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Teaching optics

Possibilities of aberration correction in a single spectacle lens

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Spectacle-wearers make up a considerable part of present-day society, so spectacles are one of the most popular optical instruments — very simple instruments since they are in fact single lenses. On the other hand their mode of operation and the demands for imaging quality are very specific. Therefore, spectacle lenses are interesting objects for aberration analysis and are excellent examples for illustrating purposes while teaching geometrical optics. Typically, the spectacle lens is fixed at some distance in front of the eye. When looking at distant objects, the eye rotates around its centre of rotation while the field-of-view angle is limited practically only by the spectacle frame. The chief ray, connecting point of fixation and centre of eye pupil always passes through the eyeball centre of rotation. Therefore we can assume that spectacle lens has shifted output pupil while preserving a relatively large field of view. Consequently, it is important to correct field aberrations, in particular astigmatism. It is interesting to investigate relationships between spherical aberration, coma and field curvature each as a function of output pupil shift and to point out that it is possible to correct fully astigmatism and minimise spherical aberration or coma.

1. Introduction

For about 700 years spectacles have been used for correction of such vision defects as myopia, hypermetropia, astigmatism or presbyopia. Except for very few cases, single lenses — mainly of spherical or toroidal surfaces — have been used to this aim. Only recently, aspherical surfaces are applied as well.

As in any other optical instrument, the imaging quality is of major importance when considering spectacle lens design. Typically, image quality is expressed in terms of geometrical aberrations (in particular, the III-order Seidel aberrations) and chromatic aberration. These aberrations depend on parameters describing the lens and imaging conditions, such as the lens surfaces radii of curvature, the lens thickness, refractive index and Abbe number of the lens material, maximum field and aperture angles as well as object distance and location of input pupil. Some of the above parameters depend on the way in which spectacle lenses are used (*e.g.*, object distance, aperture and field angle location of input pupil), the others are determined by available technology (*e.g.*, index of refraction, Abbe number). There are also additional requirements such as minimum and maximum acceptable lens thickness. All these factors determine the frames which the optimum lens design has to fit in.

First spectacle lenses had a form of simple plano-convex magnifying glasses (R. Bacon, *Opus Maius*, *ca.* 1268), then the negative lenses began to be used as well.

For many years the shape of spectacle lenses was not a result of any theoretical calculations, but rather stemmed from experiment and intuition. First theoretical solutions are due to W. H. Wollaston, who, in 1804, obtained a patent for meniscus spectacle lenses. In the following years the problem of optimum spectacle lenses and their aberrations was investigated by F. Ostwald (1898), S. Czapski (1893), M. Tscherning (1904), A. R. Percival (1910–1920), L. C. Martin (1910), J. Petzval, J. Southal (1937), and others. We will also mention Polish opticians, T. Wagnerowski, J. Gutkowski, W. H. Melanowski and J. Bartkowska [1]–[5].

In spite of the fact that nowadays spherical lenses are being frequently replaced by lenses with aspheric surfaces, the problem of how to optimize a single spherical lens seems to be still interesting. Moreover, while teaching optics it is necessary to illustrate the theoretical consideration on aberration correction with relatively simple, but evident examples. Spherical spectacle lenses may be very useful as such examples. Their construction and specific demands for imaging conditions give an opportunity for especially careful analysis of aberration correction. Their example is simple enough to be understood even by a beginner in optical design, but on the other hand, a number of changeable parameters (radii of curvature, output pupil shift and object distance) enable us to perform valuable analysis of aberrations.

2. Demands for the construction of spectacle lens

The main parameter of a spectacle lens is its focusing power Φ measured in dioptres. Its value depends on the refractive error of eye to be corrected. The refractive power itself, however, does not determine univocally the construction parameters of the lens. Assuming that the lens is spherical (and we will consider only such lenses in this paper) it is necessary to determine the radii of curvature ρ_1 and ρ_2 of its two surfaces, index of refraction n and Abbe number ν. Choice the above-mentioned parameters is a basic part of the lens design process.

While designing the lens a number of factors have to be taken into account. The three main criteria of a good quality spectacle lens are as follows:

– quality of an image formed with the lens,
– esthetic reasons and wearing comfort,
– technological reasons.

In this paper, we will concentrate only on the first of these criteria. The imaging quality is typically described in terms of aberrations, in particular III-order Seidel aberrations, such as spherical aberration, coma, distortion, field curvature and astigmatism as well as chromatic aberration. The amounts of particular aberrations depend on the construction parameters of the lens and the aperture and field angles. The latter are determined by the imaging geometry, *i.e.*, the location of an object point and input pupil which, in turn, depends on the manner in which a person wears his/her spectacles. Typically, the spectacle frame holds lenses some distance from the eyes in a fixed position. While looking straight ahead the line of sight (which with some approximation is an extension of the eye's optical axis) intersects the lens in its optical centre.

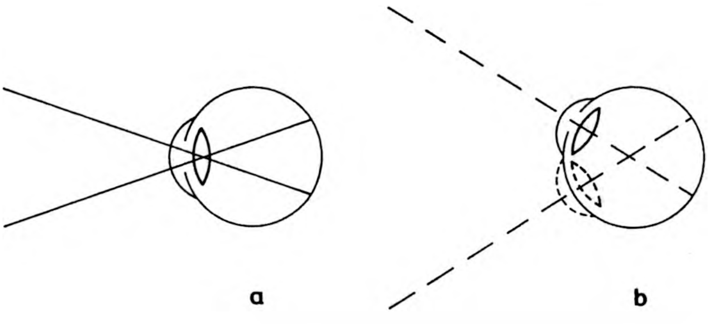

Fig. 1. Off-axis object viewed through spectacle lens: **a** — field of view, **b** — field of sight.

If the eye is at rest, then we see some part of the object space limited by the extension of retina. This is called "field of view" (Fig. 1**a**). However, the density of fotosensitive cells (rodes and cones) is high enough to give good vision only in relatively small central region of the retina called yellow spot. Therefore, while observing an extended scene the eye instinctively "scans" the object space thus allowing to form sharp images of each detail of the observed object on the yellow spot. The direction of the line of sight changes thanks to rotation of the eyeball about its centre. The part of the object space seen due to the rotation of the eyeball but with head fixed is called "field of sight" (Fig. 1**b**).

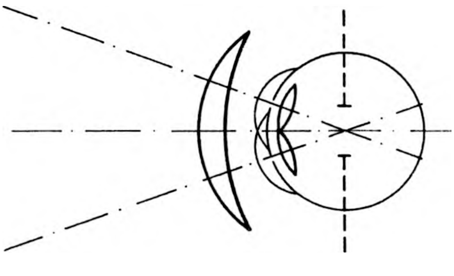

Fig. 2. Aperture diaphragm in the optical system composed of the eye and spectacle lens.

Principal rays drawn from the different object points of the whole field of sight intersect in the eyeball centre of rotation. We can recall here the definition of the aperture stop of the optical system (limiting the aperture angle of the light bundle entering it). According to it the principal rays drawn at different field angles with respect to the optical axis intersect in the centre of input pupil. Therefore we can assume that the optical system composed of motionless spectacle lens and rotating eye has an input pupil located in the eyeball centre of rotation. In other words, the spectacle lens has the input pupil shifted inwards by a distance *d*. This is illustrated in Fig. 2.

The distance from the spectacle lens to input pupil depends on the method of holding this lens in front of the eye. Typical spectacle frames fix the lens about 12–13 mm from the outer surface of the cornea. The average distance from the cornea to the eyeball centre of rotation also equals about 12–13 mm. We can, therefore, assume that in the typical case the input pupil of the spectacle lens is shifted about $d = 25$ mm behind the lens.

Moreover, the optical axis of the spectacle lens is not horizontal, but bent by the so-called pantoscopic angle (about 10°). It follows from the fact that our line of sight is very seldom strictly horizontal. More often we look somehow downwards "in front of our feet". Maximum angle between the optical axis of the spectacle lens and the line of sight is about 35° up and 45° down. Object location varies depending on whether the spectacles are destined for distant vision or for near vision. In the latter case it is assumed that the object distance equals approximately 25–40 cm (depending on the character of patient's work or any other activity).

In order to study the optical system composed of eye and spectacle lens in more detail let us assume that the eye is not emmetropic. It means that the far point of the eye (*i.e.*, sharp image of this point is formed on the retina without accommodation) is not located at infinity. For myopic eye the far point lies at a finite distance from the eye, for hyperopic one the far point lies behind the eye and is virtual independent on the direction of sight. When eyeball rotates its far point encircles a surface called far point sphere K_R. Similarly, we can define the near point sphere K_P. It is a surface encircled by the near point while rotating the eyeball. The near point is defined as an object point imaged sharply on the retina under maximum accommodation. Both spheres, the far point K_R and near point K_P for myopic eye are illustrated in Fig. 3. Let us note that both spheres have common centre being an eyeball centre of rotation.

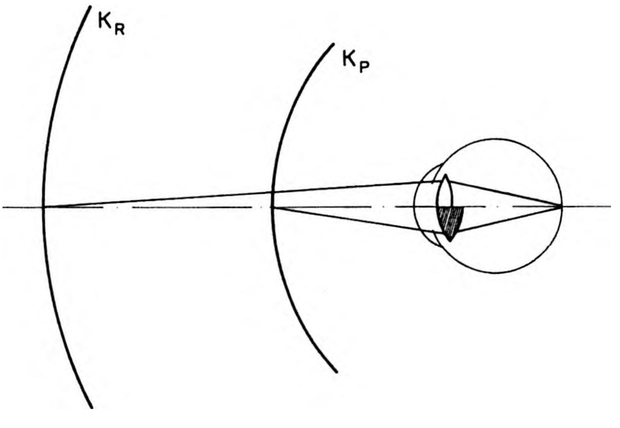

Fig. 3. Far point sphere K_R and near point sphere K_P for myopic eye.

By definition the spectacle lens (for distant vision) has to correct the imaging conditions of the eye in such a way that it should image the object point lying at infinity onto the far point of the eye. Allowing eyeball rotation means that the fixed spectacle lens should image points lying at infinity onto the far point sphere of the eye. By analogy the spectacle lens for near vision should image the points lying at some finite distance onto the near point sphere of the eye.

Light rays emerging from infinity are focused by the lens into its focal point F. In ideal conditions the rays coming from infinity at different field angles should be focused onto perfect sphere (to call it "focal sphere"). In fact, it is not true for real lens. A typical "focal" surface called Petzval–Coddington surface differs somehow from the sphere. The shape and location of the Petzval–Coddington surface depends on the lens geometry and the location of input and output pupils. As it is seen in Fig. 3, this surface can be approximated with a sphere K_F the radius of which is equal to the difference of the lens focal length and the value of the pupil shift. Sphere K_F should coincide with far point sphere K_R or near point sphere K_P for distant or near spectacles, respectively. Non-zero difference between sphere K_F and Petzval–Coddington surface means aberrations of the optical system composed of the lens and eye.

The aberrations are thus a measure of optical imaging system quality. A number of different descriptions of aberrations are used, of which we can mention wave aberrations or ray aberrations. One of the most typical aberration descriptions, called Seidel approximation, is based on developing the eiconal into a power series according to output pupil co-ordinates. The III-order coefficients of Seidel approximations describe such aberrations as spherical aberration, coma, astigmatism, *etc.*

Not all of the III-order aberrations are equally important for the spectacle lens. It is well known that spherical aberration is an aperture aberration. The aperture angle of an eye is rather small. Assuming that the iris diameter does not exceed 8 mm, and the object distance is not shorter than 20 cm we can estimate the highest aperture angle as $\omega \simeq 2°$. For such a small aperture angle spherical aberration is practically negligible. For similar reasons also coma is not very important. Distortion is an aberration which does not destroy image sharpness, so its influence on the spectacle image quality is not of major importance. Field curvature is to some extent compensated by dynamic accommodation of the eye. The most important aberration, which seriously influences the imaging quality of spectacle lens is astigmatism. As it was pointed out the field of view is rather large; maximum field angle may be as high as some 30°. Moreover, the off-axis astigmatism destroys the image in such a way that is very uncomfortable for the spectacles wearer. Concluding, we may state that not all aberrations must be corrected with equal care. Undoubtedly, the most important one is astigmatism.

Since spectacles are designed as single lenses, this paper is in fact devoted to the general discussion on the possibilities of correcting of particular aberrations of a single lens.

3. Geometrical relations

3.1. Single spherical refractive surface

In Figure 4, the imaging by a single spherical surface separating media of different refraction indices is illustrated. Let the indices of refraction be n and n', and the surface radius of curvature equal ρ. It is convenient to make use the value

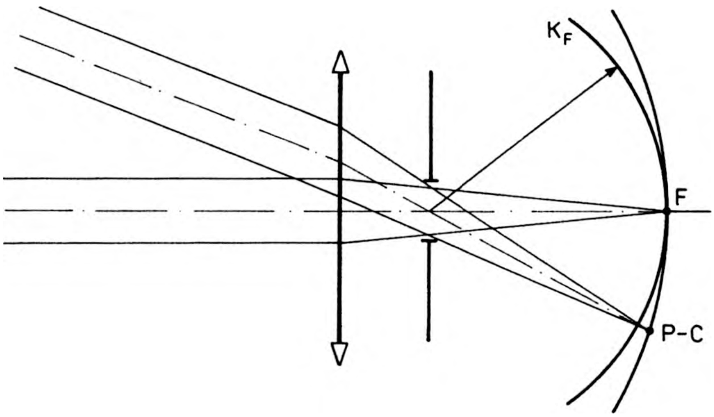

Fig. 4. Positive lens with shifted pupil and its Petzval–Coddington surface (P–C) and focal sphere K_F.

V describing the surface curvature

$$V_\rho = \frac{1}{\rho}. \tag{1}$$

Focusing power of such a surface is

$$\Phi'_\rho = (n'-n)\,V_\rho. \tag{2}$$

Imaging conditions are given by the following formulae (see notation in Fig. 4):

$$n'V' = nV' + \Phi'_\rho, \tag{3}$$

$$n'y'V' = nyV \tag{4}$$

where V and V' are the reciprocities of object and image distances, respectively:

$$V = \frac{1}{s}, \tag{5}$$

$$V' = \frac{1}{s'}. \tag{6}$$

The object and image sizes are denoted by y and y', respectively.

The wave front in the optical system output pupil is typically developed into a series according to Seidel formula. The part corresponding to the III-order aberrations is

$$W = -\frac{1}{8}S(x^2+y^2)^2 + \frac{1}{2}(C_x x + C_y y)(x^2+y^2)$$
$$-\frac{1}{4}F(x^2+y^2) - \frac{1}{2}(A_x x^2 + A_{xy}xy + A_y y^2) + \frac{1}{2}(D_x x + D_y y) \tag{7}$$

where: S, C_x, C_y, F, A_x, A_{xy}, A_y, D_x, D_y denote the III-order aberration coefficients.

For a single spherical refractive surface the above coefficients are expressed by the imaging parameters as follows:

– spherical aberration

$$S = nV(V-V_\rho)^2 - n'V'(V'-V_\rho)^2, \tag{8}$$

– coma

$$C_y = nyV^2(V-V_\rho) - n'y'V'^2(V'-V_\rho), \tag{9}$$

– astigmatism

$$A_y = nyV^3 - n'y'V'^3, \tag{10}$$

– field curvature

$$F_y = nyV^2(V-V_\rho) - n'y'V'^2(V-V_\rho), \tag{11}$$

– distortion

$$D_y = nyV^3 - n'y'V'^3. \tag{12}$$

3.2. Thin spherical lens

Spherical lens (Fig. 5) is, of course, a combination of two spherical surfaces of curvatures $V_{\rho 1}$ and $V_{\rho 2}$, and focusing powers Φ_1 and Φ_2, respectively:

$$\Phi_1 = (n-1)V_{\rho 1}, \tag{13}$$

$$\Phi_2 = (1-n)V_{\rho 2}. \tag{14}$$

By summing up the formulae (8)–(12) which describe the particular aberration coefficients for the first and second surfaces of the lens and taking into account the imaging conditions (3), (4) it is possible to derive formulae describing aberrations of the whole lens.

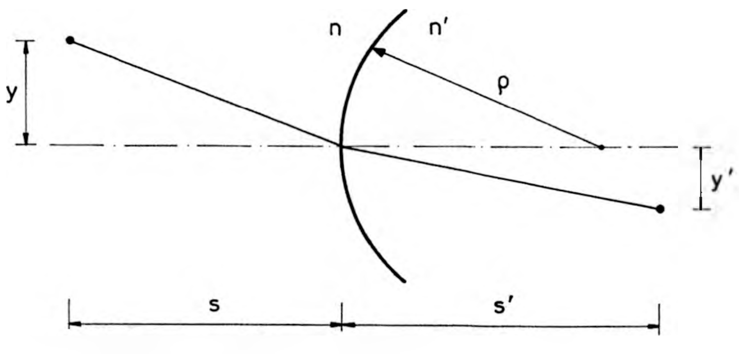

Fig. 5. Imaging geometry by a single spherical surface.

Let us assume that the point object is specified by the parameters y and V. The first surface images it into a point specified by parameters y_1' and V_1' were (see

Eqs. (3) and (4)):

$$nV_1' = V + \Phi_1, \tag{15}$$

$$ny_1'V_1' = yV. \tag{16}$$

If the lens thickness can be neglected this point acts as an object for imaging by a second surface. Therefore we can write:

$$V_1' = V_2, \tag{17}$$

$$y_1' = y_2. \tag{18}$$

Imaging by a second surface is described analogously by:

$$V' = nV_2 + \Phi_2, \tag{19}$$

$$y'V' = ny_2V_2. \tag{20}$$

From formulae (13)–(15) and (17) and (19) there result the expressions describing imaging properties of the whole lens:

$$V' = V + \Phi, \tag{21}$$

$$y'V' = yV \tag{22}$$

where:

$$\Phi = \Phi_1 + \Phi_2 \tag{23}$$

is the focusing power of the whole lens.

For convenience we can introduce the normalisation of some parameters and divide them by the focusing power of the lens Φ according to the following formulae:

$$v = V/\Phi, \tag{24}$$

$$v' = V'/\Phi, \tag{25}$$

$$\varphi_1 = \Phi_1/\Phi. \tag{26}$$

The geometrical shape of the lens is thus univocally described by a parameter

$$\varphi_1 = 1/(1-\rho_2/\rho_1). \tag{27}$$

The lens shapes for different values of parameter φ_1 are shown in Table 1.

Table 1. The shape of lens versus parameter φ_1.

Value of φ_1	$\varphi_1 < 0$	$\varphi_1 = 0$	$\varphi_1 = 0.5$	$\varphi_1 = 1$	$\varphi_1 > 1$
Lens shape	Meniscus -convex	Plano -convex	Double -concave	Plano -concave	Meniscus -concave
					

4. III-order aberrations

4.1. Spherical aberration

The coefficient describing spherical aberration of thin lens can be obtained by summing up the coefficients for both surfaces (see Eq. (8))

$$S = V(V-V_{\rho 1})^2 - nV'_1(V'_1 - V_{\rho 1})^2 + nV_2(V_2 - V_{\rho 2})^2 - V'(V - V_{\rho 2})^2. \tag{28}$$

After inserting Eqs. (15), (17), (19) we obtain

$$S = V^3 + \frac{\Phi V}{n-1}\left(\frac{2V}{n} + \frac{\Phi}{n-1}\right) - [V+\Phi]\left(V + \frac{n}{n-1}\Phi\right)^2$$
$$-2V^2 V_{\rho 1} - \frac{2\Phi V V_{\rho 1}}{(n-1)n} + \Phi V_{\rho 1}\left(\frac{2V}{n} + \frac{\Phi}{n-1}\right) + 2[V+\Phi]\left(\frac{V+n}{n-1}\Phi\right)V_{\rho 1}$$
$$-\frac{2\Phi V_{\rho 1}^2}{n} + VV_{\rho 1}^2 - [V+\Phi]V_{\rho 1}^2. \tag{29}$$

After inserting Eq. (13), introducing normalised parameters (Eqs. (24)–(26)) to (28) and suitable rearrangement we have:

$$S = \frac{\Phi^3}{(n-1)^2 n}\{(-3n^3 + 4n^2 + n - 2)v^2 + [4(n^2-1)\varphi_1 - n(3n^2 - 2n - 1)]v$$
$$-[(2+n)\varphi_1^2 - n(1+2n)\varphi_1 + n^3]\}. \tag{30}$$

Comparing the right hand side of Eq. (30) to zero should lead to the condition under which spherical aberration vanishes. It is easy to see that the resulting relationship is a quadratic equation with respect to φ_1. A real solution exists only if the discriminant of this equation is non-negative

$$\Delta = 4v(n-1)^2(v+1) + 1 - 4n \geqslant 0. \tag{31}$$

After rearranging Eq. (31), the appropriate condition is

$$4(n-1)^2 v^2 + 4(n-1)^2 v + 1 - 4n \geqslant 0. \tag{32}$$

Since the index of refraction n is always greater than 1 the above inequality holds only for values of parameter v fulfilling the relations:

$$v \leqslant -\frac{(n-1+\sqrt{n^2+2n})}{2(n-1)} \quad \text{or} \quad -\frac{(n-1-\sqrt{n^2-2n})}{2(n-1)} \leqslant v. \tag{33}$$

The values of refractive index for typical glasses are enclosed in the interval $1.4 < n < 1.8$. The possible values of parameter v fall in the hatched region of the graph presented in Fig. 6.

As one can see from this figure two regions of possible solutions exist. In one of them the values of parameter v are greater than 0. However, positive v corresponds to the object located behind the lens (imaginary object). Such a solution is not

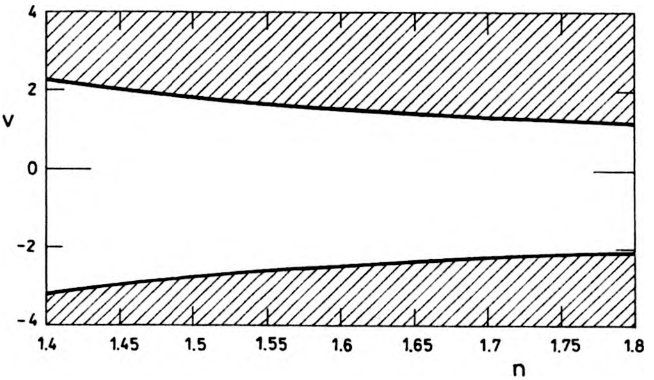

Fig. 6. Range of parameter v describing object distance for which the correction of spherical aberration is possible as a function of values of refraction index n.

interesting while considering spectacles. In the second solution $v < -2$. The object distance is then shorter than half of the lens focal length. Such a situation can be met for the reading glasses of small focusing power (object distance 25–40 cm, $\Phi < 2$ D). Unfortunately, for the most interesting case, *i.e*, when object is infinitely distant ($v = 0$) spherical aberration cannot be compensated. A single spherical spectacle lens for distant vision is always burdened with spherical aberration.

We cannot fully compensate the spherical aberration, however, there exists a possibility of its minimization. It is the case where the first derivative of following equation is equal to zero:

$$\frac{dS}{d\varphi_1} = \frac{\Phi^3}{n(n^2-1)}\left[4(n^2-1)v - 2(n+2)\varphi_1 + n(1+2n)\right]. \tag{34}$$

By comparing the right hand side of this equation to zero we obtain the well known condition for the lens of minimum spherical aberration [6], [7]

$$\varphi_1 = \frac{4(n^2-1)v + n(2n+1)}{2(n+2)}. \tag{35}$$

Table 2. Example lenses of minimum spherical aberration.

v	s [mm]	n	φ_1
0	∞	1.4	0.782
		1.5	0.875
		1.6	0.933
		1.7	1.011
		1.8	1.089
−0.25	−400	1.4	0.641
		1.5	0.679
		1.6	0.717
		1.7	0.755
		1.8	0.795

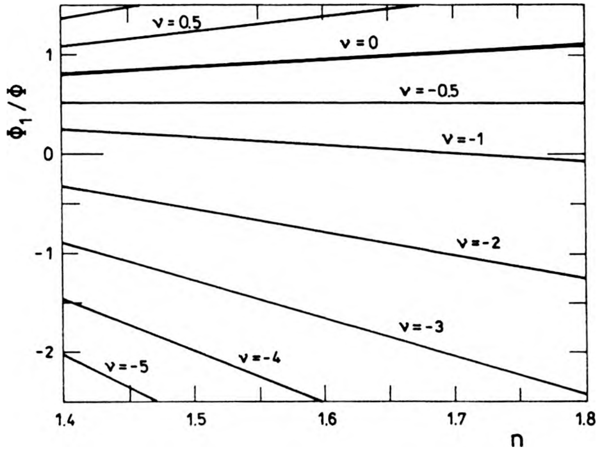

Fig. 7. Values of parameter $\varphi_1 = \Phi_1/\Phi$ describing the shape of the lens minimised spherrical aberration versus the refractive index n for different object locations: $v = 0$ — object at infinity, $v < 0$ — object in front of the lens (real), $v > 0$ — object behind the lens (imaginary).

The values of this parameter as a function of v and n are graphically presented in Fig. 7. In Table 2 the lens shape is calculated for two object distances, namely infinity (distant vision) and $s = -40$ cm (typical reading distance). It is seen from the graph and the table that for higher index of refraction the lenses of minimum spherical aberration have the first surface more convex.

The considerations presented above lead to the construction of a single lens of minimum spherical aberration. From formulas (1), (13), (16) it follows that the radii of curvature of such lens are determined by the parameter φ_1 as follows:

$$\rho_1 = \frac{n-1}{\varphi_1 \Phi}, \tag{36}$$

$$\rho_2 = \frac{1-n}{(1-\varphi_1)\Phi}. \tag{37}$$

4.2. Coma

The coefficient describing coma of thin spherical lens calculated as a sum of appropriate coefficients for both its surfaces (Eq. (9)) has the form

$$C = \omega\left[V(V-V_{\rho 1}) - \frac{V+\Phi_1}{n}\left(\frac{V+\Phi_1}{n} - V_{\rho 1}\right) + \frac{V+\Phi_1}{n}\left(\frac{V+\Phi_1}{n} - V_{\rho 2}\right) - (V+\Phi)(V+\Phi-V_{\rho 2})\right] \tag{38}$$

where ω is a field angle

$$\omega = yV. \tag{39}$$

After inserting Eqs. (13)–(26) to Eq. (39) and rearranging the latter we have

$$C = \frac{-(yV)\Phi^2}{n(n-1)}[(2n^2-n-1)v+n^2-\varphi_1(n+1)]. \tag{40}$$

From the above formula it follows that it is possible to find such a lens shape that coma vanishes. The necessary condition is

$$\varphi_1 = \frac{2n^2-n-1}{n+1}v+\frac{n^2}{n+1}. \tag{41}$$

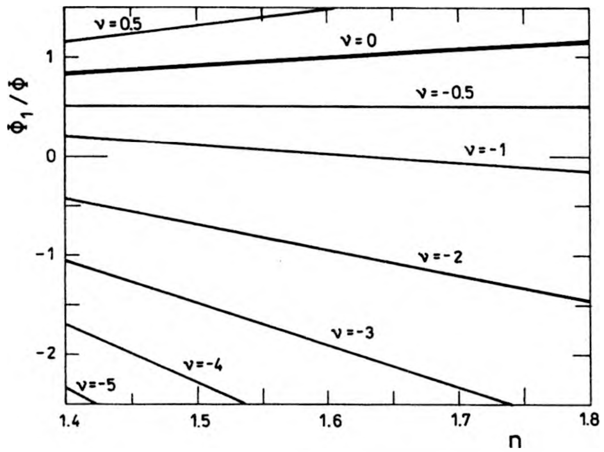

Fig. 8. Values of parameter $\varphi_1 = \Phi_1/\Phi$ determining the shape of coma-free lens as a function of value of refractive index n for different object locations: $v = 0$ – object at infinity, $v < 0$ – object in front of the lens (real), $v > 0$ – object behind the lens (imaginary).

Table 3. Example coma-fee lenses.

v	s [mm]	n	φ_1
0	∞	1.4	0.871
		1.5	0.900
		1.6	0.985
		1.7	1.070
		1.8	1.157
0.25	−400	1.4	0.658
		1.5	0.700
		1.6	0.742
		1.7	0.785
		1.8	0.829

The values of parameter φ_1 as a function of refractive index n (from the interval $1.4 < n < 1.8$) for different object locations (described by the parameter v) assuring the correction of coma are plotted in Fig. 8 and illustrated in Table 3, where two typical object distances are considered: infinity (distant vision) and $S = -40$ cm (typical reading distance). It is seen from the graph and the table, that coma-free lenses have similar shape to lenses free from spherical aberration.

4.3. Astigmatism

Starting from formula (10) applied to both surfaces of a lens and taking into account formulas (13)–(26) we obtain expression describing III-order astigmatism of a single lens

$$A = nyV^3 - n'y'V'^3. \tag{42}$$

After rearranging Eq. (42), we obtain

$$A = -(yV)^2\Phi^3. \tag{43}$$

The above relation expresses the dependence of astigmatism on the field angle $\omega = yV$. It is necessary to note that formula (43) concerns only a thin lens with input pupil in contact.

In Section 2, we pointed out that in the optical system consisting of the eye and spectacle lens the input pupil is shifted behind the lens at a relatively large distance. This fact has essential influence on the lens aberrations. Therefore, we have to take into account this pupil shift while estimating the III-order aberration coefficients. It has been shown [8] that the aberration coefficients (for the lens with shifted pupil) can be expressed by appropriate aberration coefficients of the same lens with pupil in contact as follows:

$$S_t = S, \tag{44}$$

$$C_t = C - yS, \tag{45}$$

$$A_t = A - 2y_tC + y_t^2S. \tag{46}$$

where y_t is a perpendicular shift of the pupil centre in the lens plane being a consequence of longitudinal pupil shift z_t. As can be seen from Fig. 9, y_t depends on z_t and object location. Depending on whether object point lies at infinity ($v = 0$), or at a finite distance ($v \neq 0$) the dependency between y_t and z_t is either:

$$y_t = \omega z_t, \tag{47}$$

or

$$y_t = \frac{z_t yV}{z_t V - 1}. \tag{48}$$

In the above formulas A, C and S are aberration coefficients of the lens with pupil in contact, but in appropriately shifted (y substituted by $y - y_t$) variables. Coefficient S does not depend on this shift, but formal expressions for coefficients C and A depends on the object location. For infinitely distant object the product yV in formulas (40) and (43) equals field angle ω, so the form of coefficients C and A does not change. In such a situation inserting (30), (40), (43) and (47) into (46) enables us to determine astigmatism of the lens with shifted pupil.

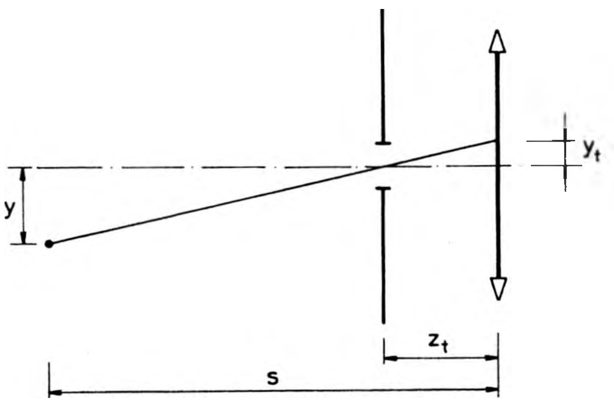

Fig. 9. Lens with the shifted input pupil — geometry relations.

From the formula below, it follows that astigmatism after pupil shift will vanish if

$$y_t = \frac{C \pm \sqrt{C^2 - SA}}{S}. \tag{49}$$

For some combination of coefficients S, C and A it is possible to find such a pupil location that astigmatism is fully compensated. To obtain such correction it is necessary to shift pupil by the calculated distance. If the object is located at infinity (for distant spectacles) it is possible to find direct formula connecting the parameter φ_1 with pupil shift z_t assuring correction of astigmatism. Inserting formulas (30), (40), (43) and (47) into (46) we find two possible values of the input pupil shift assuring full correction of astigmatism

$$z_t = \frac{n-1}{\Phi} \frac{(n+1)\varphi_1 - n^2 \pm \sqrt{\varphi_1^2 - n^2 \varphi_1}}{(n+2)\varphi_1^2 - (2n+1)n\varphi_1 + n^2}. \tag{50}$$

From formula (50) it follows that the solution exists only if the lens shape fulfils the relationships:

$$\varphi_t = \frac{\Phi_1}{\Phi} = \frac{\rho_1 - \rho_2}{\rho_1} \geqslant n^2 \quad \text{or} \quad \varphi_1 \leqslant 0 \tag{51}$$

Using formula (50) we can calculate the value of necessary shift or find out that the desired solution does not exist in each particular case. Formula (50) is more convenient after rearranging it in such a way that for given value of pupil shift it is possible to find the lens parameters assuring astigmatism correction since for the spectacle lens, the value of pupil shift is determined by the spectacle frame

$$\varphi_1 = \frac{n(2n+1)}{2(n+2)} \frac{2(n^2-1) \pm \sqrt{n^2(1-4n)\Phi^2 z_t^2 + 4(n-1)^2(1-n\Phi z_t)}}{2z_t\Phi(n+2)}. \tag{52}$$

The solution exists only if the following condition is fulfilled:

$$n^2(1-4n)\Phi^2 z_t^2+4(n-1)^2(1-n\Phi z_t)\geqslant 0 \tag{53}$$

from which we have the inequality

$$\frac{-2(n-1)[(n-1)+\sqrt{n(n+2)}]}{n(4n-1)z_t}\leqslant\Phi\leqslant\frac{-2(n-1)[(n-1)-\sqrt{n(n+2)}]}{n(4n-1)z_t}. \tag{54}$$

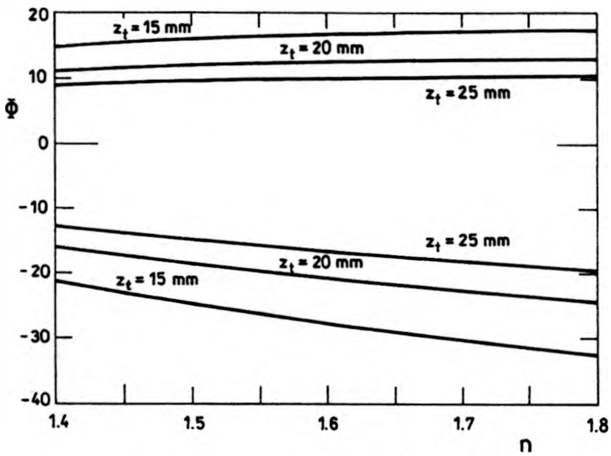

Fig. 10. Range of total focusing power Φ, where the correction of astigmatism is possible versus refractive index n for selected values of input pupil shift z_t.

It means that astigmatism can be corrected by pupil shift only for the limited range of focusing power values. In Figure 10, this range for different pupil locations versus index of refraction is presented. From Eq. (52) we can calculate the values of φ_1 describing the shape of lens with astigmatism corrected by pupil shift. Within the range given by inequality (54) two solutions exist. In the papars [3], [4] they are called Wollaston type and Ostwald type solutions, respectively. It is seen from Fig. 10 that for typical value of input pupil shift (25 mm) the lens power should not exceed +10 D. Lenses of such power (or even greater) are used in high hyperopia or for correction of aphakic eye.*

In Figure 11, the dependence of parameter φ_1 on n for several typical values of pupil shift and the lens of focal power $\Phi = +10$ D is illustrated. It can be seen that if this shift equals $z_t = 25$ mm there are no solutions for refractive index smaller than $n = 1.6$ (on the basis of III-order aberration theory). In order to obtain a solution it is necessary to assume smaller value of z_t, that is, to put the lens closer to the eye.

As numerical examples we considered three typical spectacle lenses of focusing power $\Phi = +10$ D (as discussed above), $\Phi = +2$ D (used in moderate hyperopia)

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* For example, after surgical extraction of the crystalline lens (in the case of cataract).

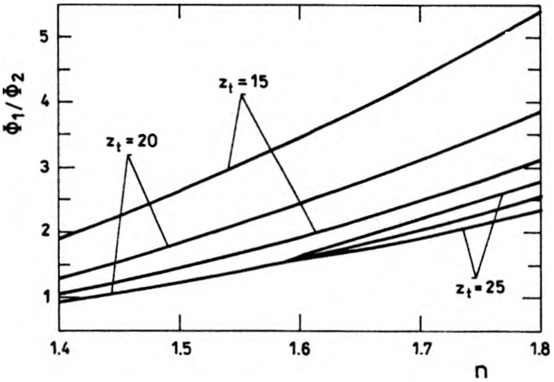

Fig. 11. Dependence parameter $\varphi_1 = \Phi_1/\Phi$ describing the lens shape assuring correction of astigmatism on index of refraction n for a few typical values of the input pupil outset z_1 and focusing power $\Phi = +10$ D and object at infinity.

and $\Phi = -2$ D (for moderate myope). In Tables 4a, b, c, the construction parameters of such lenses with compensated astigmatism are given for object distance $s = \infty$.

If an object to be observed lies at a finite distance (reading spectacles, near vision) the analytic solution of the condition $A_t = 0$ becomes too complex to be useful in practice. In such a situation the numerical methods are applicable in search for the solution.* Nowadays, thanks to fast computers and availability of a number of computer programs for symbolic calculus this makes no problem.

Table. 4a. Example astigmatism – free lenses for distant vision (object located at infinity, lens focusing power $\Phi = +10$ D, input pupil shifted 25 mm behind the lens).

n	φ_1	ρ_1 [mm]	ρ_2 [mm]
1.4		No solution	
1.5		No solution	
1.6	2.666	22.506	36.014
1.7	2.901	24.130	36.823
	3.206	21.834	31.732
1.8	3.241	24.684	35.698
	3.654	21.906	30.166

Example curves presenting the value of astigmatism versus of the lens shape (parameter φ_1) found numerically are presented in Fig. 12. The focusing power of the lens equals $\Phi = 2$ D, however the object distance is assumed to be $s = 40$ cm (typical reading distance). From the curves presented in Fig. 12 it is seen that for each case considered two solutions exist.

* There are also other possibilities of finding the solution. One of them employs the numerical tracing of a chief ray in meridional and sagittal planes (calculation of meridional and sagittal curvatures K_m and K_s). This method, also based on numerical calculation, leads to almost identical results. The other possibilty is to use approximate formulas such as given by BARTKOWSKA [5] and MELANOWSKI [4]. In this paper, however, we restricted ourselves to Seidel aberrations as the most frequently discussed.

Table 4b. Example astigmatism–free lenses for distant vision (object located at infinity, lens of focusing power $\Phi = +2$ D, input pupil shifted 25 mm behind the lens).

n	φ_1	ρ_1 [mm]	ρ_2 [mm]
1.4	4.203	47.585	62.441
	8.655	23.108	26.127
1.5	5.294	47.223	58.221
	10.705	23.354	25.760
1.6	6.444	46.555	55.107
	12.755	23.520	25.521
1.7	7.650	45.752	52.632
	14.803	23.644	25.357
1.8	8.905	44.919	50.601
	16.852	23.736	25.233

Table 4c. Example astigmatis–free lenses for distant vision (object located at infinity, lens of focusing power $\Phi = -2$ D, input pupil shifted 25 mm behind the lens).

n	φ_1	ρ_1 [mm]	ρ_2 [mm]
1.4	−2.472	80.906	57.604
	−7.259	27.552	24.216
1.5	−3.363	74.338	57.300
	−9.210	27.144	24.486
1.6	−4.308	69.638	56.528
	−11.159	26.884	24.675
1.7	−5.304	65.900	55.520
	−13.108	26.701	24.804
1.8	−6.344	63.052	54.466
	−15.057	26.566	24.921

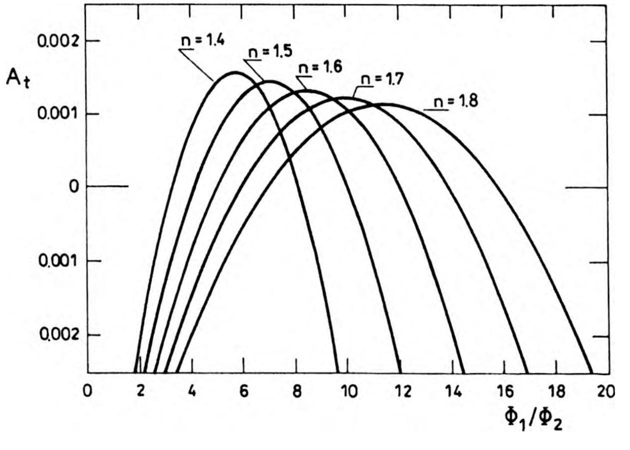

Fig. 12. Value of astigmatism versus the parameter φ_1 determining the lens shape for focusing power $\Phi = +2$ D, object distance $s = -40$ cm and selected values of refractive index n.

The values of parameter φ_1 describing the astigmatism free lenses found numerically for the lenses of focusing powers $\Phi = +10$ D, $\Phi = +2$ D, and $\Phi = -2$ D and selected indices of refraction n are collected in Tables 5a, b, c.

Table 5a. Example astigmatism–free lenses for distant vision (objcet 40 cm before the lens of focusing power $\Phi = +10$ D input pupil shifted 25 mm behind the lens).

n	φ_1	ρ_1 [mm]	ρ_2 [mm]
1.4		No solution	
1.5	1.943	25.733	53.022
	2.271	22.017	39.339
1.7	2.175	27.586	51.064
	2.724	22.026	34.803
1.7	2.444	28.642	48.476
	3.152	22.208	32.528
1.8	2.734	29.261	46.136
	3.570	22.409	31.128

Table 5b. Example astigmatism–free lenses for distant vision (object 40 cm in front of the lens of focusing power $\Phi = +2$ D input pupil shifted 25 mm behind the lens).

n	φ_1	ρ_1 [mm]	ρ_2 [mm]
1.4	3.288	60.872	87.423
	8.158	24.516	27.941
1.5	4.129	60.547	79.898
	10.084	24.792	27.521
1.6	5.023	59.725	74.571
	12.009	24.981	27.250
1.7	5.966	58.666	70.479
	13.933	25.120	27.063
1.8	6.953	57.529	67.293
	15.857	25.225	26.932

Table 5c. Example astigmatism–free lenses for distant vision (object 40 cm in front of the lens of focusing power $\Phi = -2$ D input pupil shifted 25 mm behind the lens).

n	φ_1	ρ_1 [mm]	ρ_2 [mm]
1.4	−1.557	128.452	78.217
	−6.762	29.577	25.767
1.5	−2.199	113.688	78.149
	−8.588	29.110	26.074
1.6	−2.888	103.878	77.160
	−10.413	28.810	26.286
1.7	−3.621	96.658	75.741
	−12.237	28.602	26.441
1.8	−4.393	91.054	74.170
	−14.061	28.447	26.559

5. Conclusions

From the calculations and examples considered we can conclude that single spherical lens can be successfully used as a spectacle lens. Due to the specific mode of operation (small diameter of eye pupil, rotation of eyeball) such aberrations as spherical and comma do not seriously influence the imaging quality. Correction of off-axis astigmatism is the most important task while designing spectacle lenses. This aberration can be corrected thanks to the fact that the input pupil of a system composed of spectacle lens and eye is shifted behind the lens.

The shape of the spectacle lens with astigmatism corrected by pupil shift is given by the solution of the equation determining the parameter φ_1 versus the total lens focusing power Φ and the refractive index n. For typical values of this index varying from $n = 1.4$ to $n = 1.8$ two solutions exist for small focusing powers Φ. One of them, giving greater values of the lens surface radii of curvatures, *i.e.*, the more flat lens (called Ostwald solution) is preferred. For greater focusing powers the solutions exist only if higher values of refractive index can be accepted (*e.g.*, for $\Phi > 10$ D it has to be $n > 1.6$).

The shape of astigmatism-free lens depends on the object distance. The lenses for distant vision (object located at infinity) should be slightly more bent than those designed for near vision even for the same total focusing power.

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