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Diffraction using an amplitude grating object of truncated inverted parabolic shape

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A theoretical model of an inverted parabolic function is considered to represent an amplitude object. The intensity distribution of the diffraction pattern is calculated by applying Fourier transformations and convolution operations upon the object. A computer program is constructed to plot the intensity distribution of the diffraction pattern obtained. In the computations, the maximum heights of the inverted parabolic function are taken to be 2.5, 5, and 10 μm, while the object width is kept constant at 5 μm. The theoretical results of the intensity distribution are graphically represented using MATLAB program. Finally, a discussion and conclusions are presented.

1. Introduction

The scalar theory of diffraction has been successfully applied to compute the point -spread function in coherent optical systems [1] – [4]. The calculations are based on Fourier techniques and optical communication theory [5] – [8] to investigate the image performance of the optical systems.

The precedent publications have considered only the circular and annular apertures. Recently, we have studied different modified apertures [9] – [12] in order to improve the resolution of the optical confocal microscopes. Among these apertures, one aperture has four-fold symmetry and another has eight-fold symmetry [9]. Also, conic [10] and quadratic apertures [11], [12] are investigated. The author [13] has calculated the transfer function of black-and-white concentric equally spaced annuli in confocal imaging systems.

In this study, a theoretical model of a limited comb function having an inverted truncated parabolic shape is assumed. The intensity distribution of the diffraction pattern is calculated taking into consideration the height and width of the amplitude object. A Fortran computer program is constructed to compute the intensity distribution. The theoretical results are plotted, using MATLAB program, followed by a discussion and conclusions.

2. Theoretical analysis

An amplitude object is illuminated by a coherent plane wave emitted from a He-Ne laser at $\lambda = 6328$ Å. This object assumes an inverted parabolic shape having a truncated comb function to represent it. It has a width $d = 2x_0$ and maximum height σ_m. The diffraction pattern is recorded in the Fourier plane of coordinates (u, v). This plane is situated in the focal plane of the converging Fourier lens L in order to record a far field diffraction pattern. A treatment in one dimension is considered, where N is the total number of scattering periods along x-coordinate.

Now, the transmitted complex amplitude for the inverted parabolic object can be represented mathematically as follows:

$$g(x) = \left(\frac{a}{s}\right)\sum_{n=-N/2}^{n=N/2} \sigma_m \left\{1 - \left[\frac{x - nd}{x_0}\right]^2\right\}. \tag{1}$$

The letter a represents the incident amplitude and the letter s is given to represent the effective cross-sectional area of the beam incident upon the object. Equation (1) can be rewritten, making use of convolution operations, as follows:

$$g(x) = \left(\frac{a}{s}\right)\sigma_m\left[1 - \left(\frac{x}{x_0}\right)^2\right] \otimes \sum_{n=-N/2}^{n=N/2} \delta(x - nd). \tag{2}$$

The symbol $\otimes$ is taken to represent the convolution operation and the summation is to represent the limited comb function in one dimension.

The complex amplitude of the diffracted image formed in the focal plane of the converging lens L is obtained by operating the Fourier transform upon Eq. (1) as follows:

$$G(u) = \mathrm{FT}\,[g(x)] = \int_{-\infty}^{\infty} g(x)\exp\left(\frac{-jkxu}{f}\right)\mathrm{d}x \tag{3}$$

where u-coordinate in the Fourier plane corresponds to x-coordinate in the object plane and $k = 2\pi/\lambda$ is the propagation constant. Making the substitution of Eq. (2) to Eq. (3), using convolution operations that transform the convolution product into a simple product through Fourier transform operation, then we obtain

$$G(u) = \left(\frac{a}{s}\right)\sigma_m \mathrm{FT}\left[1 - \left(\frac{x}{x_0}\right)^2\right]\mathrm{FT}\left\{\sum_{n=-N/2}^{n=N/2} \delta(x - nd)\right\}. \tag{4}$$

The transformation of the limited comb function in Eq. (4) is immediately solved applying the Fourier transform to yield this result

$$\mathrm{FT}\left\{\sum_{n=-N/2}^{n=N/2} \delta(x - nd)\right\} = \sum_{n=-N/2}^{n=N/2} \exp\left[-j\left(\frac{k}{f}\right)(nd)u\right]. \tag{5}$$

The other transformation in Eq. (4) is solved as follows:

$$\mathrm{FT}\left[1 - \left(\frac{x}{x_0}\right)^2\right] = \int_{-\infty}^{\infty}\left[1 - \left(\frac{x}{x_0}\right)^2\right]\exp\left(\frac{-jkxu}{f}\right)\mathrm{d}x. \tag{6}$$

This integration is divided into two separate integrals:

$$I_1 = \int \exp\left(\frac{-jkxu}{f}\right) \mathrm{d}x, \tag{7}$$

$$I_2 = \int x^2 \exp\left(\frac{-jkxu}{f}\right) \mathrm{d}x \tag{8}$$

The first definite integral is easily solved as

$$I_1 = \int_{-x_0}^{x_0} \exp\left(\frac{-jkxu}{f}\right) \mathrm{d}x = -\left(\frac{f}{jku}\right) \exp\left(\frac{-jkux}{f}\right)\Bigg|_{-x_0}^{x_0}$$

$$= j\left(\frac{f}{ku}\right)\left[\exp\left(\frac{-jkux_0}{f}\right) - \exp\left(\frac{jkux_0}{f}\right)\right] = 2\left(\frac{f}{ku}\right) \sin\left(\frac{kux_0}{f}\right).$$

Knowing that $k = 2\pi/\lambda$ and defining the sinc function

$$\mathrm{sinc}(x) = \left(\frac{\sin(\pi x)}{\pi x}\right)$$

we get

$$I_1 = 2x_0 \,\mathrm{sinc}\left(\frac{2ux_0}{\lambda f}\right), \tag{9}$$

The second integral is solved, using integration by parts, as follows:

$$I_2 = \int x^2 \exp\left(\frac{-jkxu}{f}\right) \mathrm{d}x = -\left(\frac{f}{jku}\right) \int x^2 \mathrm{d}\left[\exp\left(-\frac{jkxu}{f}\right)\right]$$

$$= j\left(\frac{f}{ku}\right)\left[x^2 \exp\left(\frac{-jkxu}{f}\right) - 2\int x \exp\left(\frac{-jkxu}{f}\right) \mathrm{d}x\right]$$

$$= j\left(\frac{f}{ku}\right)\left\{x^2 \exp\left(\frac{-jkxu}{f}\right) - 2\left(\frac{-f}{jku}\right)\int x \,\mathrm{d}\left[\exp\left(\frac{-jkxu}{f}\right)\right]\right\}$$

$$= j\left(\frac{f}{ku}\right)\left\{x^2 \exp\left(\frac{-jkxu}{f}\right) - 2j\left(\frac{f}{ku}\right)\left[x \exp\left(\frac{-jkux}{f}\right)\right.\right.$$

$$\left.\left. - \int \left[\exp\left(\frac{-jkxu}{f}\right) \mathrm{d}x\right]\right\} = j\left(\frac{f}{ku}\right)\left[x^2 \exp\left(\frac{-jkxu}{f}\right) - 2j\left(\frac{f}{ku}\right) x \exp\left(\frac{-jkux}{f}\right)\right.$$

$$\left. + 2j\left(\frac{f}{ku}\right)\left(\frac{-f}{jku}\right) \exp\left(\frac{-jkux}{f}\right)\right].$$

Hence, I_2 becomes a complex quantity represented as

$$I_2 = \left(\frac{f}{ku}\right) \exp\left(\frac{-jkux}{f}\right)\left\{2x\left(\frac{f}{ku}\right) + j\left[x^2 - 2\left(\frac{f}{ku}\right)^2\right]\right\}. \tag{10}$$

Let $\zeta(x) = 2x\left(\frac{f}{ku}\right)$ and $\eta(x) = x^2 - 2\left(\frac{f}{ku}\right)^2$, then the definite integral is solved after the substitution of the upper and lower limits of integration as follows:

$$I_2 = \left(\frac{f}{ku}\right)\exp\left(\frac{-jkux}{f}\right)\left[\zeta(x) + j\eta(x)\right]\Bigg|_{-x_0}^{x_0},$$

$$\left(\frac{1}{x_0^2}\right)I_2 = \left(\frac{f}{ku}\right)\left\{4\left(\frac{f}{kux_0}\right)\left[1-\sin^2\left(\frac{kux_0}{f}\right)\right]^{1/2} + 2\left[1-2\left(\frac{f}{kux_0}\right)^2\right]\sin\left(\frac{kux_0}{f}\right)\right\}. \quad (11)$$

Equations (9) and (11) are grouped to obtain finally the result for the transformation

$$\mathrm{FT}\left[1-\left(\frac{x}{x_0}\right)^2\right] = 4x_0\left(\frac{f}{kux_0}\right)^2 \operatorname{sinc}\left(\frac{2ux_0}{\lambda f}\right) - 4\left(\frac{f}{kux_0}\right)\left[\left(\frac{f}{ku}\right)^2 - x_0^2 \operatorname{sinc}^2\left(\frac{2ux_0}{\lambda f}\right)\right]^{1/2}, \quad (12)$$

substituting Eqs. (5) and (12) in Eq. (4) we get the formula for the complex amplitude of the image

$$G(u) = 4\left(\frac{a}{s}\right)\left(\frac{\sigma_m}{x_0}\right)\left\{\left(\frac{f}{ku}\right)^2 \operatorname{sinc}\left(\frac{2ux_0}{\lambda f}\right) - \left(\frac{f}{ku}\right)\left[\left(\frac{f}{ku}\right)^2 - x_0^2 \operatorname{sinc}^2\left(\frac{2ux_0}{\lambda f}\right)\right]^{1/2}\right\} \times \sum_{n=-N/2}^{n=N/2} \exp\left[-j\left(\frac{k}{f}\right)(nd)u\right]. \quad (13)$$

The summation over the exponential term is decomposed into real and imaginary parts:

$$\sum_{n=-N/2}^{n=N/2} \exp\left[-j\left(\frac{k}{f}\right)(nd)u\right] = \sum_{n=-N/2}^{n=N/2} \cos\left[\left(\frac{k}{f}\right)(nd)u\right] - j\sin\left[\left(\frac{k}{f}\right)(nd)u\right]. \quad (14)$$

The modulus square of the complex term becomes a real quantity which will appear in the intensity distribution of the diffracted image

$$I(u) = |G(u)|^2. \quad (15)$$

Substituting the respective magnitudes from Eq. (13) and (14) to Eq. (15), we finally get for the intensity distribution of the diffracted image the following equation:

$$I(u) = \alpha\left(\frac{1}{u^4}\right)\left\{\operatorname{sinc}\left(\frac{2ux_0}{\lambda f}\right) - \left[1-\left(\frac{kux_0}{f}\right)^2 \operatorname{sinc}^2\left(\frac{2ux_0}{\lambda f}\right)\right]^{1/2}\right\}^2 \times \left\{\left|\sum_{n=-N/2}^{n=N/2} \cos\left[\left(\frac{k}{f}\right)(nd)u\right]\right|^2 + \left|\sum_{n=-N/2}^{n=N/2} \sin\left[\left(\frac{k}{f}\right)(nd)u\right]\right|^2\right\} \quad (16)$$

where α represents the maximum output intensity

$$\alpha = \left(\frac{I_0}{\pi^4}\right)\left(\frac{\sigma_m}{x_0}\right)^2\left[\frac{(\lambda f)^4}{s}\right]^2. \tag{17}$$

The ratio between the input intensity $I_0 = |a|^2$ and α is

$$\text{Ratio} = \frac{\alpha}{I_0} = \left(\frac{I}{\pi^4}\right)\left(\frac{\sigma_m}{x_0}\right)^2\left[\frac{(\lambda f)^4}{s^2}\right]. \tag{18}$$

It is clear that the parameter α is dependent upon the following:

i) the incident intensity $I_0 = |a|^2$ for monochromatic illumination,
ii) the cross-sectional area of the incident beam s,
iii) the maximum height of the object σ_m,
iv) the object width $d = 2x_0$,
v) the focal length f of the lens L.

3. Theoretical results and discussion

A computer program is constructed to represent the model of the object. This object assumes a repetitive truncated inverted parabolic shape $g(x)$ as shown in Fig. 1. Three different heights of 2.5, 5, 10 μm are taken while the width is kept constant at 5 μm.

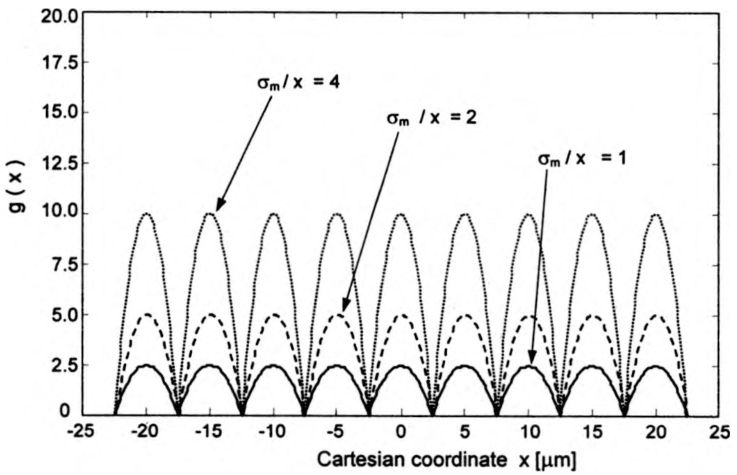

Fig. 1. Repetitive truncated inverted parabolic function to represent an amplitude object of width $2x_0$ = 5 μm at different heighst of σ_m = 2.5, 5, 10 μm.

Another Fortran computer program is designed to compute the intensity distribution of the diffracted image represented by Eq. (16). Figure 2 represents the intensity $I(u)$ at $\sigma_m/x_0 = 1$ in the range extending from $u = 0$ to $u = 10000$ μm. It is clear that the intensity shape is nearly a sinc^2 function. In Fig. 3, the magnified

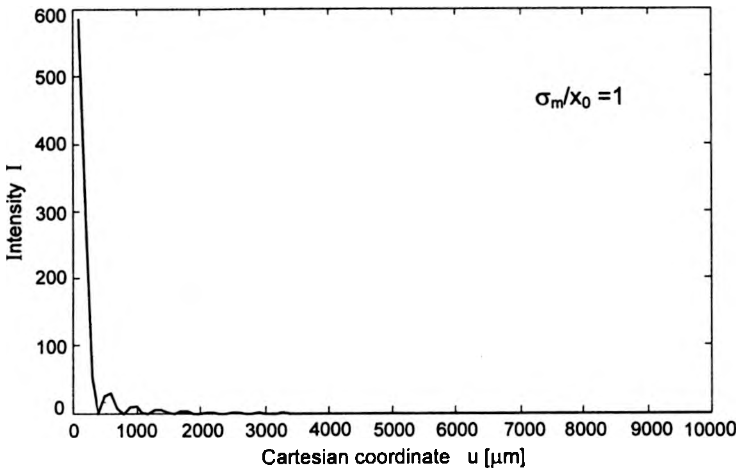

Fig. 2. Intensity distribution of the diffraction pattern *vs.* Cartesian coordinate u for an inverted parabolic shape of height $\sigma_m = 2.5$ μm and half width $x_0 = 2.5$ μm.

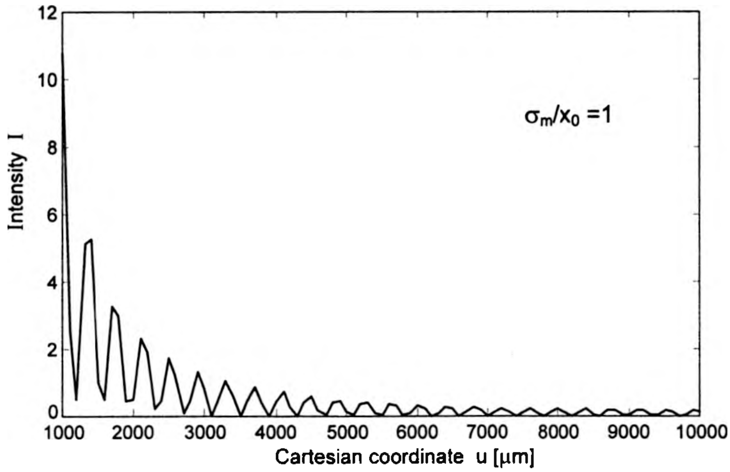

Fig. 3. Magnified portion of the intensity distribution $I(u)$ in the range from $u = 1000$ μm up to 10000 μm ($\sigma_m = 2.5$ μm and $x_0 = 2.5$ μm).

part of the intensity $I(u)$ in the range from $u = 1000$ μm up to $u = 10000$ μm is plotted. It is clear that the average width $\langle u \rangle$ is obtained from Fig. 3 in the range from $u = 5000$ μm up to 7000 μm as follows: $u = 2000/5 = 400$ μm which is in agreement with the theoretical value obtained from the linear imaging system as $u = \lambda f/(2x_0 N)$. In the object model, N is taken as $N = 65$ total number of truncated inverted parabola, $\lambda = 0.6328$ μm, $f = 20$ cm and $x_0 = 2.5$ μm. Hence, $u = 389.4$ μm. An error of only 2.65% between the computed value attained in Fig. 3 and the theoretical value obtained from the Fourier transformation is considered

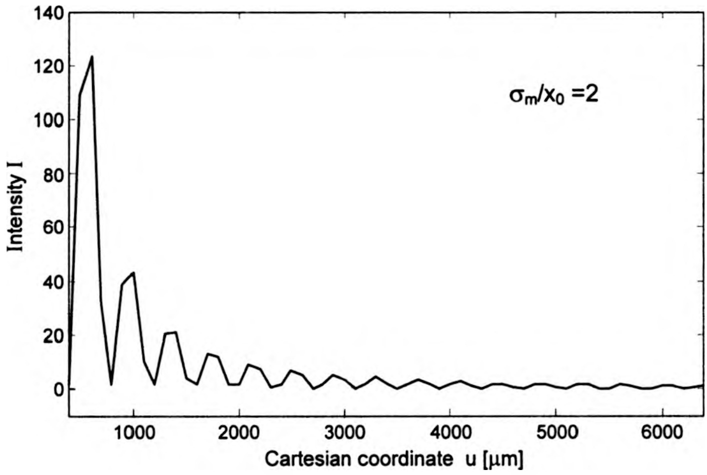

Fig. 4. Intensity distribution of the diffraction pattern *vs.* Cartesian coordinate u for an inverted parabolic shape of height $\sigma_m = 5$ μm and half width $x_0 = 2.5$ μm in the range from 300 μm up to 6300 μm.

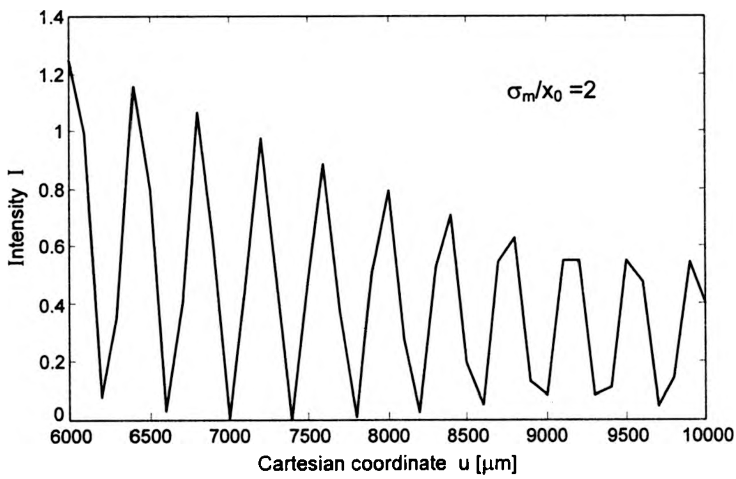

Fig. 5. Intensity distribution of the diffraction pattern *vs.* Cartesian coordinate u for an inverted parabolic shape of height $\sigma_m = 5$ μm and half width $x_0 = 2.5$ μm in the range from 6000 μm up to 10000 μm.

reasonable. A set of curves of $I(u)$ are plotted as in Figs. 4 and 5 for $\sigma_m/x_0 = 2$. The curve in Fig. 4 is drawn from $u = 300$ μm up to 6300 μm, while the other curve in Fig. 5 is plotted in the range from $u = 6000$ μm up to 10000 μm. A third set of curves is plotted as in Figs. 6–8 for $\sigma_m/x_0 = 4$. The curve plotted in Fig. 6 is for the global range of $u = 10000$ μm, the other curve in Fig. 7 is a magnified part of the intensity $I(u)$ in the range from $u = 0$ up to 3000 μm, while the curve plotted in Fig. 8 is drawn in the range from 4000 μm up to 10000 μm. It follows from Figs. 2–8, obtained from the theoretical calculations of Eq. (16) that the average

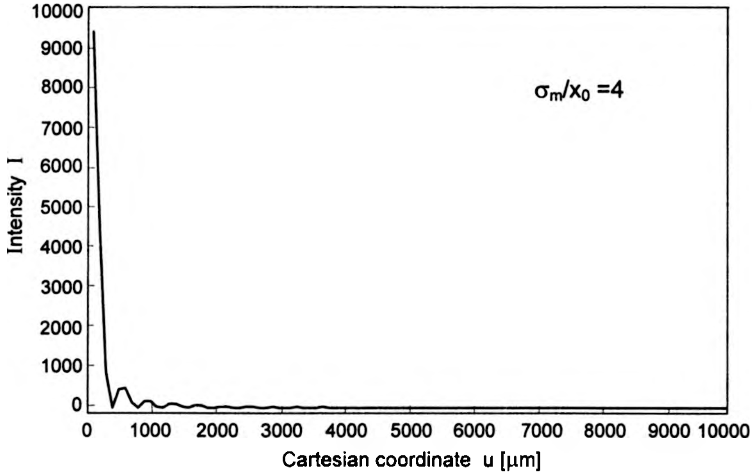

Fig. 6. Intensity distribution of the diffraction pattern *vs.* Cartesian coordinate u for an inverted parabolic shape of height $\sigma_m = 10$ μm and half width $x_0 = 2.5$ μm.

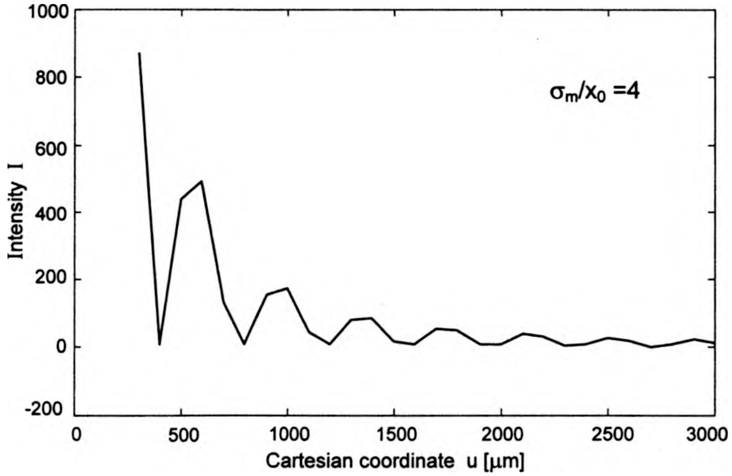

Fig. 7. Magnified portion of the intensity distribution $I(u)$ in the range from $u = 300$ μm up to 3000 μm ($\sigma_m = 10$ μm, $x_0 = 2.5$ μm).

width $\langle u \rangle = 400$ μm for the same width $2x_0$ of the truncated inverted parabolic object for different $\sigma_m = 2.5$, 5, 10 μm. Secondly, the heights of the intensity harmonics have greater values for greater heights of an object for any value of parameter u.

4. Conclusion

We have suggested a repetitive truncated inverted parabolic shape to represent an amplitude object. We have calculated the intensity distribution of the diffracted

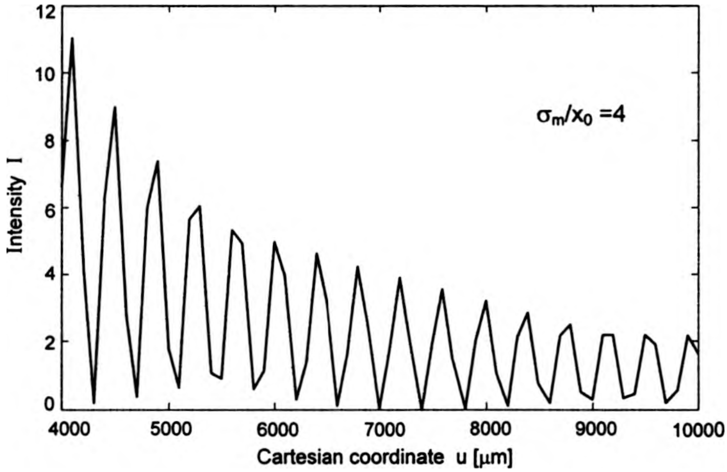

Fig. 8. Magnified portion of the intensity distribution $I(u)$ in the range from $u = 4000$ μm up to 10000 μm ($\sigma_m = 10$ μm and $x_0 = 2.5$ μm).

image using Fourier techniques and convolution operations. From the theoretical results we conclude that firstly, the intensity distribution has a nearly sinc^2 function form and the average width is computed. Secondly, the height of the amplitude σ_m only affects the heights of the intensity peaks while its shape remains unchanged. For higher values of σ_m the intensity heights increase. These concluding remarks are presented assuming the same width $2x_0$ of the object shape.

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