2}\left(b-\theta_2\right)T_1^2\right) \tag{39}$$

The quantity produced in a cycle is given by

$$Q_{m2}=\int_0^{T_1}P\left(t\right)dt=\int_0^{T_1}ka\mathrm{e}^{bt}dt\approx ka\left(T_1+\frac{1}{2}bT_1^2\right) \tag{40}$$

when $0\le\mu_1\le T_3$.

$$HD_{m2}^1=c_{2m}\int_0^{T_1}I_{m1}\left(t_1\right)\mathrm{e}^{-rt_1}dt_1+c_{2m}\int_0^{T_2}I_{m2}\left(t_2\right)\mathrm{e}^{-r\left(T_1+t_2\right)}dt_2-\left(c_{2m}\int_0^{T_3}I_r\left(t_3\right)\mathrm{e}^{-rt_3}dt_3\right)\sum_{i=0}^{n-1}\mathrm{e}^{-irT_5}$$

$$=c_{2m}\int_0^{T_1}I_{m1}\left(t_1\right)\mathrm{e}^{-rt_1}dt_1+c_{2m}\left(\int_{T_1}^{\mu}I_{m2}\left(t_2\right)\mathrm{e}^{-r\left(T_1+t_2\right)}dt_2+\int_{\mu}^{T}I_{m3}\left(t_2\right)\mathrm{e}^{-r\left(\mu+t_2\right)}dt_2\right)$$

$$-c_{2m}\left(\int_0^{\mu_1}I_{r1}\left(t_3\right)\mathrm{e}^{-rt_3}dt_3+\int_{\mu_1}^{T_3}I_{r2}\left(t_3\right)\mathrm{e}^{-rt_3}dt_3\right)\sum_{i=0}^{n-1}\mathrm{e}^{-irT_5}$$

From Sect. 3.3, Case V, and Eq. (56).

$$\begin{aligned}HD_{m2}^1=c_{2m}&\left(\frac{1}{2}\left(k-1\right)aT_1^2+a\left(\mu-T_1\right)\mathrm{e}^{-rT_1}\left(T_2+\mu\left(b+\theta_2\right)\left(T-\mu\right)\right)\right.\\&\left.+aT_2\left(T-\mu\right)\mathrm{e}^{\left(b-r\right)\mu}\right)\\&-nc_{2m}\left(a\mu_1\left(T_3+\mu_1\left(b+\theta_3\right)\left(T_3-\mu_1\right)\right)+aT_3\left(T_3-\mu_1\right)\left(1+b\mu_1\right)\right)\end{aligned} \tag{41}$$

when $T_3\le\mu_1\le T_5$.

$$HD_{m2}^{2} = c_{2m}\int_{0}^{T_1} I_{m1}(t_1)\,\mathrm{e}^{-rt_1}dt_1 + c_{2m}\int_{0}^{T_2} I_{m2}(t_2)\,\mathrm{e}^{-r(T_1+t_2)}dt_2 - \left(c_{2m}\int_{0}^{T_3} I_r(t_3)\,\mathrm{e}^{-rt_3}dt_3\right)\sum_{i=0}^{n-1}\mathrm{e}^{-irT_5}$$

$$HD_{m2}^{2} = c_{2m}\int_{0}^{T_1} I_{m1}(t_1)\mathrm{e}^{-rt_1}dt_1 + c_{2m}\left(\int_{T_1}^{\mu} I_{m2}(t_2)\mathrm{e}^{-r(T_1+t_2)}dt_2\right.$$
$$\left. + \int_{\mu}^{T} I_{m3}(t_2)\mathrm{e}^{-r(\mu+t_2)}dt_2\right) - \left(c_{2m}\int_{0}^{T_3} I_r(t_3)\,\mathrm{e}^{-rt_3}dt_3\right)\sum_{i=0}^{n-1}\mathrm{e}^{-irT_5} \tag{42}$$

$$HD_{m2}^{2} = c_{2m}\left(\frac{1}{2}(k-1)aT_1^2 + a(\mu-T_1)e^{-rT_1}\left(T_2+\mu(b+\theta_2)(T-\mu)\right)\right.$$
$$\left. + aT_2(T-\mu)\mathrm{e}^{(b-r)\mu}\right) - nac_{2m}\left(T_3^2+\frac{1}{2}(b+\theta_3)T_3^2\right)$$

From Sect. 33, Case VI, and Eq. (72)

$$IT_{m2} = c_m Q_{m2} \approx c_m ka\left(T_1 + \frac{1}{2}bT_1^2\right) \tag{43}$$

The present value of total cost per unit time over a cycle is

$$TC_{m2}^{1} = \frac{1}{T}\left(SE + HD_{m2}^{1} + IT_{m2}\right), \qquad T_1 \le \mu \le T,\ \neq 0 \le \mu_1 \le T_3 \tag{44}$$

$$TC_{m2}^{2} = \frac{1}{T}\left(SE + HD_{m2}^{2} + IT_{m2}\right), \qquad T_1 \le \mu \le T,\ T_5 \ge \mu_1 \ge T_3 \tag{45}$$

3.3. Retailer's system for storing finished goods (when $0 \le \mu_1 \le T_3$)

The change in the retailer's inventory level is depicted in Fig. 4a, b. Since $P > d$, we assumed that the initial delivery to the retailer's inventory system is made at $t_3 = 0$. Part of the stock delivered is used to satisfy previous order, leaving a balance of MI_r units in the initial inventory.

During time period T_3, the inventory level decreases due to demand and deterioration. At $t_3 = T_3$, the inventory level is zero. During the time period T_4, part of the shortage is backlogged and part of it results in lost sales. Only the backlogged items are replaced by the next delivery. There are *n* deliveries in the $T = T_1 + T_2$ time period.

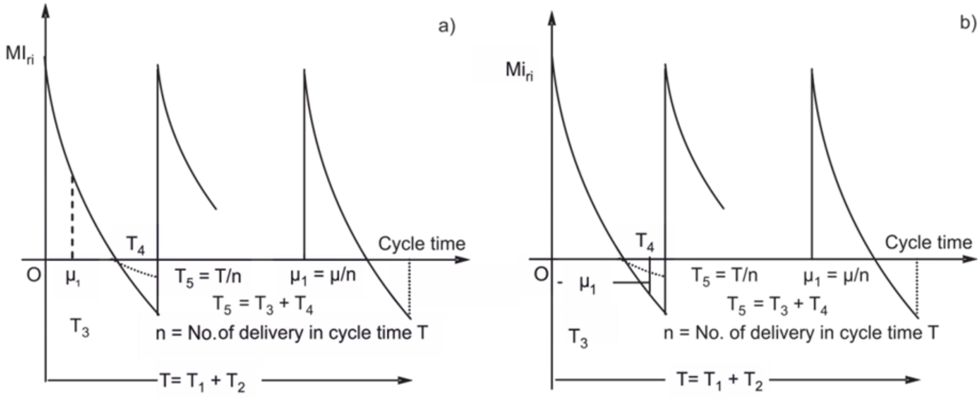

Fig. 4. Retailer's inventory system: a) $0 \le \mu_1 \le T_3$, b) $T_3 \le \mu_1 \le T_5$

The retailer's inventory system (Fig. 4) at any time *t* can be represented by the following differential equation;

$$\frac{dI_{ri}(t)}{dt} = -d(t) - \theta_3 I_{ri}(t), \qquad 0 \le t \le T_5 \tag{46}$$

with the boundary condition $I_{ri}(T_3) = 0$.

There are two possible relations between the parameters T_3, T_5 and μ: $0 \le \mu_1 \le T_3$ (Fig. 4a), $T_3 \le \mu_1 \le T_5$ (Fig. 4b). Each case implies a different ordering cost, holding cost and deterioration cost. Let us discuss them separately below.

Case V ($0 \le \mu_1 \le T_3$)

In this case, Eq. (46) becomes

$$\frac{dI_{r1}(t_3)}{dt_3} = -a\mathrm{e}^{bt_3} - \theta_3 I_{r1}(t_3), \qquad 0 \le t_3 \le \mu_1 \tag{47}$$

$$\frac{dI_{r2}(t_3)}{dt_3} = -a\mathrm{e}^{b\mu_1} - \theta_3 I_{r2}(t_3), \qquad \mu_1 \le t_3 \le T_3 \tag{48}$$

$$\frac{dI_{r3}(t_4)}{dt_4} = -Bae^{b\mu_1}, \qquad 0 \le t_4 \le T_4 \tag{49}$$

with the boundary conditions $I_{r2}(T_3) = 0$ and $I_{r3}(0) = 0$.

The solutions of Eqs. (47)–(49) are

$$I_{r1}(t_3) = \frac{ae^{(b+\theta_3)\mu_1}}{\theta_3(b+\theta_3)}\left((b+\theta_3)e^{\theta_3(T_3-\mu_1)} - b\right)e^{-\theta_3 t_3} - \frac{ae^{bt_3}}{(b+\theta_3)}, \qquad 0 \le t_3 \le \mu_1 \tag{50}$$

$$I_{r2}(t_3) = \frac{ae^{b\mu_1}}{\theta_3}\left(e^{\theta_3(T_3-t_3)} - 1\right), \qquad \mu_1 \le t_3 \le T_3 \tag{51}$$

$$I_{r3}(t_4) = -Bae^{b\mu_1}t_4, \qquad 0 \le t_4 \le T_4 \tag{52}$$

Based on Fig. 4 (a), the retailer's maximum inventory level is $MI_{r1} = I_{r1}(0)$

$$MI_{r1} = \frac{ae^{(b+\theta_3)\mu_1}}{\theta_3(b+\theta_3)}\left((b+\theta_3)e^{\theta_3(T_3-\mu_1)} - b\right) - \frac{a}{(b+\theta_3)} \tag{53}$$

The quantity supplied to the retailer per delivery is

$$Q_{r1} = MI_{r1} + Bae^{b\mu_1}T_4 \tag{54}$$

Delivery has an initial order cost (c_{1r}) incurred at the start of the delivery. The present value of ordering cost is

$$OR = c_{1r} \tag{55}$$

Inventory is held during the time period T_3. The present value of the holding cost is

$$\begin{aligned} HD_{r1} &= c_{2r}\int_0^{T_3} I_r(t_3)e^{-rt_3}dt_3 = c_{2r}\left(\int_0^{\mu_1} I_{r1}(t_3)e^{-rt_3}dt_3 + \int_{\mu_1}^{T_3} I_{r2}(t_3)e^{-rt_3}dt_3\right) \\ HD_{r1} &= c_{2r}\left(a\mu_1\left(T_3 + \mu_1(b+\theta_3)(T_3-\mu_1)\right) + ae^{b\mu_1}(T_3-\mu_1)\left(T_3 + \frac{1}{2}\theta_3 T_3^2\right)\right) \end{aligned} \tag{56}$$

Shortage occurs during the time period T_4. The present value of backlog cost is

$$BA_1 = c_3 \int_0^{T_4} \left(-I_{r3}\left(t_4\right)\right) \mathrm{e}^{-r(T_3+t_4)} dt_4 \approx Ba\mathrm{e}^{b\mu_1} c_3 \left(\frac{1}{2}\left(1 - rT_3\right)T_4^2 - \frac{1}{3} rT_4^3 \right) \tag{57}$$

Lost sales occur during the time period T_4. During this time period, the complete shortage is $\int_0^{T_4} d\left(t\right) dt$ and the partial backlogging is $\int_0^{T_4} Bd\left(t\right) dt$. The difference between them equals the amount of lost sales. The present value of the cost of lost sales is

$$LS_1 = c_4 \int_0^{T_4} \left(d\left(t\right) - Bd\left(t\right)\right) \mathrm{e}^{-r(T_3+t_4)} dt_4 \approx \left(1 - B\right) a\mathrm{e}^{b\mu_1} c_4 \left(\left(1 - rT_3\right) T_4 - \frac{1}{2} rT_4^2 \right) \tag{58}$$

The item cost includes loss due to deterioration, as well as the costs of the items sold. Because the order is carried out at $t = 0$ and $t = T_3 + T_4$, the present value of the item cost is

$$\begin{aligned} IT_{r1} &= c_r MI_{r1} + c_r Ba\mathrm{e}^{b\mu_1} T_4 \mathrm{e}^{-r(T_3+T_4)} \\ &\approx c_r \left(a\left(T_3 + \mu_1\left(b + \theta_3\right)\left(T_3 - \mu_1\right)\right) + Ba\mathrm{e}^{b\mu_1} T_4 \left(1 - r\left(T_3 + T_4\right)\right)\right) \end{aligned} \tag{59}$$

The present value of the total cost of a delivery is the sum of the ordering cost (OR), the holding cost (HD_r), the backlog cost (BA), the lost sale cost (LS) and the item cost (IT_r). The present value of the total cost per unit time for a single delivery is $TC_r^!$

$$TC_r^! = \frac{1}{T}\left(OR_r + HD_{r1} + BA_1 + LS_1 + IT_{r1}\right) \tag{60}$$

There are n deliveries per cycle. The fixed time interval between the deliveries is $T_5 = T/n$. The present value of the total cost per unit time over the cycle at $t = 0$ is

$$TC_{r1} = TC_r^! \sum_{i=0}^{n-1} \mathrm{e}^{-irT_5} = \left(\frac{OR_r + HD_{r1} + BA_1 + LS_1 + IT_{r1}}{T} \right) \left(\frac{1 - \mathrm{e}^{-rT}}{1 - \mathrm{e}^{-rT_5}} \right) \tag{61}$$

Case VI ($T_3 \le \mu_1 \le T_5$)

In this case, Eq. (46) becomes

$$\frac{dI_{r1}\left(t_3\right)}{dt_3} = -a\mathrm{e}^{bt_3} - \theta_3 I_{r1}\left(t_3\right), \qquad 0 \le t_3 \le T_3 \tag{62}$$

$$\frac{dI_{r2}\left(t_4\right)}{dt_4} = -Ba\mathrm{e}^{bt_4}, \qquad T_3 \le t_3 \le \mu_1 \tag{63}$$

$$\frac{dI_{r3}\left(t_4\right)}{dt_4} = -Ba\mathrm{e}^{b\mu_1}, \qquad \mu_1 \le t_4 \le T_5 \tag{64}$$

with the boundary conditions $I_{r1}(T_3)$ 0, $I_{r2}(0) = 0$, and $I_{r2}(\mu_1) = I_{r3}(\mu_1)$.

The solutions of Eqs. (62)–(64) are

$$I_{r1}\left(t_3\right) = \frac{a\mathrm{e}^{(b+\theta_3)T_3}}{\left(b+\theta_3\right)}\mathrm{e}^{-\theta_3 t_3} - \frac{a\mathrm{e}^{bt_3}}{\left(b+\theta_3\right)}, \qquad 0 \le t_3 \le T_3 \tag{65}$$

$$I_{r2}\left(t_4\right) = \frac{Ba}{b}\left(1-\mathrm{e}^{bt_4}\right), \qquad T_3 \le t_3 \le \mu_1 \tag{66}$$

$$I_{r3}\left(t_4\right) = Bab\mu_1^2 - Bat_4\mathrm{e}^{b\mu_1}, \qquad \mu_1 \le t_4 \le T_5 \tag{67}$$

Based on Fig. 4b, the retailer's maximum inventory level is $MI_{r2} = I_{r1}\left(0\right)$

$$MI_{r2} = a\left(T_3 + \frac{1}{2}(b+\theta_3)T_3^2\right) \tag{68}$$

The quantity supplied to the retailer per delivery is

$$Q_{r2} = MI_{r2} + Ba\left(\left(\mu_1 - T_3\right) + \frac{1}{2}b\left(\mu_1^2 - T_3^2\right) + e^{b\mu_1}\left(T_5 - \mu_1\right)\right) \tag{69}$$

$$HD_{r2} = c_{2r}\int_0^{T_3} I_r\left(t_3\right)\mathrm{e}^{-rt_3}dt_3 = ac_{2r}\left(T_3^2 + \frac{1}{2}\left(b+\theta_3\right)T_3^3\right) \tag{70}$$

$$\begin{aligned}
BA_2 &= c_3\int_0^{T_4}\left(-I_{r3}\left(t_4\right)\right)\mathrm{e}^{-r(T_3+t_4)}dt_4 \\
&= c_3\left(\int_{T_3}^{\mu_1}\left(-I_{r2}\left(t_4\right)\right)\mathrm{e}^{-r(T_3+t_4)}dt_4 + \int_{\mu_1}^{T_5}\left(-I_{r3}\left(t_4\right)\right)\mathrm{e}^{-r(T_3+t_4)}dt_4\right) \\
&\approx c_3\left(Bae^{-rT_3}\left(\mu_1^2 - T_3^2\right) + Bae^{b\mu_1}e^{-rT_3}\left(\frac{1}{2}\left(T_5^2 - \mu_1^2\right) - \frac{1}{3}r\left(T_5^3 - \mu_1^3\right)\right)\right. \\
&\quad \left. - Bab\mu_1^2\left(T_5 - \mu_1\right)\left(1 - rT_5 - r\mu_1\right)\right)
\end{aligned} \tag{71}$$

$$LS_2 = c_4 \int_0^{T_4} \left(d\left(t\right) - Bd\left(t\right)\right) \mathrm{e}^{-r\left(T_3+t_4\right)} dt_4 \approx \left(1-B\right) a \mathrm{e}^{bT_3} c_4 \left(T_5 - T_3 + b\mu_1\left(T_5 - \mu_1\right)\right) \quad (72)$$

$$IT_{r2} = c_r MI_{r2} + \left(c_r \int_0^{T_4} Bd\left(t\right)dt \right) \mathrm{e}^{-r\left(T_3+T_4\right)} \approx c_r \left(a\left(T_3 + \frac{1}{2}\left(b+\theta_3\right)T_3^2 \right) \right.$$

$$\left. + Ba\left(T_5 - T_3 + b\mu_1\left(T_5 - \mu_1\right)\right)\left(1 - r\left(T_3 + T_4\right)\right)\right) \quad (73)$$

The present value of the total cost incurred by a single delivery is the sum of the ordering cost (*OR*), the holding cost (HD_r), the backlog cost (*BA*), lost sale cost (*LS*) and the item cost (IT_r). The present value of the total cost incurred per unit time by a single delivery is

$$TC_r^{!!} = \frac{1}{T}\left(OR_r + HD_{r2} + BA_2 + LS_2 + IT_{r2}\right) \quad (74)$$

There are *n* deliveries per cycle. The fixed time interval between the deliveries T_5 is T/n. Thus, the present value of the total cost per unit time over a cycle at $t = 0$ is

$$TC_{r2} = TC_r^{!!} \sum_{i=0}^{n-1} \mathrm{e}^{-irT_5} = \left(\frac{OR_r + HD_{r2} + BA_2 + LS_2 + IT_{r2}}{T} \right) \left(\frac{1 - \mathrm{e}^{-rT}}{1 - \mathrm{e}^{-rT_5}} \right) \quad (75)$$

It is obvious that the results of Section 3 are not enough to derive the total system cost for this case. Thus the determination of the total cost, system cost requires the further examination of the ordering relations between the time parameters T_1, T_2, T_3, T_4, T_5, T, μ_1 and μ. Now we have all the quantities needed to formulate the total system cost and proceed with its optimization.

4. The optimal replenishment policy

The results in the previous sub-section lead to the following total system cost over the time interval [0, *T*];

Case A, $0 \le \mu_1 \le T_3$

$$TC_h = \begin{cases} TC_1 & 0 < \mu \le T_1 \quad \text{(a)} \\ TC_2 & T_1 \le \mu < T \quad \text{(b)} \end{cases} \quad (76)$$

$$TC_1 = TC_{w1} + TC_{m1}^1 + TC_{r1}, TC_2 = TC_{w2} + TC_{m2}^1 + TC_{r2}$$

Case B, $T_3 \le \mu_1 \le T_5$

$$TC_g = \begin{cases} TC_3 & 0 < \mu \le T_1 \quad \text{(a)} \\ TC_4 & T_1 \le \mu < T \quad \text{(b)} \end{cases} \tag{77}$$

$$TC_3 = TC_{w1} + TC_{m1}^2 + TC_{r1},\ \ TC_4 = TC_{w2} + TC_{m2}^2 + TC_{r2}$$

and the problem is $\min TC\left(T_1^*\right)$.

We have used a second degree polynomial to approximate an exponential function. The resulting model with a single supplier, a single retailer and a single manufacturer is developed to derive the optimal production policy and lot size. Since $T = T_1 + T_2$, $T_5 = T/n$, and it is assumed that $T_4 = \alpha T_3$, where $0 < \alpha << 1$, its solution requires separately studying each of the branches and then combining the results to obtain the optimal policy. It is easy to check that $TC(T_1)$ is continuous at the point μ. The first order condition for a minimum of $TC_1(T_1)$ is

$$\frac{dTC_1\left(T_1\right)}{dT_1} = 0$$

$$\begin{aligned} &are^{b\mu}c_{2m}T_1^2 + \left(2kac_{2w} + a\left(k-1\right)c_{2m}\mathrm{e}^{\mu(b-r)} - 2ac_{2m}\mathrm{e}^{b\mu}\right)T_1 \\ &\quad + \left(ka\mu^2 c_{2w}\left(b+\theta_1\right) + kac_w\left(1 + b\mu + \mu\theta_1\right)\right. \\ &\quad \left. + kac_m\left(1+b\mu\right) + 2ac_{2m}\mathrm{e}^{b\mu}T - ac_{2m}\mathrm{e}^{b\mu}T^2\right) = 0 \end{aligned}$$

Suppose the derivative $\dfrac{dTC_1\left(T_1\right)}{dT_1} = 0$ at $T_{1.1}$ with $0 \le T_{1.1} \le T$. For this we have

$$\left(\frac{\partial^2 TC_1\left(T_1\right)}{d\partial_1^2}\right)_{T_1 = T_{1.1}} = 2are^{b\mu}c_{2m}T_1 + \left(2kac_{2w} + a\left(k-1\right)c_{2m}\mathrm{e}^{\mu(b-r)} - 2ac_{2m}e^{b\mu}\right) > 0$$

which shows the convexity of the function TC.

Solution procedure

The problem is to determine the value of n and T_1 that minimize the TC. Since the number of deliveries per order, n, is a discrete variable, the following procedure is proposed to determine the optimal production policy:

Step 1. Since the number of deliveries, n, is an integer value, we start by choosing an integer value of $n \geq 1$.

Step 2. Determine the first derivatives of TC_i (i = 1, 2, 3, 4) with respect to T_1 and equate them to zero.

Step 3. Find the optimal value of T_1 for given n, which is denoted by $T_1^*(n)$ or simply T_1^* when there is no ambiguity regarding the value of n. The total cost in this case is given by $TC(n, T_1^*)$.

Step 4. Repeat steps 1–3 for all the possible values of n until the minimum TC is found such that $TC(n^* - 1, T_1) \geq TC(n^*, T_1^*)$ and $TC(n^*, T_1^*) \leq TC(n^* + 1, T_1)$.

5. Numerical examples and sensitivity analysis

In this section, we provide some numerical examples to illustrate the theoretical results obtained in the previous sections. In addition, we also carry out a sensitivity analysis for the effect of the values of the most important parameters on the optimal order quantity and total system cost.

Example 1

The input parameters are: c_{1w} = \$ 100 per order, c_{1m} = \$ 90 per order, c_{1r} = \$ 50 per order, c_{2w} = \$ 1per unit per week, c_{2m} = \$ 5 per unit per week, c_{2r} = \$ 6 per unit per week, c_w = \$ 10 per unit, c_m = \$ 15 per unit, c_r = \$ 20 per unit, $\theta_1 = 0.05$, $\theta_2 = 0.06$, $\theta_3 = 0.09$, $B = 0.8$, $r = 0.06$, c_3 = \$ 15 per unit, c_4 = \$ 35 per unit, k =3, a =1, b = 2, T = 20 weeks, μ = 1 week, $\alpha = 0.2$.

Using Eq. (76a), we find the optimal values of T_1 = 5.10 weeks, and T_2 = 14.90 weeks for n =5 , $\mu_1 = 0.2$, T_3 = 3.33 weeks, T_4 = 0.67 weeks and the optimal value of the total system cost is TC_1 = \$ 244.10. We can see that the results we have found from this analysis satisfy the condition of convexity and the conditions of Eq. (76a) such as $0 \leq \mu_1 \leq T_3$, $0 \leq \mu \leq T_1$.

Using Eq. (76b), we found the optimal values of T_1 = 3.52 weeks, and T_2 = 16.48 weeks for n = 4, μ_1 = 2, T_3 = 4.16 weeks, T_4 = 0.83, and the optimal value of the total system cost TC_1 = \$ 31,17. Here, $0 \leq \mu_1 \leq T_3$, $T_1 \leq \mu \leq T$.

Example 2

In this, example, $T_3 \leq \mu_1 \leq T_5$, the input parameters are the same as in Example 1. Using Eq. (77a), we found the optimal values of T_1 = 4.11 weeks, and T_2 = 15.89 weeks for n =5, $\mu = 2$, $\mu_1 = 0.4$, $T_3 = 3.33$ weeks, $T_4 = 0.67$ weeks and the optimal value of the total system cost is TC_1 = \$ 477.67. This means that there is no feasible solution in this case that satisfies both of the conditions $T_3 \le \mu_1 \le T_5$ and $0 \le \mu \le T_1$.

Using the data from Example 1, the sensitivity analysis is performed to explore the effect of changes in some of the model parameters (a, b, μ, T) on the optimal policy (i.e. on the optimal order quantity and optimal total system cost). The results are presented in Table 1 and some interesting findings are summarized as follows:

- Increases in the first demand parameter (a) have no impact on the optimal production time, while the total system cost also increases. But increases in the second demand parameter (b) lead to a decrease in the production time and an increase in the total system cost.
- Increases in the time parameter (μ) lead to a decrease in T_1 and an increase in the total system cost. Increases in the parameter T lead to increases in both T_1 and the total system cost.
- The changes in the optimal total system cost indicate that the model is highly sensitive to the changes on a, b, μ and T_1.

Table 1. Sensivity analysis

Parameter change	Percentage	T_1	TC_1	Parameter change	Percentage	T_1	TC_1
μ	–50	8.03	–29.91	a	–50	0.00	–48.77
	–25	4.11	–15.16		–25	0.00	–24.59
	25	–4.31	14.75		25	0.00	24.18
	50	–8.62	29.09		50	0.00	48.36
T	–50	–73.72	–29.91	b	–50	8.23	–31.14
	–25	–36.47	–13.11		–25	4.41	–15.98
	25	36.07	9.42		25	–4.31	16.93
	50	71.76	15.98		50	–8.62	32.78

6. Concluding remarks

A model for a three echelon inventory with deteriorating items and a ramp type demand rate and ramp type production rate under inflation is studied. In this model, the retailer is allowed to have shortages which are partially backlogged. The model assumes an individual deterioration rate for each party. The possible ordering relations between the time parameters lead to four different situations. The optimal production policy was derived for one of them. Convexity was also proved for one case. An easy to use algorithm to find the optimal production policy and optimal production time is presented. Some numerical examples are studied to illustrate the proposed model. The sensitivity of the solution to changes in the value of different parameters has also been discussed. Here the retailer's shipment time is completely independent of the production time, it is dependent on the cycle time T, number of shipments n and the factor α. this means that we can choose the value of n. The proposed model can be used to determine the total system cost when all the parties work together, together with the optimal production time. This paper may be extended by using a two-parameter Weibull distribution to model the deterioration rate. A very interesting extension would be to permit delays.

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