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An Approach to the Direct Recovery in Coherent Imaging

In the papers [1]–[5] the principle of the so-called direct recovery problem in incoherent imaging have been considered. The present paper presents an attempt to modify the incoherent recovery procedure so that it may also be applied to the coherent case.

Suppose that a coherent object is imaged by an imaging system A and next scanned with the help of an observing system B, the latter consisting of an imaging stage and an integrating element E (see Fig. 1). The measurement representation of the image obtained by locating the observing system in a number of points in the image is not identical with the intensity distribution in the object or image and thus the last two have to be recovered. Hereafter, an idea of the recovering procedure is proposed for the case when no a priori information about the object is available before the measurement and the object itself is of amplitude modulation type.

1. Mechanism of an Observed Measurement Point Creation

It is clear that an observed measurement point understood as the result of measurement taken with the observing system at a given point a of its location with respect to the image plane (see Fig. 1) appears as a consequence of two successive transformations:

I. Transformation by the imaging system A

$$U_{\mathrm{im}}(\boldsymbol{p}) = \int_{P_0} U_{\mathrm{ob}}(\boldsymbol{a}) K_{\mathrm{im}}(\boldsymbol{a},\boldsymbol{p})\, d\boldsymbol{a} \tag{1}$$

resulting in the image intensity distribution

$$I_{\mathrm{im}}(\boldsymbol{p}) = \left|\int_{P_0} U_{\mathrm{ob}}(\boldsymbol{a}) K_{\mathrm{im}}(\boldsymbol{a},\boldsymbol{p})\, d\boldsymbol{a}\right|^2, \tag{2}$$

where

$U_{\mathrm{ob}}(\boldsymbol{a})$ denotes the complex amplitude distribution in the object plane P_0,

$K_{\mathrm{im}}(\boldsymbol{a},\boldsymbol{p})$ is the amplitude impuls response of the imaging system,

$U_{\mathrm{im}}(\boldsymbol{p})$ is the complex amplitude distribution in the image plane P (see Fig. 1).

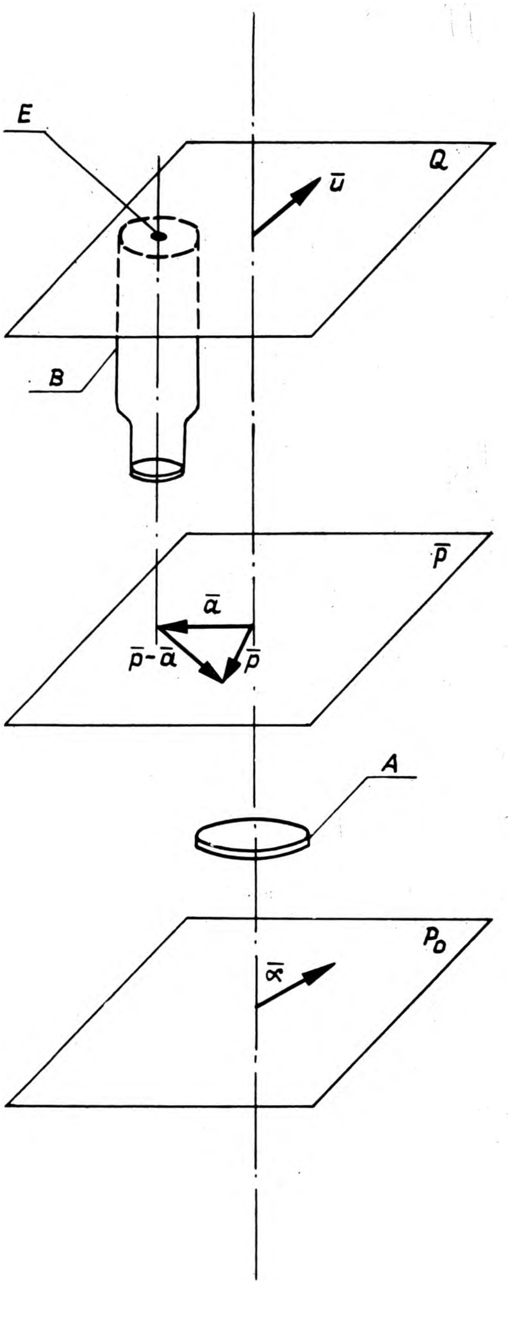

Fig. 1. Notation:
$\boldsymbol{a} = (\alpha, \beta)$ – radius vector in the object plane P_0, $d\boldsymbol{a} = d\alpha d\beta$,
$\boldsymbol{p} = (p, q)$ – radius vector in the image plane P,
$d\boldsymbol{p} = dp dq$,
$\boldsymbol{a} = (a, b)$ – position vector of the observing system,
$\boldsymbol{u} = (u, s)$ – radius vector in the observation plane A,
$d\boldsymbol{u} = du dv$.

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II. Transformation by the observing system.

It is convenient to distinguish two steps of this transformation:

a) Transformation by the imaging part of the observing system. Here, the complex amplitude distribution in the observation plane Q is given by

$$U_{\mathrm{obs}}(u, a) \int_{P} = U_{\mathrm{im}}(p) K_{\mathrm{obs}}(p-a, u) dp =$$

$$\int\int_{P\,P_0} U_{\mathrm{ob}}(a) K_{\mathrm{im}}(a, p) K_{\mathrm{obs}}(p-a, u) da dp \tag{3}$$

the corresponding intensity distribution being defined by

$$I_{\mathrm{obs}}(u, a) = |U_{\mathrm{obs}}(u, a)|^2 =$$

$$= |\int\int_{P\,P_0} U_{\mathrm{ob}}(a) K_{\mathrm{im}}(a, p) K_{\mathrm{obs}}(p-a, u) da dp|^2.$$

b) Transformation by the integrating element

$$x(a) = \int_{E} I(a, u) du =$$

$$\int\int\int_{E\,P\,P_0} U_{\mathrm{ob}}(a) U^*_{\mathrm{ob}}(a') K_{\mathrm{im}}(a, p) K^*_{\mathrm{im}}(a', p') \times$$

$$\times K_{\mathrm{obs}}(p-a, u) K^*_{\mathrm{obs}}(p'-a, u) da da' dp dp' du, \tag{4}$$

where $K_{\mathrm{obs}}(p-a, u)$ is the coherent impuls response of the imaging part of the observing system, $U_{\mathrm{obs}}(u, a)$ is the complex amplitude distribution in the observation plane and E denotes the integrating element (see Fig. 1).

Defining

$$\varphi_c(p-a, p, a) = \int_{E} K_{\mathrm{obs}}(p-a, u) K^*_{\mathrm{obs}}(p'-a, u) du \tag{5}$$

as a coherent instrumental function we can rewrite (4) in the form

$$x(a) = \int\int_{P\,P_0} U_{\mathrm{ob}}(a) U^*_{\mathrm{ob}}(a') K_{\mathrm{im}}(a, p) K^*_{\mathrm{im}}(a', p') \times$$

$$\times \varphi_c(p-a, p-a) da da' dp dp' \tag{6}$$

and call it an observed image point.

It is striking that (similarly to the incoherent case [5]) there is no immediate connection between the observed image $x(a)$, being a measurement representation and the intensity or amplitude distribution in the image.

2. General Problem Formulation

Suppose that a series of the observed image points $x(a_k)$ for $k = 1, \dots, N$ is given as a result of scanning the image with the observing system. Find the corresponding values of the intensity in the image-plane for the same set of $k = 1, \dots, N$ being called the real image points.

It is evident, that the problem can not be solved in a unique way. Hereafter, we are going to determine the limiting values of the real image points, which would still be consistent with the given set $x(a_k)$ of the observed image points.

For this purpose we have to accept some complex amplitude distributions, which would be believed to be extreme admissible ones and show that they are consistent with the measurement results. Let us assume for the upper bound object amplitude distribution

$$U_{\mathrm{ob}}(a) = \sum^{N} b_n \delta(a-a_n), \tag{7}$$

where $a_n = \frac{a_n}{\gamma}$, γ — magnification of the imaging system and for the lower bound object amplitude distribution

$$U_{\mathrm{ob}}(a) = \sum_{n=1}^{N} b'_n \delta(a-a'_n), \tag{8}$$

where a'_n — points located exactly in between the respective points set a_n (see Fig. 2).

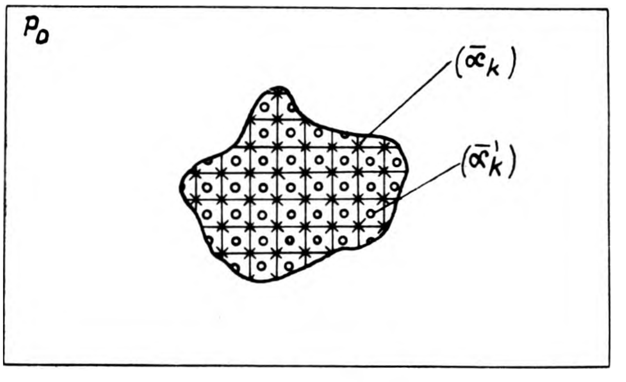

Fig. 2. A region in the object plane to be recovered. An illustration of the mutual position of $a_k = \frac{a_k}{\gamma}$ (where γ — magnification of the imaging system) and a'_k

Substituting to (6) the equations (7) and (8) we obtain

$$x(a_k) = \sum_{nm}^{N} b_n b^*_m \int_{P_0} K_{\mathrm{im}}(a_n, p) K^*_{\mathrm{im}}(a_n, p') \times$$

$$\times \varphi_c(p-a, p'-a) dp dp'$$

and

$$x(a_k) = \sum_{nm}^{N} b'_m b'^*_m \int_{P_0} K_{\mathrm{im}}(a'_n, p) K^*_{\mathrm{im}}(a'_n, p) \times$$

$$\times \varphi_c(\boldsymbol{p}-\boldsymbol{a}, \boldsymbol{p}'-\boldsymbol{a})\,d\boldsymbol{p}\,d\boldsymbol{p}'.$$

Calling

$$A_{nmk} = \int_{P_0} K_{\mathrm{im}}(\boldsymbol{a}_n, \boldsymbol{p}) K^*_{\mathrm{im}}(\boldsymbol{a}_m, \boldsymbol{p}')\varphi_c(\boldsymbol{p}-\boldsymbol{a}, \boldsymbol{p}'-\boldsymbol{a})\,d\boldsymbol{p}\,d\boldsymbol{p}' \tag{9}$$

the upper bound reconstruction matrix for coherent imaging and

$$A'_{nmk} = \int_{P_0} K_{\mathrm{im}}(\boldsymbol{a}'_n, \boldsymbol{p}) K^*_{\mathrm{im}}(\boldsymbol{a}'_m, \boldsymbol{p}')\varphi_c(\boldsymbol{p}-\boldsymbol{a}, \boldsymbol{p}'-\boldsymbol{a})\,d\boldsymbol{p}\,d\boldsymbol{p}' \tag{10}$$

the lower bound reconstruction matrix for coherent imaging we get two systems of equations

$$x(\boldsymbol{a}_k) = \sum_{mn}^{N} b_n b^*_m A_{nmk}, \quad k = 1, \ldots, N \tag{11}$$

$$x(\boldsymbol{a}_k) = \sum_{nm}^{N} b'_n b'^*_m A_{nmk}, \tag{12}$$

which have to be solved with respect to b_n and b'_n accordingly. Note that A_{nmk} contain, by definition, all the information about the imaging system, the observing system and the way of scanning and, therefore, may be treated as known quantities.

3. Amplitude Modulation Objects

We shall present a procedure of solving the recovery problem for a special case of amplitude modulation objects. Representing b_n in the form

$$|b_n| e^{i\beta n} \tag{13}$$

we may assume that the phases β_n are determined by the illuminating system and are therefore quantities known in principle. For the special case when a plane wave is used for illumination all the β_n are the same. Then (11) and (12) take the forms

$$x(\boldsymbol{a}_k) = \sum_{mn}^{N} |b_n||b_m| A_{nmk}, \tag{14}$$

$$x(\boldsymbol{a}_k) = \sum_{mn}^{N} |b'_n||b'_m| A'_{nmk} \quad k = 1, \ldots, N \tag{15}$$

respectively.

For the sake of simplicity we shall restrict our attention to the upper bound recovery problem as formulated in (14), the solving procedure for the lower bound recovery being highly similar*, though not identical.

*) Some differences in the procedure due to the lack of complete symmetry in the two problems will be discussed elsewhere.

4. Solving Procedure

The solving procedure will be based on the following properties of the matrix elements

$$A_{nmk} = A^*_{mnk}, \tag{16}$$

which may be readily proved from (9) and

$$\begin{matrix} A_{nmk} \to 0 \\ |n-k| \to \infty \\ |m-k| \to \infty \end{matrix} \tag{17}$$

which is intuitively acceptable (a rigorous proof is complex and will be supplied in a separate paper being now prepared for publication).

On the base of (16) and (17) the following procedure of successive approximations may be proposed:

a) The zero order approximation.

We reject all the mixed terms in (14)*, getting as the zero order approximation the system of equations

$$x(\boldsymbol{a}_k) = \sum_{n=1}^{N} |b^0_n|^2 A_{nnk} \quad k = 1, \ldots, N. \tag{18}$$

Note that (18) is a system of linear equations with respect to $|b'_n|^2$ with real coefficients A_{nnk} (see (16)) and may therefore, be solved in a routine way. Thus, we obtain the values $|b^0_n|$ for $n = 1, \ldots, N$ as a solution of the zero order approximation of the upper bound recovery problem.

b) The first order approximation.

Defining

$$|b^1_n| = |b^0_n| + \varepsilon^1_n, \tag{19}$$

where ε^1_n are the correcting terms of the first order, and substituting (19) to the equations system (14) we may write after small rearrangements

$$x(\boldsymbol{a}_k) = \sum_{m=n}^{N} [|b^0_n|^2 + (\varepsilon^1_n)^2] A_{nnk} + 2\sum_{m=n}^{N} \varepsilon^1_n |b^0_n| A_{nnk} + \sum_{\substack{nm \\ n\neq m}}^{N} |b^0_n||b^0_m| \operatorname{Re}\{A_{nmk}\} + \sum_{\substack{nm \\ n\neq m}}^{N} [2|b^0_n| + \varepsilon^1_n]\varepsilon^1_m \operatorname{Re}\{A_{nmk}\}.$$

Assuming the correcting terms ε^1_n to be small enough so that

$$|b^0_n|^2 \gg |\varepsilon^1_n|^2 \quad \text{and} \quad 2|b^0_n| \gg |\varepsilon^1_n|$$

and noting that $\sum_{n}^{N} |b^0_n|^2 A_{nnk} = x(\boldsymbol{a}_k)$ we obtain the following relationship for ε^1_n

$$Z_k = \sum \varepsilon^1_n S_{nk}, \quad k = 1, \ldots, N \tag{20}$$

where

*) Compare (17).

$$Z_k = -\sum_{nm}^{N} |b_n^0|\,|b_m^0|\,\mathrm{Re}\{A_{nmk}\}, \quad (21)$$

$$S_{nk} = 2\Big\{|b_n^0|\,A_{nnk} + \sum_{m}^{N}{}' |b_m^0|\,\mathrm{Re}\{A_{nmk}\}\Big\} \quad (22)$$

are the known quantities. Thus, to determine the first order correcting terms $\varepsilon_n^1, n = 1, \ldots, N$ we have to solve the system (18) of equations, which is again linear and real. After solving it and substituting the solution to the equations (19) we get the first order approximation. By repeating the procedure t times we determine the successive correcting terms ε_n^t and consequently find the respective solutions of t-order for (14) in the form

$$|b_n^t| = |b_n^{(t-1)}| + \varepsilon_n^t. \quad (23)$$

As mentioned before the lower bound recovery problem may be solved in a similar way resulting in

$$|b_n'^t| = |b_n'^{(t-1)}| + \varepsilon_n'^t \quad (24)$$

as an estimation of the lower bound values.

The next step in the recovery procedure will be the determination of the image intensity distributions associated with (24) and (25), respectively. Intensity distribution resulting from the upper bound procedure may be estimated by successive substituting (23) to (13), (13) to (7) and (7) to (2). Then we get

$$I_{\mathrm{im}}^{(\max)}(\boldsymbol{p}) = \sum_{nm}^{N} b_n^t b_m^{t*} K_{\mathrm{im}}(\boldsymbol{a}_n, \boldsymbol{p}) K_{\mathrm{im}}^*(\boldsymbol{a}_m, p), \quad (25)$$

which has the property that for $\boldsymbol{p} = \boldsymbol{a}_k$

$$I(\boldsymbol{a}_k) = \sum b_n^t b_m^{t*} K_{\mathrm{im}}(\boldsymbol{a}_n, \boldsymbol{a}_n) K_{\mathrm{im}}^*(\boldsymbol{a}_m, \boldsymbol{u}_k) \quad (26)$$

takes the largest possible a posteriori values consistent with the given set of the observed image points $x(a_k)$ determined by scanning.

By a similar argument we can estimate the image intensity distribution attributed with the lower bound formulation of the recovery procedure, which gives

$$I_{\mathrm{im}}^{(\min)}(\boldsymbol{p}) = \sum_{nm}^{N} b_n'^t b_m'^{t*} K_{\mathrm{im}}(\boldsymbol{a}_n', \boldsymbol{a}_k) K_{\mathrm{im}}(\boldsymbol{a}_m', \boldsymbol{a}_k) \quad (27)$$

with the property that for $\boldsymbol{p} = \boldsymbol{a}_k$

$$I_{\mathrm{im}}^{(\min)}(\boldsymbol{a}_k) = \sum_{nm}^{N} b_n'^t b_m'^{t*} K_{\mathrm{im}}(\boldsymbol{a}_n', \boldsymbol{a}_k) K_{\mathrm{im}}(\boldsymbol{a}_m', \boldsymbol{a}_k) \quad (28)$$

gives the smallest possible a posteriori values consistent with the same observed image representation $x(a_k)$, $k = 1, \ldots, N$.

Defining

$$I_{im}(p) = \frac{I_{\mathrm{im}}^{(\max)}(\boldsymbol{p}) + I_{\mathrm{im}}^{(\min)}(\boldsymbol{p})}{2} \quad (29)$$

as the recovered intensity distribution in the image, we can accept the formula

$$\Delta I\boldsymbol{p}(\boldsymbol{a}_k) = \pm \tfrac{1}{2}[I_{\mathrm{im}}^{(\max)}(\boldsymbol{p}) - I_{\mathrm{im}}^{(\min)}(\boldsymbol{p})] \quad (30)$$

as a measure of the recovery errors at the points $\boldsymbol{p} = \boldsymbol{a}_k$.

5. Concluding Remarks

The purpose of the present paper was to present only the principal ideas of the recovery problem for coherent imaging, reduced to the case of amplitude modulation subjects illuminated by a plane wave. A more fundamental treatment of the problem is now being prepared for publication.

References

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[5] Wilk I. Optica Applicata, Vol. 1, No. 2, p. 37.