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Use of computer algebra system for recalculation of the fifth-order aberrations of GRIN media

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The methodology of calculation of the fifth-order aberrations of gradient-index (GRIN) media by application of computer symbolic calculations (computer algebra) is presented. The software package applied was REDUCE. The results obtained enabled us to correct the results known from the literature. The designed computer program has also been added. A special computer program has also been designed.

1. Main points of the theory

The procedure for evaluating the fifth-order abberations of GRIN media has been described by Gupta *et al.* [1]. Unfortunately, a few of the coefficients derived there are incorrect. The aim of this paper is to demonstrate how to apply computer algebra software for calculations of aberrations of GRIN media. For simplicity, let us recall the main points from Gupta's paper.

The optical ray path is described by the well known Hamilton's equations [1]:

$$\begin{aligned} \dot{X} &= 2P\frac{\partial \mathrm{H}}{\partial V}, \quad \dot{P} = -2X\frac{\partial \mathrm{H}}{\partial U}, \\ \dot{Y} &= 2Q\frac{\partial \mathrm{H}}{\partial V}, \quad \dot{Q} = -2Y\frac{\partial \mathrm{H}}{\partial U} \end{aligned} \tag{1}$$

where X and Y denote transverse coordinates of the ray and P and Q are their optical direction cosines corresponding to the x and y axes, respectively. The dot placed over any function symbol indicates differentiation with respect to $z\left(\dot{X} = \frac{\mathrm{d}X}{\mathrm{d}z}\ldots\right)$. Coordinate z is measured along the axis of symmetry. Further, $U = X^2+Y^2$, $V = P^2+Q^2$ and H denotes Hamiltonian, which has the following simple form derived by Luneburg [2]:

$$\mathrm{H} = -\sqrt{n^2(U,z)-V}, \quad (n - \text{refractive index}). \tag{2}$$

Expanding X and P in ascending powers of ray parameters we obtain

$$\begin{aligned} \dot{X} &= \dot{X}_1+\dot{X}_3+\dot{X}_5+\ldots, \\ \dot{P} &= \dot{P}_1+\dot{P}_3+\dot{P}_5+\ldots, \end{aligned} \tag{3}$$

where the subscript i denotes the i-th order term. In Equations (3) X_3 and X_5 are x and y components of the third- and fifth-order aberrations, respectively.

Further expanding Hamiltonian in the Taylor series we get

$$H = H_0 + H_{10}U + H_{01}V + \frac{1}{2}H_{20}U^2 + \dots \tag{4}$$

where

$$H_{ij} = \left.\frac{\partial^{i+j}H}{\partial U^i \partial V^j}\right|_{u=v=0}.$$

Now, inserting Eqs. (3) and (4) into Eqs. (1) and equating terms of equal order we get:

$$\dot{X}_1 = 2H_{01}P_1, \tag{5a}$$

$$\dot{X}_3 = 2H_{01}P_3 + \tilde{X}_3, \tag{5b}$$

$$\dot{X}_5 = 2H_{01}P_5 + \tilde{X}_5, \tag{5c}$$

$$\dot{P}_1 = -2H_{10}X_1, \tag{5d}$$

$$\dot{P}_3 = -2H_{10}X_3 + \tilde{P}_3, \tag{5e}$$

$$\dot{P}_5 = -2H_{10}X_5 + \tilde{P}_5 \tag{5f}$$

where:

$$\tilde{X}_3 = 2(H_{11}U_1 + H_{02}V_1)P_1,$$

$$\tilde{X}_5 = 2(H_{11}U_1 + H_{02}V_1)P_3 + 4(H_{11}U_{13} + H_{02}V_{13})P_1 + (H_{12}U_1^2 + 2H_{12}U_1V_1 + H_{03}V_1^2)P_1,$$

$$\tilde{P}_3 = -2(H_{20}U_1 + H_{11}V_1)X_1,$$

$$\tilde{P}_5 = -2(H_{20}U_1 + H_{11}V_1)X_3 - 4(H_{20}U_{13} + H_{11}V_{13})X_3 - (H_{30}U_1^2 + 2H_{21}U_1V_1 + H_{12}V_1^2)X_1,$$

with:

$$U_1 = X_1^2 + Y_1^2, \quad V_1 = P_1^2 + Q_1^2, \quad U_{13} = X_1X_3 - Y_1Y_3, \quad V_{13} = P_1P_3 + Q_1Q_3.$$

Aberrations will be expressed in terms of two paraxial rays: an axial one (h, ϑ) and the field one (H, θ). These two rays have the following properties:

$$h(z_0) = 0, \quad h(\zeta) = 1, \quad H(z_0) = 1 \text{ and } H(\zeta) = 0,$$

where z_0 and ζ localise object and reference planes, respectively. Therefore, for general ray with coordinates (x_0, y_0) and (ξ, η) in object and reference planes, respectively, we have:

$$\begin{aligned} X_1 &= x_0H + \xi h, & Y_1 &= y_0H + \eta h, \\ P_1 &= x_0\theta + \xi\vartheta, & Q_1 &= y_0\theta + \eta\vartheta. \end{aligned} \tag{6}$$

Paraxial rays satisfy Hamilton's equations (5a) and (5e). Therefore, for the axial ray, in particular we may write:

$$\dot{h} = 2H_{01}\vartheta, \quad \dot{\vartheta} = -2H_{10}h. \tag{7}$$

The evaluation procedure of aberration X_5 can be carried out in steps described below. The difference of Eqs. (5b) and (5e) gives

$$\frac{\mathrm{d}}{\mathrm{d}z}(X_3\vartheta - hP_3) = \tilde{X}_3\vartheta - \tilde{P}_3 h. \tag{8}$$

Hence (8) results in

$$P_3 = \frac{\vartheta}{h}x_3 - \frac{2}{h}I_1 \tag{9a}$$

where

$$I_1 = \frac{1}{2}\int_{z_0}^{z}(\tilde{x}_3\vartheta - \tilde{P}_3 h)\mathrm{d}z. \tag{9b}$$

Inserting Eq. (9a) into Eq. (5b) and integrating the above equation we obtain

$$X_3 = h\int_{\zeta}^{z}\left[-\frac{\dot{h}}{\vartheta h^2}I_1 + \frac{1}{h}\tilde{X}_3\right]\mathrm{d}z. \tag{10}$$

Analogically, the difference of Eqs. (5c) and (5f) gives

$$\frac{\mathrm{d}}{\mathrm{d}z}[X_5\vartheta - hP_5] = [\tilde{x}_5\vartheta - \tilde{P}_5 h]. \tag{11}$$

As in the paraxial image plane localised at z_1, $h(z_1) = 0$, the integration of Eq. (11) finally gives

$$X_5 = \frac{1}{\vartheta(z_1)}\int_{z_0}^{z_1}[\tilde{x}_5\vartheta - \tilde{P}_5 h]\mathrm{d}z. \tag{12}$$

Resuming we see that in order to evaluate aberration X_5 we have to go through the following steps:

- estimate the refractive index distribution n,
- build the Hamiltonian H,
- fix parameters of the ray $(x_0 y_0, \zeta, \eta)$,
- solve paraxial equations to obtain h, ϑ, and θ,
- evaluate Taylor coefficients $H_{ij}ij$,
- evaluate formulae (9b), (9a), (10) and finally (12).

An analogous evaluation procedure is true for Y_5.

2. Results and conclusions

A computer program for evaluation of the fifth-order aberrations has been designed (see Appendix).

With the help of the program developed, aberrations of the fifth-order have been evaluated for GRIN medium with the refractive index of the form

$$n = n_0\left(1-\frac{1}{2}a^2U+\frac{1}{2}ba^4U^2+ga^6U^3\right). \tag{13}$$

For this refractive index distribution we have

$$z_1 = \frac{m\pi}{a} \quad \text{and} \quad \zeta = \frac{(2m-1)\pi}{2a}, \quad (m \text{ — natural number}),$$

and aberrations of the fifth-order have the following compact form:

$$\begin{aligned} X_5(x_0,r,\varphi) = {} & Sr^5\cos\varphi+(B_1+B_2\cos 2\varphi)x_0r^4+x_0^2r^3(C_1+C_2\cos^2\varphi)\cos\varphi \\ & +x_0^3r^2(D_1+D_2\cos 2\varphi)+x_0^4r\cos\varphi\, E_{\mathrm{coeff}}+Fx_0^5, \end{aligned} \tag{14a}$$

$$\begin{aligned} Y_5(x_0,r,\varphi) = {} & Sr^5\sin\varphi+x_0r^4G\sin 2\varphi+x_0^2\rho^3\sin\varphi(H_1+H_2\cos^2\varphi) \\ & +x_0^3r^2P\sin 2\varphi+x_0^4rQ\sin\varphi \end{aligned} \tag{14b}$$

where:

$$r = \sqrt{\xi^2+\eta^2}, \qquad \varphi = A\sin\left(\frac{\eta}{\xi}\right)$$

and $S, B_1, B_2 \ldots$ are aberration coefficients. The exact forms of all these coefficients have been extracted from $X_5(x_0,r,\varphi)$ and $Y_5(x_0,r,\varphi)$ in the following way:

$$S = X_5(0, 1, 0),$$

$$B_1 = \left(\frac{\partial X_5(x_0, 1, \pi/4)}{\partial x_0}\right)_{x_0=0}, \qquad B_2 = \left(\frac{\partial X_5(x_0, 1, 0)}{\partial x_0}\right)_{x_0=0} - B_1,$$

$$C_2 = \frac{1}{2\sin^2(\pi/4)}\left(\frac{\partial^2 X_5(1,r,0)}{\partial r^2} - \frac{1}{\cos(\pi/4)}\frac{\partial^2 X_5(1,r,\pi/4)}{\partial r^2}\right)_{r=0},$$

$$C_1 = \frac{1}{2}\left(\frac{\partial^2 X_5(1,r,0)}{\partial r^2}\right)_{r=0} - C_2,$$

$$D_1 = \frac{1}{2}\left(\frac{\partial^2 X_5(1,r,\pi/4)}{\partial r^2}\right)_{r=0}, \qquad D_2 = \frac{1}{2}\left(\frac{\partial^2 X_5(1,r,0)}{\partial r^2}\right)_{r=0} - D_1,$$

$$E_{\mathrm{coeff}} = \left(\frac{\partial X_5(1,r,0)}{\partial r}\right)_{r=0}, \qquad F = X_5(1,0,1),$$

$$G = \left(\frac{\partial Y_5(x_0, 1, \pi/4)}{\partial x_0} \right)_{r_0 = 0},$$

$$H_1 = \frac{1}{2} \left(\frac{\partial^2 Y_5(x_0, 1, \pi/2)}{\partial x_0^2} \right)_{r_0 = 0},$$

$$H_2 = \left[\frac{1}{2 \sin(\pi/4)} \left(\frac{\partial^2 Y_5(x_0, 1, \pi/4)}{\partial x_0^2} \right)_{r_0 = 0} - H_1 \right] / \cos^2(\pi/4),$$

$$P = \frac{1}{2} \left(\frac{\partial^2 Y_5(1, r, \pi/4)}{\partial r^2} \right)_{r = 0}, \quad Q = \left(\frac{\partial Y_5(1, r, \pi/2)}{\partial r} \right)_{r = 0}.$$

Comparing all the results obtained with those known from [1] one can easily observe that coefficients: B_1, B_2, D_1 and D_2 are not the same. The correct results should be as follows:

$$B_1 = a^4 m \pi^2 \left(\frac{3}{2} b^2 m - \frac{15}{16} b^2 - bm + \frac{29}{32} b + \frac{5}{32} m - \frac{55}{256} \right),$$

$$B_2 = a^4 m \pi^2 \left(\frac{3}{4} b^2 m - \frac{3}{4} b^2 - \frac{7}{8} bm + \frac{1}{2} b + \frac{15}{64} m - \frac{5}{64} \right),$$

$$D_1 = a^4 m \pi^2 \left(\frac{5}{4} b^2 m - \frac{3}{2} b^2 - \frac{9}{8} bm + \frac{11}{8} b + \frac{9}{64} m - \frac{5}{16} \right),$$

$$D_2 = a^4 m \pi^2 \left(b^2 m - \frac{3}{4} b^2 - \frac{3}{4} bm + \frac{1}{2} b + \frac{1}{4} m - \frac{5}{64} \right),$$

The remaining coefficients are the same as those published in [1].

An analogous calculation procedure has been extended for seventh-order approximation [3].

Summing up we notice that REDUCE has proved to be a powerful tool for calculation of aberrations of GRIN media [4].

Appendix

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% PROGRAM FOR CALCULATIONS OF 5-th ORDER ABERRATIONS ( x - COMPONENTS )
% OF GRIN MEDIA WITH AXIAL SYMMETRY

%                    " REFRACTIVE INDEX "
n:=no*(1-1/2*a^2*u+1/2*b*a^4*u^2+g*a^6*u^3)$


%                " HAMILTONIAN "
HAMIL:= -(n^2-v)^(1/2)$
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%               " TAYLOR COEFFICIENTS "
H20:=SUB(u=0,v=0,DF(HAMIL,u,2,v,0))$  H11:=SUB(u=0,v=0,DF(HAMIL,u,1,v,1))$
H02:=SUB(u=0,v=0,DF(HAMIL,u,0,v,2))$   H30:=SUB(u=0,v=0,DF(HAMIL,u,3,v,0))$
H21:=SUB(u=0,v=0,DF(HAMIL,u,2,v,1))$   H12:=SUB(u=0,v=0,DF(HAMIL,u,1,v,2))$
H03:=SUB(u=0,v=0,DF(HAMIL,u,0,v,3))$

%          " PARAXIAL RAYS "

h:=SIN(a*z)$  hD:=COS(a*z)$   th:=no*a*COS(a*z)$  thD:=-no*a*SIN(a*z)$
z0:=0$      z1:=2*m*PI/a$       dz:=(4*m-3)*PI/(2*a)$
x1:=xo*hD+k*h$                  p1:=xo*thD+k*th$
y1:=eta*h$                      q1:=eta*th$
U1:=x1**2+y1**2$                V1:=p1**2+q1**2$

%               " FURTHER   EVALUATIONS "
let COS(2*m*PI)=1; let SIN(2*m*PI)=0;
let COS((2*m*PI-PI)/2)=0; let (SIN((2*m*PI-PI)/2))^2=1;
let SIN((4*m-3)*PI/(2))=1;  let COS((4*m-3)*PI/(2))=0;
on div; on exp, off allfac;

f1:=(H11*U1+H02*V1)*p1*th+(H20*U1+H11*V1)*x1*h$    c1:=INT(f1,z)$
 c1:=c1-SUB(z=0,c1)$
f1:=(H02*V1+H11*U1)*(p1/h)$      f2:=DF(h,z)/(h**2*th)*c1$   w1:=INT(f1-f2,z)$
x3:=2*h*(w1-SUB(z=dz,ws))$          P3:=(th/h)*x3-(2/h)*c1$

f1:=(H11*U1+H02*V1)*q1*th+(H20*U1+H11*V1)*y1*h$        c1:=INT(f1,z)$
f1:=(H02*V1+H11*U1)*(q1/h)$      f2:=DF(h,z)/(h**2*th)*c1$   w1:=INT(f1-f2,z)$

y3:=2*h*(w1-SUB(z=dz,w1))$   Q3:=(th/h)*y3-(2/h)*c1$
U13:=x1*x3+y1*y3$  V13:=p1*P3+q1*Q3$
fx5a:=INT((H11*U1+H02*V1)*P3*th,z)$   fx5a:=SUB(z=z1,fx5a)-SUB(z=0,fx5a)$
fx5b:=INT((H20*U1+H11*V1)*x3*h,z)$   fx5b:=SUB(z=z1,fx5b)-SUB(z=0,fx5b)$
fx5c:=2*INT((h11*U13*p1*th+h02*V13*p1*th),z)$    fx5c:=SUB(z=z1,fx5c)-SUB(z=0,fx5c)$
fx5d:=2*INT((H20*U13+H11*V13)*x1*h,z)$      fx5d:=SUB(z=z1,fx5d)-SUB(z=0,fx5d)$
fx5e:=1/2*INT((H21*U1**2+2*H12*U1*V1+H03*V1**2)*p1*th,z)$
fx5e:=SUB(z=z1,fx5e)-SUB(z=0,fx5e)$
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```
fx5f:=1/2*INT((H30*U1**2+2*H21*U1*V1+H12*V1**2)*x1*h,z)$
fx5f:=SUB(z=z1,fx5f)-SUB(z=0,fx5f)$
%clear f1,w1,w1,x3,p3,c1,f2,y3,q3,u13,v13,u1,v1;
fx5:=(fx5a+fx5b+fx5c+fx5d+fx5e+fx5f)$
k:=r*COS(fi)$ eta:=r*SIN(fi)$
on allfac$

%                " TOTAL ABERRATION   X5 "
x5:=2/SUB(z=z1,th)*fx5$
pause$

%                " ABERRATION COEFFICIENTS "
S:=SUB(xo=0,r=1,fi=0,x5);
B1:=SUB(xo=0,r=1,fi=PI/4,DF(x5,xo));  B2:=SUB(xo=0,r=1,fi=0,DF(x5,xo))-B1;
C1:=SQRT(2)*(SUB(xo=0,r=1,fi=PI/4,DF(x5,xo,2))-1/2*SUB(xo=0,r=1,fi=0,DF(x5,xo,2));
C2:=SUB(xo=0,r=1,fi=0,DF(x5,xo,2))-SQRT(2)*SUB(xo=0,r=1,fi=PI/4,DF(x5,xo,2));
D1:=0.5*SUB(xo=1,r=0,fi=PI/4,DF(x5,r,2));
D2:=0.5*SUB(xo=1,r=0,fi=0,DF(x5,r,2))-D1;
Ecoeff:=SUB(xo=1,r=0,fi=0,DF(x5,r));
F:=SUB(xo=1,r=0,x5);
showtime;
end;
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